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# Exit Times for Some Processes with Normally Distributed Noise



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## Preface

The work on this thesis was carried out at the Department of Mathematics at Åbo Akademi University during the years 2004-2014, with some intermissions. During this time, I have always found the staff of the department friendly and supportive, and I am happy to have had the opportunity to work in such a pleasant environment. In particular, my supervisor, professor emeritus Göran Högnäs, has always been encouraging, and without his support, this thesis would never have been finished.

I would like to thank professor Linda J. S. Allen for reviewing my thesis, and professor Timo Koski for taking the time both to review and to act as an opponent.

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I would also like to thank my friends and my family, in particular Adde and Ida, for making my life fun and always being there for me.

Brita Jung



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## List of papers

This thesis consists of a summary and the following papers:

- I. B. Ruths, Exit times for past-dependent systems, *Surveys of Applied and Industrial Mathematics (Obozrenie prikladnoy i promyshlennoy matematiki)* 15 (2008), 1, 25-30.
- II. G. Högnäs, B. Jung, Analysis of a stochastic difference equation: Exit times and invariant distributions, *Fasc. Math.* 44 (2010), 69-74.
- III. B. Jung, The Cauchy-Frullani integral formula extended to double integrals, *Math. Sci.* 37 (2012), 1-6.
- IV. B. Jung, Exit times for multivariate autoregressive processes, *Stoch. Proc. Appl.* 123 (2013), 8, 3052-3063.
- V. B. Jung, Simulations of a model for the Northern Spotted Owl, submitted to *Natur. Resource Modeling*.

Paper **II**, contribution of the authors:

The main idea of the paper, which is the comparison of exit times with return times, so that Cogburn's result can be applied to exit times, is due to G. Högnäs. B. Jung carried out the work and wrote the paper.

## Abstrakt

Denna avhandling handlar om metoder för att hitta begränsningar för det asymptotiska beteendet hos en förväntad uthoppstid från ett område omkring en fixpunkt för processer som har normalfördelad störning. I huvudsak behandlas olika typer av autoregressiva processer. Fyra olika metoder används. En metod som använder principen för stora avvikelser samt en metod som jämför uthoppstiden med en återkomsttid ger övre begränsningar för den förväntade uthoppstiden. En martingalmetod och en metod för normalfördelade stokastiska variabler ger undre begränsningar. Metoderna har alla både förtjänster och nackdelar. Genom att kombinera de olika metoderna får man de bästa resultaten. Vi får fram gränsvärdet för det asymptotiska beteendet hos en uthoppstid för den multivariata autoregressiva processen, samt motsvarande gränsvärde för den univariata autoregressiva processen av ordning  $n$ .



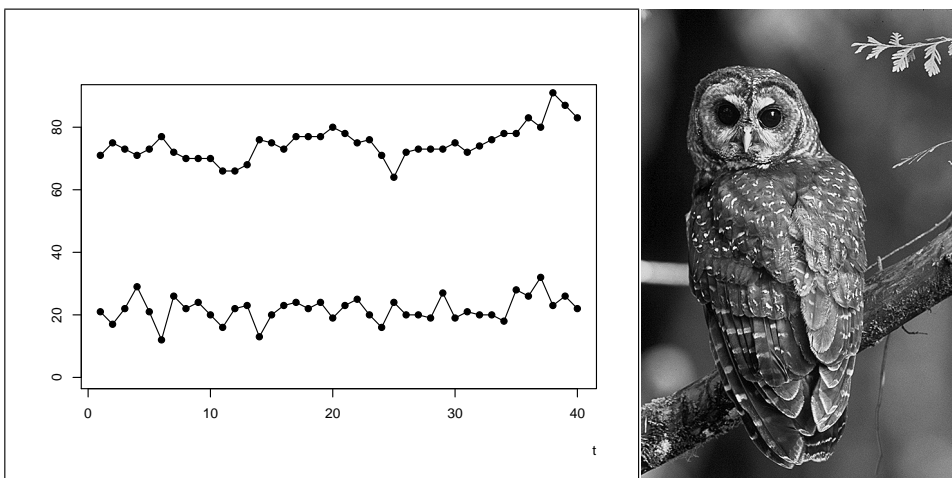


# 1 Introduction

When trying to look into the future, one may find a mathematical model a very useful tool. This holds in many situations, but let us for the moment focus on the prediction of the future of some biological population. This could be the growth of a population of salmon in a fish farm or some culture of bacteria, or prediction of the survival or extinction of some animal population.

A population can be modelled by a sequence of real numbers, which describe the number of individuals in each generation, or the density of the population in each year. It is natural that the density or the number of individuals should depend on the value in the previous time step. One can also have vector-valued sequences. Often, deterministic models are used, but nature is not deterministic. To make a model more true to life, one can introduce an element of randomness in it. However, this adds difficulty to the analysis of the model as well.

Figure 1: A simulation of 40 years of a bivariate model of the Northern Spotted Owl. (Photograph by U.S. Fish and Wildlife Service, J. and K. Hollingsworth.)



(a)  $P_t$ , the number of pairs (above) and  $S_{m,t}$ , the number of single males (below).

(b) The Northern Spotted Owl

As an example of a population model, consider figure 1. Here, we see the short-term behaviour of a bivariate model of the Northern Spotted Owl (for details of the model, see Allen et. al ([1]) or Jung (paper V)). At

a given time (year)  $t$ , the population is described by a vector  $(P_t, S_{m,t})^T$ , where  $P_t$  is the number of pairs of owls and  $S_{m,t}$  is the number of single male owls.  $(P_{t+1}, S_{m,t+1})^T$  then depends on the previous step  $(P_t, S_{m,t})^T$  and on a random variable as well. For the parameter values chosen in our simulation, the process stays in the neighbourhood of the fixed point  $(71.44, 20.98)$ . This means that the population consists of about 71 pairs of owls, about 21 single male owls and about 21 single female owls (since the numbers of male and female owls are assumed to be the same in this model).

The owl population in that simulation survived for at least forty years, but, as we know from real life, in the long run all populations become extinct. The question is how long the time until extinction is.

Since models describing real populations (such as the model for the Northern Spotted Owl) are often very complicated, we will now restrict our study to much simpler models, to be able to obtain some theoretical results. This is the reason why the focus in this thesis lies on autoregressive processes of different types (univariate and multivariate), with normally distributed noise. For these processes, we consider methods that can be used for obtaining bounds of the asymptotics of the expected exit time from a set around the fixed point of the process.

This summary is based on papers **I-V** in the list on page 2. In chapter 2, we use the large deviation principle to get upper bounds of the expected exit times of some autoregressive processes as  $\varepsilon$  becomes small. The chapter is based on papers **I** and **IV**. In chapter 3, which is based on papers **I** and **III**, and also contains previously unpublished material, we consider a martingale method that gives lower bounds for some models. Chapter 4 is about a method for lower bounds for normally distributed variables and is based on paper **IV**. In chapter 5, which is based on paper **II**, we study a method that connects the exit time with the stationary distribution of the process. Paper **V**, which is a simulation study of the owl population mentioned above, is discussed briefly in the final chapter.

## 2 The large deviation method

In this chapter, which is based on papers **I** and **IV**, we use the large deviation principle to get an upper bound of an exit time. Consider first a family of past-dependent processes of the form

$$X_t = f(X_{t-1}) + \varepsilon \xi_t, \quad t \geq 1, X_0 = x_0, \quad (1)$$

where  $X_t \in \mathbb{R}$ ,  $f : \mathbb{R} \mapsto \mathbb{R}$  is a continuous function with a fixed point at the origin (that is,  $f(0) = 0$ ),  $\varepsilon$  is a small positive parameter and  $\{\xi_t\}_{t \geq 1}$  is a sequence of independent and identically distributed standard normal random variables (with mean 0 and variance 1). The starting point of the process is  $X_0 = x_0 \in (-1, 1)$ , and we will study the time until the process leaves the interval  $(-1, 1)$ . Thus, the exit time  $\tau$  is defined by

$$\tau := \min\{t \geq 1 : |X_t| \geq 1\}. \quad (2)$$

As an example, we illustrate the process in the case when  $f(x) = ax$  in figure 2. When  $|a| < 1$ , the process stays near the origin for a while, and then exits from the interval  $(-1, 1)$ . Obviously, the exit time grows larger as the parameter  $\varepsilon$  gets smaller (and if  $\varepsilon$  were to be 0, no exit at all would take place).

By using the large deviation principle we will get bounds on the asymptotic behaviour of the expectation of the exit time  $\tau$  as  $\varepsilon$  approaches zero. That is, we will have a bound on the rate at which  $E\tau$  grows as  $\varepsilon \rightarrow 0$ .

