DISSERTATIO MATHEMATICA,

SPECIMEN

DESCRIPTIONIS ORGANICÆ LINEARUM CURVARUM

SISTENS.

QUAM

CONS. AMPLISS. FACULT. PHILOS. ABOËNS

AUCTOR

Mag. GUSTAVUS GABR. HÅLLSTRÖM,

PHYSICES DOCENS,

&

RESPONDENS

JACOBUS JOHANNES ERLING,

Stip. Reg. Borealis,

PUBLICE EXAMINANDAM PROPONUNT

In Auditorio Majori die 10 Novembris 1798,

horis a. m consvetis.

PARS II.

ABOÆ,

Typis FRENCKELLIANIS

9.

THESES.

Thes. I. Grammatices studium, ut & ad puritatem & perspicuitatem orationis conservandam vehementer confert, ita in nulla lingva impune negligitur.

Thes. II. Cum neminem alios recte instituere posse constet, nisi qvi ipse habeat distinctam rerum, qvas propositurus est, cognitionem; ita tironibus hoc negotium sapienter non committi, facile quisque deprehendit.

Thes. III. Qvi rem bene callet, de qva sibi dicendum est, is de inopia dicendorum qveri haud cogetur. Qvare futuris Oratoribus Horatii consilium identidem inculcandum remur: *Scribendi recte, sapere est & principium & fons.*

Thes. IV. Etiamsi diversæ Lineæ curvæ perpetuo diversas præbeant æqvationes inter Coordinatas, non tamen semper ex diversa æquationum forma ad diversitatem Curvarum, quæ his æquationibus indicantur, vicissim concludere licet.

Thes. V. In tot punctis Linea qvævis curva a recta secari potest, quot continet unitates numerus, ordinem, ad quem refertur Curva, exprimens. Non autem numerus intersectionum, qvas cum linea curva descripta facere comperimus rectam, ordinem ad quem illa pertinet, semper indigitat.

Thes. VI. Si linea recta data in plano quovis ita sibi semper parallela movetur, ut summa distantiarum terminorum ejus a puncto aliqvo fixo, in plano illo sumto, semper sit constans; qvodvis lineæ illius punctum sub hocce motu Ellipsin describet.

in æqv. $y = -\frac{(\alpha.\overline{a+b}+\beta.\overline{1-ab})x+\gamma(1-ab)}{2(\alpha-\beta a)}$, prodit

$y = \mathrm{P}m = -\frac{\alpha\gamma(a^2+2a^2b^2+b^2)+\beta\gamma(a-b)(1-ab)}{(\alpha.\overline{a-b}+\beta.\overline{1-ab})^2} \mp$

$\mp \frac{\gamma(\alpha.\overline{a+b}+\beta.\overline{1-ab})\sqrt{a(\alpha b-\beta)(\alpha-\beta a)(1+a^2)(1+b^2)}}{(\alpha-\beta a)(\alpha.\overline{a-b}+\beta.\overline{1+ab})^2}$. Hinc autem habetur $\mathrm{PQ} = \pm(\mathrm{AQ}-\mathrm{AP}) =$

$\mp \frac{2\gamma\sqrt{a(\alpha b-\beta)(\alpha-\beta a)(1+a^2)(1+b^2)}}{(\alpha.\overline{a-b}+\beta.\overline{1+ab})^2}$, & $\pm(\mathrm{QB}-\mathrm{P}m) =$

$\pm \frac{\gamma(\alpha.\overline{a+b}+\beta.\overline{1-ab})\sqrt{a(\alpha b-\beta)(\alpha-\beta a)(1+a^2)(1+b^2)}}{(\alpha-\beta a)(\alpha.\overline{a-b}+\beta.\overline{1+ab})^2}$; erit-que ergo $Tg\,\mathrm{QB}m = \frac{\mathrm{PQ}}{\mathrm{QB}-\mathrm{P}m} = \mp \frac{2(\alpha-\beta a)}{\alpha(a+b)+\beta(1-ab)}$. Eſt vero præterea $Tg\,\mathrm{AGF} = \frac{\mathrm{AF}}{\mathrm{AG}} = b$, & $Tg\,\mathrm{LEA} = \frac{\mathrm{AL}}{\mathrm{AE}} = \frac{\alpha a+\beta}{\alpha-\beta a}$. His angulis ſic inventis, habetur angulus Aſymptotorum LBF = AGF + LEA, adeoque $Tg\,\mathrm{LBF} = \frac{\alpha(a+b)+\beta(1-ab)}{\alpha(1-ab)-\beta(a+b)}$. Biſecetur hic ang. LBF recta BN, quæ itaque erit ſemiaxis transverſus Hyperbolæ MNS, & habebitur ang. QBN = 90° − LEA + ½ LBF = 90° − ½ (LEA − AGF), quo invento, atque ducta ex centro B ad punctum Hyperbolæ m, ubi eam Applicata Pm tangit, ſemidiametro Bm, reperitur quoque ang. NBm = QBN − QBm. Eſt

 itaque

itaque $\mathrm{Tg\,LBN} = \left(\dfrac{-1 \pm \sqrt{1+\mathrm{Tg\,LBF}^2}}{\mathrm{Tg\,LBF}} =\right)$

$= \dfrac{-\alpha(1-ab)+\beta(a+b) \pm \sqrt{(\alpha^2+\beta^2)(1+a^2)(1+b^2)}}{\alpha(a+b)+\beta(1-ab)}$. & posito brevitatis causa $\dfrac{\alpha a+