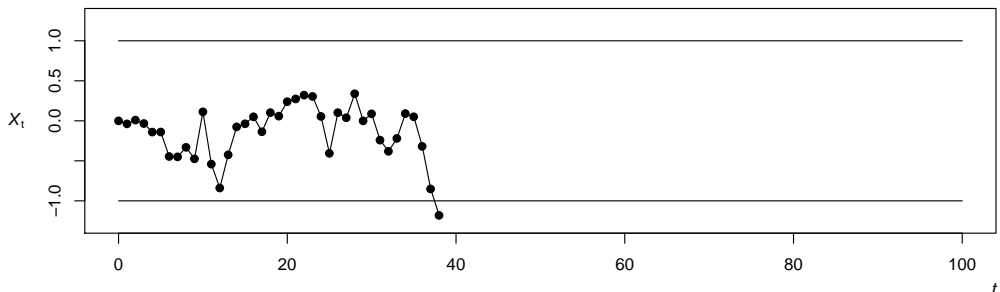
We begin by defining the large deviation principle. The following definition is the one used by Varadhan ([15]):

**Definition.** *A family of probability measures  $\{P_\varepsilon\}$  on the Borel subsets of a complete separable metric space  $Z$  satisfies the large deviation principle with rate of speed  $q(\varepsilon)$  and a rate function  $I$  if there exists a function  $I$  from  $Z$  into  $[0, \infty]$  such that  $0 \leq I(z) \leq \infty$  for all  $z \in Z$ ,  $I$  is lower semicontinuous, the set  $\{z : I(z) \leq m\}$  is compact in  $Z$  for all  $m < \infty$  and*

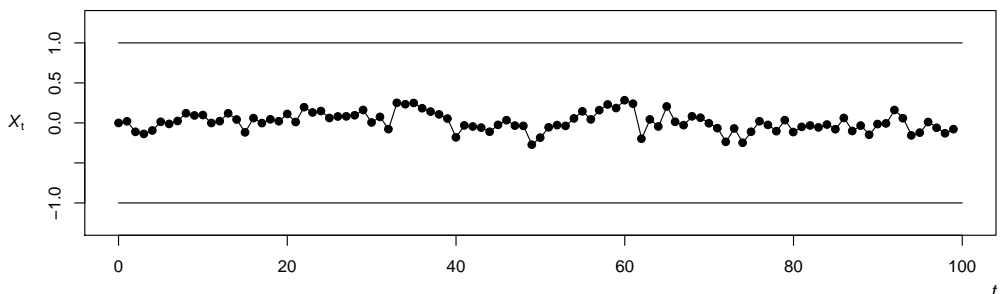
$$\begin{aligned} \limsup_{\varepsilon \rightarrow 0} q(\varepsilon) \log P_\varepsilon(C) &\leq - \inf_{z \in C} I(z) \text{ for every closed set } C \subset Z, \\ \liminf_{\varepsilon \rightarrow 0} q(\varepsilon) \log P_\varepsilon(G) &\geq - \inf_{z \in G} I(z) \text{ for every open set } G \subset Z. \end{aligned}$$

Now, for the family of processes defined in equality 1, it has been shown by Klebaner and Liptser ([8]) that a large deviation principle holds with rate

Figure 2: Sample paths of the process  $X_t = aX_{t-1} + \varepsilon\xi_t$  for  $t \geq 1$ ,  $X_0 = 0$ , for  $a = 0.5$  and  $\varepsilon = 0.4$  and  $0.1$ , respectively.



(a) For  $\varepsilon = 0.4$ , the process exits from the interval  $(-1, 1)$  at the time  $t = 38$ .



(b) For  $\varepsilon = 0.1$ , the process is less volatile and no exit takes place during the first 100 steps.

of speed  $q(\varepsilon) = \varepsilon^2$  and rate function

$$J(\bar{u}) = \begin{cases} \frac{1}{2} \sum_{t=1}^{\infty} (u_t - f(u_{t-1}))^2, & \text{when } u_0 = x_0, \\ \infty, & \text{otherwise.} \end{cases} \quad (3)$$

When we know that the large deviation principle holds, we can calculate the limit of an expression of the form  $q(\varepsilon) \log P(B)$  for a set  $B$ , by computing infima of the rate function. For the exit time  $\tau$ , one can show that

$$E\tau \leq \frac{2M}{\inf_{x_0 \in [-1,1]} P_{x_0}(\tau \leq M)}, \text{ and} \quad (4)$$

$$E\tau \geq \frac{1}{\sup_{x_0 \in [-1,1]} P_{x_0}(\tau \leq M)}, \quad (5)$$

for any positive integer  $M$  (for details, see paper **I**). Here, the index  $x_0$  in  $P_{x_0}$  denotes that the starting point of the process is  $x_0$ . The exit time  $\tau$  is

smaller than or equal to  $M$  if and only if  $(X_0, X_1, \dots) \in B$ , where

$$B = \{(u_0, u_1, \dots) | u_0 = x_0 \text{ and } |u_t| \geq 1 \text{ for some } t, 1 \leq t \leq M\}.$$

By using these inequalities for the choice  $M = 1$ , and thus minimizing the rate function when  $|u_1| \geq 1$ , we get the following result for any function  $f$  that satisfies the conditions:

**Theorem 2.1.** *(For a proof, see paper I.) For the past-dependent process defined in equality 1,*

$$\limsup_{\varepsilon \rightarrow 0} \varepsilon^2 \log E\tau \leq \frac{1}{2}, \text{ and} \tag{6}$$

$$\liminf_{\varepsilon \rightarrow 0} \varepsilon^2 \log E\tau \geq \frac{1}{2} \left(1 - \max_{x \in [-1, 1]} |f(x)|\right)^2. \tag{7}$$

For the function  $f(x) \equiv 0$ , these upper and lower bounds coincide, and it follows that  $\lim_{\varepsilon \rightarrow 0} \varepsilon^2 \log E\tau = 1/2$ . This can of course also be calculated directly, since the process is then just a sequence of independent and identically distributed standard normal random variables.

By specifying the function  $f$ , and using larger values of  $M$ , we will get better bounds. To be able to determine the infimum of the rate function explicitly, we need to choose  $f$  in a simple way.

## 2.1 The autoregressive process

The choice  $f(x) = ax$  in equality 1 gives what is called the autoregressive process of order one, defined by

$$X_t = aX_{t-1} + \varepsilon\xi_t, \quad t \geq 1, \quad X_0 = x_0, \tag{8}$$

where  $X_t \in \mathbb{R}$ ,  $\varepsilon > 0$ ,  $x_0$  is the starting point of the process, and  $\{\xi_t\}_{t \geq 1}$  is a sequence of independent and identically distributed standard normal random variables. We make the assumption that  $|a| < 1$ . Then the process has a stationary distribution which is normal with mean 0 and variance  $\varepsilon^2/(1-a^2)$ . For the exit time

$$\tau = \min\{t \geq 1 : |X_t| \geq 1\}, \tag{9}$$

we have the following result:

**Lemma 2.2.** *For the autoregressive process, defined in equality 8,*

$$\limsup_{\varepsilon \rightarrow 0} \varepsilon^2 \log E\tau \leq \frac{1 - a^2}{2}. \quad (10)$$

Proof: Shown by Klebaner and Liptser in [8]. The proof is also included in paper **I**. It uses the upper bound in inequality 4 for an arbitrarily large value of  $M$ .

It turns out that using the lower bound in inequality 5 for larger values of  $M$  gives no better bound than the one already achieved for  $M = 1$  in inequality 7. That is, we get the bound

$$\liminf_{\varepsilon \rightarrow 0} \varepsilon^2 \log E\tau \geq \frac{(1 - |a|)^2}{2}, \quad (11)$$

for the autoregressive process.

We note the correspondence between the variance  $\varepsilon^2/(1 - a^2)$  of the stationary distribution of the autoregressive process and the bound of the upper limit in lemma 2.2. It will be seen later, that this bound is sharp.

## 2.2 The multivariate autoregressive process

In this section, which is based on paper **IV**, we extend the large deviation method to a multivariate case. We consider a multivariate version of the autoregressive process, where

$$X_t = AX_{t-1} + \varepsilon\xi_t, t \geq 1, X_0 = x_0, \quad (12)$$

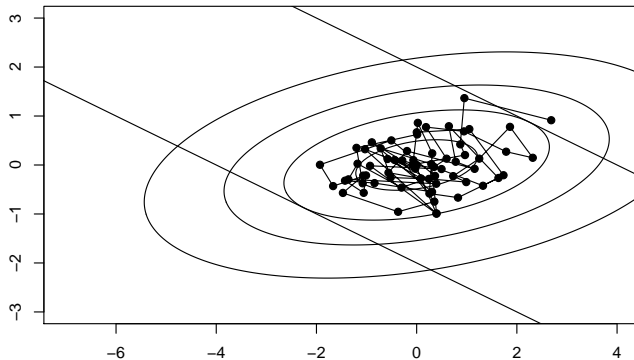
where  $X_t \in \mathbb{R}^d$ ,  $A$  is a real  $d \times d$  matrix,  $\varepsilon > 0$  and  $\{\xi_t\}_{t \geq 1}$  is a sequence of independent and identically distributed multivariate normal random variables with mean zero and covariance matrix  $I$  (the identity matrix). We make the assumption that all eigenvalues of  $A$  have absolute values that are smaller than one. The process then has a stationary distribution which is multivariate normal with the zero vector as mean, and the covariance matrix  $\varepsilon^2\Sigma_\infty$ , where  $\Sigma_\infty$  satisfies the equation

$$\Sigma_\infty = A\Sigma_\infty A^T + I. \quad (13)$$

In the univariate case, we considered exits from the interval  $(-1, 1)$ . The corresponding exit time for the multivariate autoregressive process is

$$\tau = \min\{t \geq 1 : |c^T X_t| \geq 1\}, \quad (14)$$

Figure 3: A sample path of the bivariate autoregressive process when  $A = \begin{pmatrix} 0.7 & 1 \\ 0 & 0.5 \end{pmatrix}$  and  $\varepsilon = 0.5$ , and the level curves of the stationary distribution. The vector  $c$  is chosen as  $(-0.25, -0.5)$  and the exit time is  $\tau = 65$ .



where  $c$  is a vector in  $\mathbb{R}^d$ ,  $c \neq (0, \dots, 0)^T$ . In figure 3, we see a sample path of a bivariate autoregressive process and its exit from a set of this type.

For the univariate autoregressive process, we used the large deviation principle shown by Klebaner and Liptser ([8]). We can show that a large deviation principle holds for the family of multivariate autoregressive processes as well:

**Theorem 2.3.** *(For a proof, see paper IV.) For the multivariate autoregressive process defined in equality 12, a large deviation principle holds with rate of speed  $q(\varepsilon) = \varepsilon^2$  and rate function*

$$I(y_0, y_1, \dots) = \frac{1}{2} \sum_{t=1}^{\infty} (y_t - Ay_{t-1})^T (y_t - Ay_{t-1}), \quad (15)$$

where  $y_0 = x_0$ ,  $y_1, y_2, \dots \in \mathbb{R}^d$ .

As could be expected, the rate function is the corresponding multivariate version of the rate function in equality 3 in the univariate case. In the univariate case we used inequality 4 to prove an upper bound. In this case, the corresponding inequality is

$$E\tau \leq \frac{2M}{\inf_{|c^T x_0| < 1} P_{x_0}(\tau \leq M)}, \quad (16)$$



where  $\tau$  is defined in equality 14. By using this upper bound for an arbitrarily large  $M$ , we can prove the following bound on the upper limit:

**Lemma 2.4.** *(For a proof, see paper IV.) For  $\tau$  defined as in equality 14, we have the following bound of the upper limit:*

$$\limsup_{\varepsilon \rightarrow 0} \varepsilon^2 \log E\tau \leq \frac{1}{2c^T \Sigma_\infty c}, \quad (17)$$

where  $c$  is the vector in the definition of  $\tau$  and  $\varepsilon^2 \Sigma_\infty$  is the covariance matrix of the stationary distribution.

We note that the bound depends on the covariance matrix of the stationary distribution, as in the univariate case. We will show later that the bound is sharp.

## 2.3 The autoregressive process of order $n$

In the previous section, we extended the autoregressive process to the multi-dimensional case. Now, we consider another extension of the autoregressive process. This is the univariate autoregressive process of order  $n$ , where  $X_t$  depends on the  $n$  previous steps, instead of only the last one. This process is defined by the recursion formula

$$X_t = b_1 X_{t-1} + \dots + b_n X_{t-n} + \varepsilon \xi_t, t \geq n, X_0 = x_0, \dots, X_{n-1} = x_{n-1}, \quad (18)$$

where  $X_t \in \mathbb{R}$ ,  $b_1, \dots, b_n$  are real parameters,  $\varepsilon > 0$  and  $\{\xi_t\}_{t \geq n}$  is a sequence of i.i.d. standard normal random variables. The exit time considered is

$$\tau = \min\{t \geq n : |X_t| \geq 1\} \quad (19)$$

(as in the case with the autoregressive process of order 1). We can use the large deviation method here as well. This is easiest if we see the process as a multivariate process. Let

$$Y_t = (X_t, \dots, X_{t-n+1})^T. \quad (20)$$

Then the process can be written as

$$Y_t = B Y_{t-1} + \varepsilon (\xi_t, 0, \dots, 0)^T, t \geq n, Y_{n-1} = (x_{n-1}, \dots, x_0)^T, \quad (21)$$

where

$$B = \begin{pmatrix} b_1 & b_2 & \cdots & b_n \\ 1 & 0 & \cdots & 0 \\ 0 & \ddots & 0 & 0 \\ 0 & \cdots & 1 & 0 \end{pmatrix}. \quad (22)$$

We make the assumption that the parameters  $b_1, \dots, b_n$  are such that all eigenvalues of the matrix  $B$  are smaller than one in absolute value. Then the process defined by equality 21 has a stationary distribution, which is multivariate normal with mean  $(0, \dots, 0)^T$  and covariance matrix  $\varepsilon^2 \Sigma_\infty$ , where  $\Sigma_\infty$  satisfies the equation

$$\Sigma_\infty = B \Sigma_\infty B^T + (1, 0, \dots, 0)^T (1, 0, \dots, 0). \quad (23)$$

Since the multivariate process has this stationary distribution, the original univariate autoregressive process of order  $n$  has a stationary distribution which is (univariate) normal with mean 0 and variance  $\varepsilon^2 \sigma^2$ , where

$$\sigma^2 = \sum_{k=0}^{\infty} (B_{11}^k)^2, \quad (24)$$

where  $B_{11}^k$  denotes the element at the first row and the first column of the matrix  $B^k$ .

This multivariate process is similar to the multivariate autoregressive process. The same arguments as in that case can be used to show that the family of processes satisfies a large deviation principle:

**Theorem 2.5.** *(For a proof, see paper IV.) For the family of probability measures induced by the multivariate process  $\{Y_t\}_{t \geq n-1}$ , a large deviation principle holds with rate of speed  $\varepsilon^2$  and rate function*

$$I(y_{n-1}, y_n, \dots) = \begin{cases} \frac{1}{2} \sum_{t=n}^{\infty} ((y_t - B y_{t-1})_1)^2 & \text{if } (y_t - B y_{t-1})_k = 0 \\ & \forall k = 2, \dots, n, \forall t \geq n \\ & \text{and } y_{n-1} = (x_{n-1}, \dots, x_0)^T \\ \infty, & \text{otherwise,} \end{cases}$$

where  $y_{n-1}, y_n, \dots \in \mathbb{R}^n$  and  $(y_t - B y_{t-1})_k$  denotes the  $k$ :th element of the vector  $y_t - B y_{t-1}$ .

Now, for the exit time  $\tau$  defined in equality 19, one can prove the following bound of the upper limit by using the inequality

$$E\tau \leq \frac{2M}{\inf_{|x_0|<1, \dots, |x_{n-1}|<1} P_{x_0, \dots, x_{n-1}}(\tau \leq M)}, \quad (25)$$

for an arbitrarily large  $M$ :

**Lemma 2.6.** *(For a proof, see paper IV.) For the autoregressive process of order  $n$ ,*

$$\limsup_{\varepsilon \rightarrow 0} \varepsilon^2 \log E\tau \leq \frac{1}{2\sigma^2},$$

where  $\varepsilon^2\sigma^2$  is the variance of the stationary distribution of the process.

Again, the bound depends on the variance of the stationary distribution, and, again, the bound is sharp (the corresponding lower bound will be shown later).

### 3 The martingale method

In this chapter, which is partly based on papers I and III, and partly previously unpublished, we consider a method for lower bounds of the asymptotics of an exit time. The method involves a certain martingale, which was introduced by Novikov in [11]. The basis of the method is the Cauchy-Frullani integral formula from calculus: If the function  $f$  is such that the integral

$$\int_{\varepsilon}^A \frac{f(x)}{x} dx$$

exists for all positive  $\varepsilon$  and  $A$ , and the limits

$$f(0) := \lim_{\varepsilon \downarrow 0} \int_{\varepsilon}^A \frac{f(x)}{x} dx \quad \text{and} \quad f(\infty) := \lim_{A \rightarrow \infty} \int_{\varepsilon}^A \frac{f(x)}{x} dx$$

exist, then

$$\int_0^{\infty} \frac{f(ax) - f(bx)}{x} dx = (f(\infty) - f(0)) \log \frac{a}{b}.$$

(For a proof, see for example Ostrowski ([12]).) What is useful to us here, is that the value of the integral does not depend on the function  $f$ , but only on its values at zero and at infinity.

### 3.1 A martingale for the autoregressive process

This section is based on paper **I**, and the details can be found there. Let  $\{X_t\}_{t \geq 0}$  be the univariate autoregressive process, where

$$X_t = aX_{t-1} + \varepsilon\xi_t, t \geq 1, X_0 = x_0, \quad (26)$$

where  $0 < a < 1$ . The case  $-1 < a < 0$  is analogous, and the case  $a = 0$  is not treated at all with this method. Let

$$\varphi(u) := \frac{u^2\varepsilon^2}{2(1-a^2)}. \quad (27)$$

Define the process  $\{N_t\}_{t \geq 0}$  by

$$N_t := \int_0^\infty \frac{\cosh(uX_t) - \cosh(ux_0)}{u} e^{-\varphi(u)} du - t \log \frac{1}{a}, \forall t \geq 0. \quad (28)$$

One can show, by using the Cauchy-Frullani integral formula, that the process  $\{N_t\}_{t \geq 0}$  is a martingale. This implies the following for the exit time  $\tau$  from  $(-1, 1)$  defined in equality 9:

$$E\tau \log \frac{1}{a} = E \left( \int_0^\infty \frac{\cosh(uX_\tau) - \cosh(ux_0)}{u} e^{-\varphi(u)} du \right) \quad (29)$$

$$\geq \int_0^\infty \frac{\cosh(u) - \cosh(ux_0)}{u} e^{-\varphi(u)} du, \quad (30)$$

where the inequality holds because  $|X_\tau| \geq 1$ . Now, the integral on the right hand side in inequality 30 does not depend on the process  $\{X_t\}_{t \geq 0}$  at all, and straightforward calculations give the lower bound in the following lemma:

**Lemma 3.1.** (For a proof, see paper **I**.) For the autoregressive process  $\{X_t\}_{t \geq 0}$ , and the exit time  $\tau = \min\{t \geq 1 : |X_t| \geq 1\}$ ,

$$\liminf_{\varepsilon \rightarrow 0} \varepsilon^2 \log E\tau \geq \frac{1-a^2}{2}. \quad (31)$$

Recall that the corresponding upper bound was given in lemma 2.2. Thus, we know that this lower bound is sharp.

## 3.2 A Cauchy-Frullani double integral

The martingale used in the previous section was tailored to fit the autoregressive process. We would like to use the same method for a multivariate process as well, but then a multivariate version of the Cauchy-Frullani integral formula is needed. In paper **III**, the following bivariate version of the formula was constructed, where

$$g_A(u) = |\det(u, Au)|, \quad (32)$$

and  $(u, Au)$  is the matrix with columns  $u$  and  $Au$ :

**Theorem 3.2.** (For a proof, see paper **III**). Assume that  $u = (u_1, u_2)^T \in \mathbb{R}^2$ ,  $A$  is a real  $2 \times 2$  matrix with non-real eigenvalues  $\lambda(\cos \alpha \pm i \sin \alpha)$ , where  $\sin \alpha \neq 0$  and  $\lambda > 0$ , and the continuous function  $f : \mathbb{R}^2 \mapsto \mathbb{R}$  is such that the limits  $f(\infty) := \lim_{|u| \rightarrow \infty} f(u)$  and  $f(0) := \lim_{|u| \rightarrow 0} f(u)$  exist. Then

$$\int \int_{\mathbb{R}^2} \frac{f(Au) - f(u)}{g_A(u)} du_1 du_2 = (f(\infty) - f(0))C_A, \quad (33)$$

where  $C_A = 2\pi \log \lambda / (\lambda |\sin \alpha|)$ , if  $f$  is such the integral is convergent.

When using the Cauchy-Frullani double integral formula to construct a martingale, we need the following corollary:

**Corollary 3.3.** Under the assumptions in theorem 3.2,

$$\int \int_{\mathbb{R}^2} \frac{f(A^t u) - f(u)}{g_A(u)} du_1 du_2 = t(f(\infty) - f(0))C_A, \quad (34)$$

for any integer  $t \geq 1$ .

Proof: One can write  $f(A^t u) - f(u)$  as the telescoping sum

$$f(A^t u) - f(A^{t-1} u) + f(A^{t-1} u) - \dots + f(Au) - f(u).$$

Then

$$\int \int_{\mathbb{R}^2} \frac{f(A^t u) - f(u)}{g_A(u)} du_1 du_2 = \sum_{i=1}^t \int \int_{\mathbb{R}^2} \frac{f(A^i u) - f(A^{i-1} u)}{g_A(u)} du_1 du_2,$$

where each integral in the sum equals  $(f(\infty) - f(0))C_A$  according to theorem 3.2.

### 3.3 A martingale for the bivariate autoregressive process

Now that we have a Cauchy-Frullani double integral formula, we consider the bivariate autoregressive process  $\{X_t\}_{t \geq 0}$ , where

$$X_t = AX_{t-1} + \varepsilon \xi_t, X_0 = x_0, \quad (35)$$

where  $X_t \in \mathbb{R}^2 \forall t \geq 0$ ,  $A$  is a real  $2 \times 2$  matrix,  $\varepsilon$  is a positive parameter and  $\{\xi_t\}_{t \geq 1}$  is an i.i.d. sequence of bivariate standard normal random variables. For  $t \geq 1$ ,  $X_t$  can be written as

$$X_t = A^t x_0 + \varepsilon \sum_{i=0}^{t-1} A^i \xi_{t-i}. \quad (36)$$

It is then easy to see that  $X_t$  has a bivariate normal distribution with mean  $A^t x_0$  and covariance matrix  $\varepsilon^2 \Sigma_t$ , where

$$\Sigma_t = \sum_{i=0}^{t-1} A^i (A^T)^i. \quad (37)$$

One can also write the matrix  $\Sigma_t$  with a recursion formula,

$$\Sigma_t = A \Sigma_{t-1} A^T + I, \quad (38)$$

where  $I$  is the identity matrix. We make the assumption that the eigenvalues of  $A$  are smaller than one in absolute value, so that the bivariate autoregressive process has a stationary distribution which is bivariate normal with mean  $(0, 0)^T$  and covariance matrix  $\varepsilon^2 \Sigma_\infty$ , where  $\Sigma_\infty$  is

$$\Sigma_\infty = \sum_{i=0}^{\infty} A^i (A^T)^i. \quad (39)$$

The matrix  $\Sigma_\infty$  is also the solution of the equation

$$\Sigma_\infty = A \Sigma_\infty A^T + I. \quad (40)$$

Now that we are going to use a Cauchy-Frullani integral with the denominator  $g_{A^T}(u)$ , we must make the additional assumption that the eigenvalues of  $A$  are non-real. The following results have not been published elsewhere, so the proofs are included here for completeness.

**Lemma 3.4.** *When  $\{X_t\}_{t \geq 0}$  is the bivariate autoregressive process and  $A$  has only non-real eigenvalues with absolute value smaller than one,*

$$N_t := \iint_{\mathbb{R}^2} \frac{\cosh(u^T X_t) - \cosh(u^T x_0)}{g_{A^T}(u)} e^{-\frac{1}{2}\varepsilon^2 u^T \Sigma_\infty u} du_1 du_2 + tC_{A^T}, \quad (41)$$

(where  $C_{A^T}$  is the appropriate constant for a Cauchy-Frullani integral with the matrix  $A^T$ ) is a martingale.

Proof: Clearly,  $N_t$  is  $\mathcal{F}_t$ -measurable when the filtration is  $\mathcal{F}_t = \sigma(X_s, s \leq t)$ . We will now show that  $E|N_t|$  is finite. We have

$$\begin{aligned} E|N_t| &\leq \iint_{\mathbb{R}^2} \frac{E(\cosh(u^T X_t)) - 1}{g_{A^T}(u)} e^{-\frac{1}{2}\varepsilon^2 u^T \Sigma_\infty u} du_1 du_2 \\ &\quad + \iint_{\mathbb{R}^2} \frac{\cosh(u^T x_0) - 1}{g_{A^T}(u)} e^{-\frac{1}{2}\varepsilon^2 u^T \Sigma_\infty u} du_1 du_2 + t|C_{A^T}|. \end{aligned} \quad (42)$$

The second integral on the right hand side is positive. To show that it is bounded from above, split the integration area  $\mathbb{R}^2$  into the sets  $\{g_{A^T}(u) \leq 1\}$  and  $\{g_{A^T}(u) > 1\}$ . Consider the integral over  $\{g_{A^T}(u) \leq 1\}$ . The area is limited and the integrand is positive. The integrand is bounded in a neighbourhood of the origin, because

$$\cosh(u^T x_0) - 1 = \frac{(u^T x_0)^2}{2} + O((u^T x_0)^4) \leq \frac{\|u\|^2 \|x_0\|^2}{2} + O(\|u\|^4), \quad (43)$$

and

$$g_{A^T}(u) = |\det(u, A^T u)| = \|u\|^2 g_{A^T} \left( \frac{u}{\|u\|} \right) \geq \|u\|^2 \min_{\|u\|=1} g_{A^T}(u). \quad (44)$$

Thus

$$\frac{\cosh(u^T x_0) - 1}{g_{A^T}(u)} \leq \frac{\|x_0\|^2}{2 \min_{\|u\|=1} g_{A^T}(u)} + O(\|u\|^2), \quad (45)$$

so it is bounded near the origin, and we get that

$$\iint_{g_{A^T}(u) \leq 1} \frac{\cosh(u^T x_0) - 1}{g_{A^T}(u)} e^{-\frac{1}{2}\varepsilon^2 u^T \Sigma_\infty u} du_1 du_2 \leq K, \quad (46)$$

for a positive constant  $K$  that does not depend on  $\varepsilon$ . For the integral over the area  $\{g_{AT}(u) > 1\}$ , we have

$$\begin{aligned}
& \iint_{g_{AT}(u) > 1} \frac{\cosh(u^T x_0) - 1}{g_{AT}(u)} e^{-\frac{1}{2}\varepsilon^2 u^T \Sigma_\infty u} du_1 du_2 \\
& \leq \iint_{\mathbb{R}^2} \cosh(u^T x_0) e^{-\frac{1}{2}\varepsilon^2 u^T \Sigma_\infty u} du_1 du_2 \\
& = \frac{2\pi}{\varepsilon^2 \sqrt{|\Sigma_\infty|}} \iint_{\mathbb{R}^2} \frac{\cosh(u^T x_0)}{\sqrt{(2\pi)^2 |\frac{1}{\varepsilon^2} \Sigma_\infty^{-1}|}} e^{-\frac{1}{2}\varepsilon^2 u^T \Sigma_\infty u} du_1 du_2 \\
& = \frac{2\pi}{\varepsilon^2 \sqrt{|\Sigma_\infty|}} E(\cosh(x_0^T U)) = \frac{2\pi}{\varepsilon^2 \sqrt{|\Sigma_\infty|}} E(e^{x_0^T U})
\end{aligned}$$

for a bivariate normal random variable  $U$  with mean  $(0, 0)^T$  and covariance matrix  $(1/\varepsilon^2)\Sigma_\infty^{-1}$ . Thus,

$$\iint_{g_{AT}(u) > 1} \frac{\cosh(u^T x_0) - 1}{g_{AT}(u)} e^{-\frac{1}{2}\varepsilon^2 u^T \Sigma_\infty u} du_1 du_2 \leq \frac{2\pi}{\varepsilon^2 \sqrt{|\Sigma_\infty|}} e^{\frac{1}{2\varepsilon^2} x_0^T \Sigma_\infty^{-1} x_0}, \quad (47)$$

which is finite. We now consider the first integral on the right hand side in inequality 42. Recall that  $X_t$  has a normal distribution with mean  $A^t x_0$  and covariance matrix  $\varepsilon^2 \Sigma_t$ . Then

$$E(e^{u^T X_t}) = e^{u^T A^t x_0 + \frac{1}{2}\varepsilon^2 u^T \Sigma_t u}, \quad (48)$$

which implies that

$$E(\cosh(u^T X_t)) = \cosh(u^T A^t x_0) e^{\frac{1}{2}\varepsilon^2 u^T \Sigma_t u}. \quad (49)$$

For the first integral on the right hand side in inequality 42, we then have

$$\begin{aligned}
& \iint_{\mathbb{R}^2} \frac{E(\cosh(u^T X_t)) - 1}{g_{AT}(u)} e^{-\frac{1}{2}\varepsilon^2 u^T \Sigma_\infty u} du_1 du_2 \\
& = \iint_{\mathbb{R}^2} \frac{\cosh(x_0^T (A^T)^t u) e^{\frac{1}{2}\varepsilon^2 u^T \Sigma_t u} - 1}{g_{AT}(u)} e^{-\frac{1}{2}\varepsilon^2 u^T \Sigma_\infty u} du_1 du_2 \\
& = \iint_{\mathbb{R}^2} \frac{h((A^T)^t u, x_0) - h(u, x_0)}{g_{AT}(u)} du_1 du_2 \\
& \quad + \iint_{\mathbb{R}^2} \frac{\cosh(u^T x_0) - 1}{g_{AT}(u)} e^{-\frac{1}{2}\varepsilon^2 u^T \Sigma_\infty u} du_1 du_2, \quad (50)
\end{aligned}$$



where

$$\begin{aligned} h(u, x_0) &= \cosh(x_0^T u) e^{-\frac{1}{2}\varepsilon^2 u^T \Sigma_\infty u}, \\ h((A^T)^t u, x_0) &= \cosh(x_0^T (A^T)^t u) e^{-\frac{1}{2}\varepsilon^2 ((A^T)^t u)^T \Sigma_\infty (A^T)^t u}, \end{aligned}$$

and we have used that  $\Sigma_t - \Sigma_\infty = -A^t \Sigma_\infty (A^T)^t$ . Now, the first integral on the right hand side in equation 50 is finite because it is a Cauchy-Frullani integral, and the second integral was already shown to be finite. Thus, we have shown that  $E|N_t| < \infty$ . It also holds that

$$\begin{aligned} & E(N_{t+1} - N_t | \mathcal{F}_t) \\ &= \iint_{\mathbb{R}^2} \frac{\cosh((A^T u)^T X_t) e^{\frac{1}{2}\varepsilon^2 u^T \Sigma_2 u} - \cosh(u^T X_t)}{g_{A^T}(u)} e^{-\frac{1}{2}\varepsilon^2 u^T \Sigma_\infty u} du_1 du_2 + C_{A^T} \\ &= \iint_{\mathbb{R}^2} \frac{h(A^T u, X_t) - h(u, X_t)}{g_{A^T}(u)} du_1 du_2 + C_{A^T} = 0, \end{aligned}$$

because we get a Cauchy-Frullani double integral that equals  $-C_{A^T}$  (we have  $\lim_{|u| \rightarrow \infty} h(u, X_t) = 0$  and  $\lim_{u \rightarrow 0} h(u, X_t) = 1$ ). Thus,  $\{N_t\}_{t \geq 0}$  is a martingale, and the proof is finished.

In the following lemma, we will use this martingale to get a lower bound of the exit time

$$\tau = \min\{t \geq 1 : |c^T X_t| \geq 1\}, \quad (51)$$

where  $c$  is a vector in  $\mathbb{R}^2$ ,  $c \neq (0, 0)^T$ . (This is the exit time used in chapter 2, where an upper bound was achieved.)

**Lemma 3.5.** *If the starting point  $x_0$  of the bivariate autoregressive process satisfies*

$$x_0^T \Sigma_\infty^{-1} x_0 - \frac{1}{c^T \Sigma_\infty c} < 0,$$

we have

$$\liminf_{\varepsilon \rightarrow 0} \varepsilon^2 \log E\tau \geq \frac{1}{2c^T \Sigma_\infty c}, \quad (52)$$

where  $\varepsilon^2 \Sigma_\infty$  is the covariance matrix of the stationary distribution of the process  $\{X_t\}_{t \geq 0}$ .

Proof: Since  $\{N_t\}_{t \geq 0}$  is a martingale and  $\tau$  is a stopping time with respect to the filtration  $\mathcal{F}_t$ , the stopped process  $\{N_{\tau \wedge t}\}_{t \geq 0}$  (where  $\tau \wedge t = \min(\tau, t)$ ) is also a martingale. Thus,  $EN_{\tau \wedge t} = EN_0 = 0$ , which means that

$$E(\tau \wedge t) = \frac{1}{|C_{A^T}|} E \left( \iint_{\mathbb{R}^2} \frac{\cosh(u^T X_{\tau \wedge t}) - \cosh(u^T x_0)}{g_{A^T}(u)} e^{-\frac{1}{2}\varepsilon^2 u^T \Sigma_\infty u} du_1 du_2 \right).$$

Here,  $-C_{A^T}$  is written as  $|C_{A^T}|$  for clarity, since  $C_{A^T}$  is always negative under our assumption that the eigenvalues of  $A$ , and thus also those of  $A^T$ , have absolute values smaller than one. Now, let  $t \rightarrow \infty$ . It is known that  $E\tau < \infty$  (since  $\tau \leq \min\{t \geq 1 : |\xi_t| \geq 2\}$ , which has finite expectation) and  $\tau \wedge t \uparrow \tau$ , so by the monotone convergence theorem,  $\lim_{t \rightarrow \infty} E(\tau \wedge t) = E\tau$ . For the right hand side, we use Fatou's lemma and get

$$\begin{aligned} \lim_{t \rightarrow \infty} E & \left( \iint_{\mathbb{R}^2} \frac{\cosh(u^T X_{\tau \wedge t}) - \cosh(u^T x_0)}{g_{A^T}(u)} e^{-\frac{1}{2}\varepsilon^2 u^T \Sigma_\infty u} du_1 du_2 \right) \\ &= \lim_{t \rightarrow \infty} E \left( \iint_{\mathbb{R}^2} \frac{\cosh(u^T X_{\tau \wedge t}) - 1}{g_{A^T}(u)} e^{-\frac{1}{2}\varepsilon^2 u^T \Sigma_\infty u} du_1 du_2 \right) \\ & \quad + \iint_{\mathbb{R}^2} \frac{1 - \cosh(u^T x_0)}{g_{A^T}(u)} e^{-\frac{1}{2}\varepsilon^2 u^T \Sigma_\infty u} du_1 du_2 \\ &\geq E \left( \iint_{\mathbb{R}^2} \frac{\cosh(u^T X_\tau) - 1}{g_{A^T}(u)} e^{-\frac{1}{2}\varepsilon^2 u^T \Sigma_\infty u} du_1 du_2 \right) \\ & \quad + \iint_{\mathbb{R}^2} \frac{1 - \cosh(u^T x_0)}{g_{A^T}(u)} e^{-\frac{1}{2}\varepsilon^2 u^T \Sigma_\infty u} du_1 du_2 \\ &= E \left( \iint_{\mathbb{R}^2} \frac{\cosh(u^T X_\tau) - \cosh(u^T x_0)}{g_{A^T}(u)} e^{-\frac{1}{2}\varepsilon^2 u^T \Sigma_\infty u} du_1 du_2 \right). \end{aligned}$$

Thus, we have the inequality

$$E\tau \geq \frac{1}{|C_{A^T}|} E \left( \iint_{\mathbb{R}^2} \frac{\cosh(u^T X_\tau) - \cosh(u^T x_0)}{g_{A^T}(u)} e^{-\frac{1}{2}\varepsilon^2 u^T \Sigma_\infty u} du_1 du_2 \right). \quad (53)$$

Consider the integral on the right hand side.

$$\begin{aligned} E & \left( \iint_{\mathbb{R}^2} \frac{\cosh(u^T X_\tau) - \cosh(u^T x_0)}{g_{A^T}(u)} e^{-\frac{1}{2}\varepsilon^2 u^T \Sigma_\infty u} du_1 du_2 \right) \\ &= E \left( \iint_{\mathbb{R}^2} \frac{\cosh(u^T X_\tau) - 1}{g_{A^T}(u)} e^{-\frac{1}{2}\varepsilon^2 u^T \Sigma_\infty u} du_1 du_2 \right) \\ & \quad + \iint_{\mathbb{R}^2} \frac{1 - \cosh(u^T x_0)}{g_{A^T}(u)} e^{-\frac{1}{2}\varepsilon^2 u^T \Sigma_\infty u} du_1 du_2. \end{aligned} \quad (54)$$

The first integral on the right hand side in equality 54 is greater than or equal to

$$\iint_{\Lambda_i} \frac{\cosh(u^T X_\tau) - 1}{g_{AT}(u)} e^{-\frac{1}{2}\varepsilon^2 u^T \Sigma_\infty u} du_1 du_2 \quad (55)$$

for any set  $\Lambda_i \subseteq \mathbb{R}^2$ , because the integrand is positive. Now we will choose to integrate over a certain set  $\Lambda_i$ . Consider the following four squares:

$$\Lambda_i = \{u \in \mathbb{R}^2 : u = \frac{1}{\varepsilon^2 c^T \Sigma_\infty c} c + (p, q)^T\}, \quad (56)$$

where  $c$  is the vector in the definition of  $\tau$  and

$$\begin{aligned} p, q &\in [0, 1] & \text{for } i = 1, \\ p &\in [0, 1], q \in [-1, 0] & \text{for } i = 2, \\ p &\in [-1, 0], q \in [0, 1] & \text{for } i = 3, \\ p, q &\in [-1, 0] & \text{for } i = 4. \end{aligned}$$

Each of the squares has area 1, and a corner in the point  $(1/(\varepsilon^2 c^T \Sigma_\infty c))c$ . One of these squares is such that

$$|u^T X_\tau| \geq \frac{1}{\varepsilon^2 c^T \Sigma_\infty c} |c^T X_\tau|$$

on that square. For which one of the squares this holds, depends on the sign of  $c^T X_\tau$  and on the signs of the two elements of  $X_\tau$ . This is shown in the following way: We know that  $|c^T X_\tau| \geq 1$  (because an exit takes place at  $t = \tau$ ). If  $c^T X_\tau \leq -1$  and  $X_{\tau,1} > 0$  and  $X_{\tau,2} < 0$ , choose  $\Lambda_3$ . For  $(u_1, u_2)^T \in \Lambda_3$ ,  $u_1 < (1/(\varepsilon^2 c^T \Sigma_\infty c))c_1$  and  $u_2 > (1/(\varepsilon^2 c^T \Sigma_\infty c))c_2$ , so

$$u_1 X_{\tau,1} + u_2 X_{\tau,2} < \frac{1}{\varepsilon^2 c^T \Sigma_\infty c} c_1 X_{\tau,1} + \frac{1}{\varepsilon^2 c^T \Sigma_\infty c} c_2 X_{\tau,2} \quad (57)$$

$$= \frac{1}{\varepsilon^2 c^T \Sigma_\infty c} c^T X_\tau = \frac{-1}{\varepsilon^2 c^T \Sigma_\infty c} |c^T X_\tau|, \quad (58)$$

which implies that

$$|u^T X_\tau| \geq \frac{1}{\varepsilon^2 c^T \Sigma_\infty c} |c^T X_\tau|. \quad (59)$$

One can go through all of the eight possible cases in a similar way. In all of the cases we can choose the suitable set  $\Lambda_i$  and get

$$\begin{aligned} & \iint_{\Lambda_i} \frac{\cosh(u^T X_\tau) - 1}{g_{AT}(u)} e^{-\frac{1}{2}\varepsilon^2 u^T \Sigma_\infty u} du_1 du_2 \\ & \geq (\cosh(\frac{1}{\varepsilon^2 c^T \Sigma_\infty c} |c^T X_\tau|) - 1) \iint_{\Lambda_i} \frac{e^{-\frac{1}{2}\varepsilon^2 u^T \Sigma_\infty u}}{g_{AT}(u)} du_1 du_2, \end{aligned} \quad (60)$$

which is greater than or equal to

$$\left( \cosh\left(\frac{1}{\varepsilon^2 c^T \Sigma_\infty c}\right) - 1 \right) \frac{e^{-\frac{1}{2}\varepsilon^2 (\frac{1}{\varepsilon^2 c^T \Sigma_\infty c} c^T + (p', q')) \Sigma_\infty (\frac{1}{\varepsilon^2 c^T \Sigma_\infty c} c + (p', q')^T)}}{g_{AT}(\frac{1}{\varepsilon^2 c^T \Sigma_\infty c} c + (p', q')^T)} \quad (61)$$

(by the mean value theorem for integrals), for some vector  $(p', q')$ , where  $|p'| < 1$  and  $|q'| < 1$ . By introducing the notation  $v$  for the constant  $1/(c^T \Sigma_\infty c)$ , we can write this expression as

$$\frac{e^{\frac{v}{2\varepsilon^2}} \left(\frac{1}{2} + \frac{1}{2} e^{-\frac{2v}{\varepsilon^2}} - e^{-\frac{v}{\varepsilon^2}}\right) e^{-v(p', q') \Sigma_\infty c} e^{-\frac{1}{2}\varepsilon^2 (p', q') \Sigma_\infty (p', q')^T}}{g_{AT}(\frac{v}{\varepsilon^2} c + (p', q')^T)} \quad (62)$$

The second integral on the right hand side in equality 54 is negative, and can be written as a sum of two integrals over  $\{g_{AT}(u) \leq 1\}$  and  $\{g_{AT}(u) > 1\}$ , respectively. As we know from inequalities 46 and 47 in the proof of lemma 3.4, it is greater than or equal to

$$-K - \frac{2\pi}{\varepsilon^2 \sqrt{|\Sigma_\infty|}} e^{\frac{1}{2\varepsilon^2} x_0^T \Sigma_\infty^{-1} x_0}, \quad (63)$$

where  $K$  is a positive constant. Adding the terms in 62 and 63 together results in the expression

$$\begin{aligned} & \frac{e^{\frac{v}{2\varepsilon^2}}}{g_{AT}(\frac{v}{\varepsilon^2} c + (p', q')^T)} \left[ \left( \frac{1}{2} + \frac{1}{2} e^{-\frac{2v}{\varepsilon^2}} - e^{-\frac{v}{\varepsilon^2}} \right) e^{-v(p', q') \Sigma_\infty c} e^{-\frac{1}{2}\varepsilon^2 (p', q') \Sigma_\infty (p', q')^T} \right. \\ & \quad \left. - g_{AT}(\frac{v}{\varepsilon^2} c + (p', q')^T) \left( K + \frac{2\pi}{\varepsilon^2 \sqrt{|\Sigma_\infty|}} e^{\frac{1}{2\varepsilon^2} x_0^T \Sigma_\infty^{-1} x_0} \right) e^{-\frac{v}{2\varepsilon^2}} \right] \end{aligned} \quad (64)$$

Now, if

$$\frac{1}{2} x_0^T \Sigma_\infty^{-1} x_0 - \frac{v}{2} < 0, \quad (65)$$

that is, if

$$x_0^T \Sigma_\infty^{-1} x_0 - \frac{1}{c^T \Sigma_\infty c} < 0, \quad (66)$$

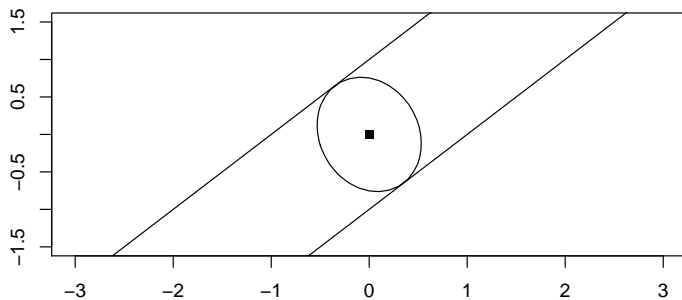
then the product of  $\varepsilon^2$  and the logarithm of expression 64 approaches  $v/2$  as  $\varepsilon \rightarrow 0$ . This implies that

$$\liminf_{\varepsilon \rightarrow 0} \varepsilon^2 \log E\tau \geq \frac{1}{2c^T \Sigma_\infty c}, \quad (67)$$

and the proof is finished.

This lower bound is sharp, considering the corresponding upper bound achieved in the more general multivariate case in lemma 2.4.

Figure 4: An illustration of the condition on  $x_0$  in inequality 66, in the case when  $A = \begin{pmatrix} -0.2 & 0.4 \\ -1 & -0.6 \end{pmatrix}$  and  $c = (1, -1)$ . The condition is satisfied if  $x_0$  is inside the ellipse. The area  $\{x \in \mathbb{R}^2 : |c^T x| < 1\}$ , from which exits take place, is also drawn.



This martingale method cannot be extended directly to the general multivariate case, since the Cauchy-Frullani integral formula does not work for odd-numbered dimensions. The matrix  $A^T$  then necessarily has a real eigenvalue and the denominator  $g_{A^T}(u)$  in the Cauchy-Frullani integral is zero along the corresponding eigenvector.

## 4 A lower bound for normal random variables

This chapter is about another method of getting lower bounds for certain exit times. It is based on paper **IV**. For a sequence of normally distributed random variables with mean zero and bounded variance, we have the following theorem:

**Theorem 4.1.** *(For a proof, see paper **IV**.) Let  $\{Y_t\}_{t \geq 1}$  be a sequence of normally distributed random variables, all with mean 0 and assume that*

$$\text{Var}(Y_t) \leq q(\varepsilon)\sigma^2, \quad \forall t \geq 1,$$

*for some  $\sigma^2 > 0$  and some positive function  $q(\varepsilon)$ , where  $\lim_{\varepsilon \rightarrow 0} q(\varepsilon) = 0$ . Let  $\tau := \min\{t \geq 1 : |Y_t| \geq 1\}$ . Then*

$$\liminf_{\varepsilon \rightarrow 0} q(\varepsilon) \log E\tau \geq \frac{1}{2\sigma^2}.$$

### 4.1 Application to autoregressive processes

This theorem can be applied to the multivariate autoregressive process  $\{X_t\}_{t \geq 0}$ . If the starting point of the process is assumed to be  $x_0 = (0, \dots, 0)^T$ ,  $c^T X_t$  is normally distributed with mean zero, and the variance of  $c^T X_t$  is bounded by  $\varepsilon^2 c^T \Sigma_\infty c$ , where  $\varepsilon^2 \Sigma_\infty$  is the covariance matrix of the stationary distribution of the process. Then we can use theorem 4.1 for the exit time  $\tau = \min\{t \geq 1 : |c^T X_t| \geq 1\}$ , and get the following lemma.

**Lemma 4.2.** *(For a proof, see paper **IV**.) For the multivariate autoregressive process and the exit time  $\tau$  defined in equality 14, we have the following bound of the lower limit:*

$$\liminf_{\varepsilon \rightarrow 0} \varepsilon^2 \log E\tau \geq \frac{1}{2c^T \Sigma_\infty c},$$

*if the starting point of the process is  $x_0 = (0, \dots, 0)^T$ .*

Obviously, the lower bound for the univariate autoregressive process is a special case of this result:

**Corollary 4.3.** *For the univariate autoregressive process  $\{X_t\}_{t \geq 0}$ , with  $x_0 = 0$ , and  $\tau = \min\{t \geq 1 : |X_t| \geq 1\}$ ,*

$$\liminf_{\varepsilon \rightarrow 0} \varepsilon^2 \log E\tau \geq \frac{1}{2}(1 - a^2).$$

The theorem can also be used for the autoregressive process of order  $n$ :

**Lemma 4.4.** *(For a proof, see paper IV.) For the autoregressive process of order  $n$ , when  $x_0 = \dots = x_{n-1} = 0$ ,*

$$\liminf_{\varepsilon \rightarrow 0} \varepsilon^2 \log E\tau \geq \frac{1}{2\sigma^2},$$

where  $\varepsilon^2\sigma^2$  is the variance of the stationary distribution of the process.

We note that this method gives sharp lower bounds, but that it requires the additional assumption that the processes start at the origin.

## 5 Exit times and stationary distributions

As we have seen in the previous chapters, the asymptotic behaviour of the expected exit time depends on the stationary distribution of the process. In this chapter, which is based on paper II, we explore this connection by comparing the exit time with a certain return time to a set. In 1947, Kac ([7]) showed that the expected return time of a discrete Markov chain to a point  $x$  is the reciprocal of the invariant probability  $\pi(x)$ . Cogburn ([2]) extended the result to chains on general measure spaces in 1975.

We leave the autoregressive processes for a moment and consider the more general past-dependent multivariate process  $\{X_t\}_{t \geq 0}$ , where

$$X_t = f(X_{t-1}) + \varepsilon\xi_t, t \geq 1, X_0 = x_0 \in \mathbb{R}^d. \quad (68)$$

Here,  $f$  is assumed to be a continuous and contractive function, the parameter  $\varepsilon$  is positive and  $\{\xi_t\}_{t \geq 1}$  is a sequence of independent and identically distributed random variables with mean zero and finite covariance matrix. Note that the  $\xi_t$ :s are not yet assumed to have a normal distribution. We assume that  $f$  and  $\{\xi_t\}_{t \geq 1}$  are such that the process is Harris recurrent. The process then has an invariant probability measure which we denote by  $\pi$ . We consider the exit time from a set  $\Gamma \in \mathbb{R}^d$ :

$$\tau := \min\{t \geq 1 : X_t \notin \Gamma\}. \quad (69)$$

For fixed positive  $\eta$  and  $h$ , we define the set  $A_\eta$  as

$$A_\eta = \{x \in \mathbb{R}^d : \eta < \inf_{y \in \Gamma} \|y - x\| \leq \eta + h\}. \quad (70)$$

By comparing the exit time  $\tau$  with the time of the  $M$ :th return of the process to the set  $A_\eta$ , we get the following upper bound of the expectation of  $\tau$ :

**Theorem 5.1.** (For a proof, see paper II.) There is an  $M > 0$  (that depends on  $\eta$ ) such that

$$E\tau \leq \frac{M}{\pi(A_\eta)}, \quad (71)$$

where  $\pi$  is the invariant probability measure of the process.

To get more explicit results, we need to know the stationary distribution.

## 5.1 Results for autoregressive processes

We now apply theorem 5.1 to a general multivariate autoregressive process. Let

$$X_t = RX_{t-1} + \varepsilon S\xi_t, t \geq 1, X_0 = x_0, \quad (72)$$

where  $R$  is an  $d \times d$  matrix,  $S$  is an  $d \times p$  matrix for some  $p \leq d$  and  $\{\xi_t\}_{t \geq 1}$  is an i.i.d. sequence of multivariate standard normal random variables in  $\mathbb{R}^p$ . We assume that  $R$  and  $S$  are such that the process has a stationary distribution  $\pi$ . This distribution is multivariate normal with mean  $(0, \dots, 0)^T$  and covariance matrix  $\varepsilon^2 \Sigma_\infty$ . For the exit time  $\tau$ , we have the following result:

**Theorem 5.2.** (For a proof, see paper II.) When  $\tau$  is defined as in equality 69,

$$\limsup_{\varepsilon \rightarrow 0} \varepsilon^2 \log E\tau \leq \inf_{u \in A_0} \frac{1}{2} u^T \Sigma_\infty^{-1} u, \quad (73)$$

where  $A_0 = \{x \in \mathbb{R}^d : 0 < \inf_{y \in \Gamma} \|y - x\| \leq h\}$ , and  $\varepsilon^2 \Sigma_\infty$  is the covariance matrix of the stationary distribution.

The infimum on the right hand side is attained in the point where the level curve of the density of the stationary distribution of the process touches the boundary of the set  $\Gamma$ , as illustrated in figure 5.

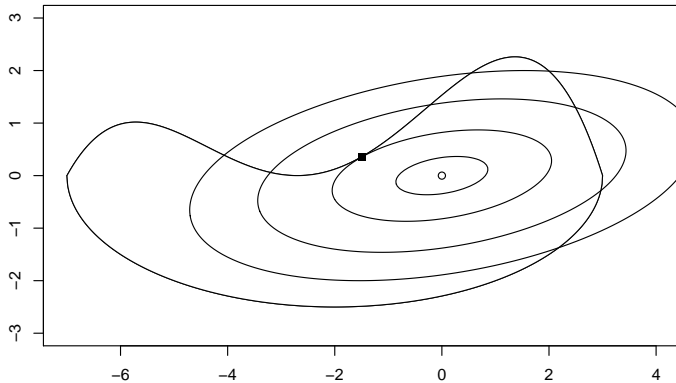
As a special case, we now assume that  $S = I$  (the identity matrix) and consider the exit time  $\tau = \min\{t \geq 1 : |c^T X_t| \geq 1\}$  (we then have the multivariate autoregressive process that was also used in chapter 2). This means that we consider exits from the set

$$\Gamma = \{x \in \mathbb{R}^d : |c^T x| < 1\}. \quad (74)$$

By using Lagrange multipliers, we can minimize  $\frac{1}{2} u^T \Sigma_\infty^{-1} u$  under the condition that  $|c^T u| = 1$  and obtain the minimum  $1/(2c^T \Sigma_\infty c)$ . Thus, we get the same



Figure 5: Example of a set  $\Gamma$ , and the level curves of the stationary distribution when  $A = \begin{pmatrix} 0.7 & 1 \\ 0 & 0.5 \end{pmatrix}$ .



upper bound as the one that was obtained by the large deviation method in lemma 2.4.

As another example, we consider the exit time from the unit sphere of the multivariate autoregressive process, that is,  $\tau = \min\{t \geq 0 : \|X_t\| \geq 1\}$ . Then  $\Gamma = \{x \in \mathbb{R}^n : \|x\| < 1\}$ . We determine the infimum by minimizing  $\frac{1}{2}u^T \Sigma_\infty^{-1}u$  under the condition that  $u^T u = 1$ . Using Lagrange multipliers for this, we get the following result:

**Corollary 5.3.** *When  $\tau$  is the exit time from the unit sphere of the multivariate autoregressive process,*

$$\limsup_{\varepsilon \rightarrow 0} \varepsilon^2 \log E\tau \leq \frac{1}{2\lambda}, \quad (75)$$

where  $\lambda$  is the largest eigenvalue of the matrix  $\Sigma_\infty$  and  $\varepsilon^2 \Sigma_\infty$  is the covariance matrix of the stationary distribution.

## 6 Results and conclusions

We have now discussed several methods to get bounds on the asymptotic behaviour of an expected exit time of a process.

In chapter 2, the large deviation method was treated. It gives some (non sharp) upper and lower bounds even in the general univariate case when

$$X_t = f(X_{t-1}) + \varepsilon\xi_t, t \geq 1, X_0 = x_0, \quad (76)$$

where the function  $f$  is not specified. In the univariate autoregressive case, when  $f(x) = ax$ , it gives a sharp upper bound, and we have extended the method so that it can be used in the multivariate autoregressive case, when  $x \in \mathbb{R}^d$  and  $f(x) = Ax$  for a  $d \times d$  matrix  $A$ . The large deviation method also gives us a sharp upper bound in the case of the autoregressive process of order  $n$ , which is really a type of multivariate process. The method in itself might be useful for a more complicated function  $f$  as well, but it is then difficult to minimize the rate function explicitly. It is not known whether the achieved upper bounds would be sharp in that case. The large deviation method has its limitations. We can only study exits from a symmetric interval around zero in the univariate case, and, in the multivariate case, exits from sets of the type

$$\{x \in \mathbb{R}^d : |c^T x| \geq 1\},$$

for a vector  $c$  that is not the zero vector. Also, we do not get sharp lower bounds. For these, we have to turn to other methods.

In chapter 3, the martingale method introduced by Novikov was explored. It involves some technical difficulties, but it gives a sharp lower bound for the asymptotics of the mean exit time of the univariate autoregressive process. We were able to extend this method to the bivariate autoregressive process, where a sharp lower bound was found, with some additional assumptions on the matrix  $A$  and on the starting point  $x_0$ . It is not possible to extend the method directly to the general multivariate case.

In chapter 4, a lower bound for normal random variables was used. The principle was first shown to me by M.M. Lifshits (by personal communication). The result holds for processes where each element is normal with mean zero and bounded variance. Thus, we get a sharp lower bound in the autoregressive cases, by assuming that the starting point is at the origin. For a more general function  $f$ , each element in the process does not have a normal

distribution, and the method is not useful. On the other hand, the method might be used with other distributions than the normal distribution, if the distribution and the function  $f$  are such that each element has the same distribution with mean zero and bounded variance.

In chapter 5, the asymptotic behaviour of the mean exit time is connected to the stationary distribution. To get explicit results with this method, we need to know the stationary distribution (which we do know, in the case of the autoregressive process). The advantage of this method is that we get results for exits from more general sets than in the previous chapters. This method only gives upper bounds, and in the autoregressive example, the upper bound is sharp.

As we have seen, a combination of the different methods gives us stronger results. We formulate a couple of them here. By combining lemma 2.4 and lemma 3.5, we have the following theorem:

**Theorem 6.1.** *For the bivariate autoregressive process  $\{X_t\}_{t \geq 0}$  and the exit time  $\tau = \min\{t \geq 1 : |c^T X_t| \geq 1\}$ ,*

$$\lim_{\varepsilon \rightarrow 0} \varepsilon^2 \log E\tau = \frac{1}{2c^T \Sigma_\infty c},$$

*if  $A$  has non-real eigenvalues with absolute value smaller than one, and the starting point  $x_0$  satisfies*

$$x_0^T \Sigma_\infty^{-1} x_0 - \frac{1}{c^T \Sigma_\infty c} < 0.$$

By combining lemma 2.4 and lemma 4.2, we get the following theorem for the exit time of a multivariate autoregressive process, starting at the origin:

**Theorem 6.2.** *For the exit time  $\tau = \min\{t \geq 1 : |c^T X_t| \geq 1\}$ , where  $\{X_t\}_{t \geq 0}$  is the multivariate autoregressive process, and  $x_0 = (0, \dots, 0)^T$ ,*

$$\lim_{\varepsilon \rightarrow 0} \varepsilon^2 \log E\tau = \frac{1}{2c^T \Sigma_\infty c},$$

*where  $\varepsilon^2 \Sigma_\infty$  is the covariance matrix of the stationary distribution of the process.*

By combining lemma 2.6 and lemma 4.4, we get the following theorem for the exit time of a univariate autoregressive process of order  $n$ , when the starting points are zeroes:

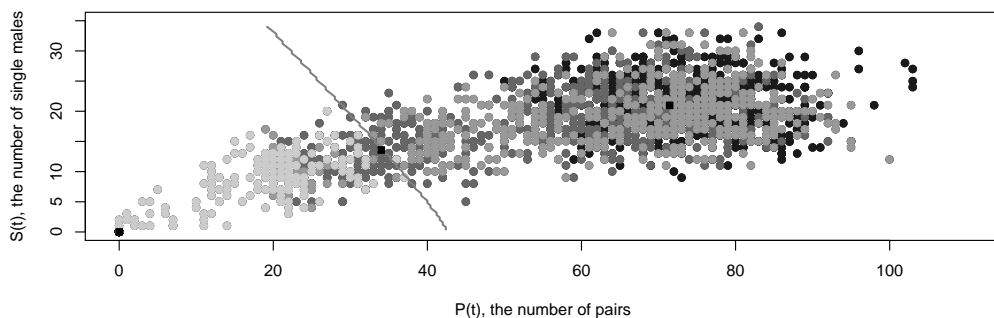
**Theorem 6.3.** For the univariate autoregressive process  $\{X_t\}_{t \geq 0}$  of order  $n$ , and the exit time  $\tau = \min\{t \geq n : |X_t| \geq 1\}$ ,

$$\lim_{\varepsilon \rightarrow 0} \varepsilon^2 \log E\tau = \frac{1}{2\sigma^2},$$

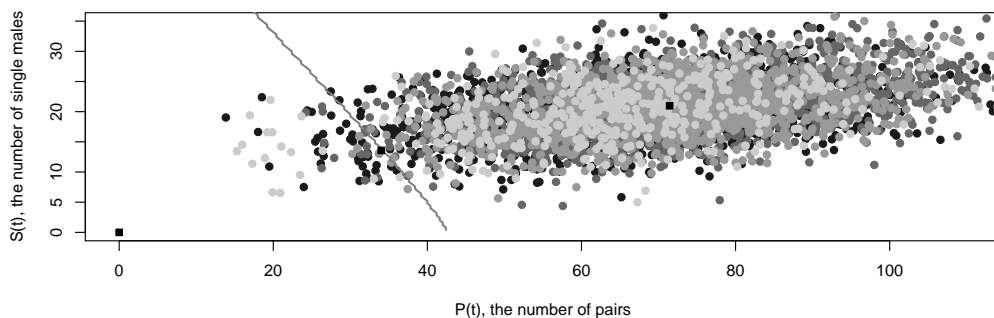
assuming that all eigenvalues of  $B$  are smaller than one in absolute value, and that  $x_0 = \dots = x_{n-1} = 0$ .

We now have some results for the asymptotic behaviour of the mean exit times for autoregressive processes. The original reason for studying exit times, was our wish to study the time until extinction of some population model. Is the autoregressive process a good approximation of such a population model?

Figure 6: A comparison of sample paths. The paths are colour coded so that the points get lighter coloured as time passes.



(a) A sample path of the branching process. Extinction took place at  $\tau = 1830$ .



(b) A sample path of the approximating bivariate autoregressive process. No extinction during the first 5000 steps.

In the introduction, a multitype branching process model for the Northern Spotted Owl was mentioned. This is a much more complicated process than the autoregressive process. Paper **V** contains simulation studies of this model, and of simpler approximations of it. In the context of this thesis, the relevant part of the paper is the one where an approximation with a bivariate autoregressive process is tried. When simulating the exit time from a relatively small area around the equilibrium, the result is very close to the one achieved for the original process. But when we simulate the time until extinction, the approximation with an autoregressive process is very bad indeed.

In figure 6 sample paths of the two processes are shown. The time until extinction of the bivariate autoregressive process is much larger than that of the original branching process.

Paper **V** illustrates that approximation with the autoregressive process may be useful near the equilibrium of the process, but does not give good results for the extinction time. We need other tools as well, to study the extinction time of a complicated process such as the model of the Northern Spotted Owl.

## References

- [1] L. J. S. Allen, J. F. Fagan, G. Högnäs, H. Fagerholm, Population extinction in discrete-time stochastic population models with an Allee effect, *J. Difference Equ. Appl.*, 11 (2005), 4-5, 273-293.
- [2] R. Cogburn, A uniform theory for sums of Markov chain transition probabilities, *Ann. Probab.* 3 (2) (1975), 191-214.
- [3] A. Dembo, O. Zeitouni, *Large Deviation Techniques and Applications*, Jones and Bartlett, Boston, 1993.
- [4] G. Högnäs, B. Jung, Analysis of a stochastic difference equation: Exit times and invariant distributions, *Fasc. Math.* 44 (2010), 69-74.
- [5] B. Jung, The Cauchy-Frullani integral formula extended to double integrals, *Math. Sci.*, 37 (2012), 1-6.
- [6] B. Jung, Exit times for multivariate autoregressive processes, *Stoch. Proc. Appl.* 123 (2013), 8, 3052-3063.
- [7] M. Kac, On the notion of recurrence in discrete stochastic processes, *Bull. Amer. Math. Soc.* 53 (1947), 1002-1010.
- [8] F. Klebaner, R. Liptser, Large deviations for past-dependent recursions, *Probl. Inf. Transm.* 32 (1996), 23-34. Revised version 2006, available at <http://arxiv.org/abs/math/0603407>
- [9] J. Lynch, J. Sethuraman, Large deviations for processes with independent increments, *Ann. Probab.* 15 (1987), 2, 610-627.
- [10] S. P. Meyn, R. L. Tweedie, *Markov Chains and Stochastic Stability*, Springer-Verlag, London, 1993.
- [11] A. A. Novikov, On the first passage time of an autoregressive process over a level and an application to a "disorder" problem, *Theory. Probab. Appl.*, 35 (1990), 2.
- [12] A. Ostrowski, *Vorlesungen über Differential- und Integralrechnung*, vol. 3, 2nd ed., Birkhäuser Verlag, Basel, 1967.

- [13] B. Ruths, Exit times for past-dependent systems, *Surveys of Applied and Industrial Mathematics (Obozrenie prikladnoy i promyshlennoy matematiki)* 15 (2008), 1, 25-30.
- [14] B. Ruths, *Stora avvikelser och uthoppstider för autoregressiva processer*, licentiate thesis (in Swedish), Åbo Akademi, 2008.
- [15] S.R.S. Varadhan, *Large Deviations and Applications*, SIAM, Philadelphia, 1984.



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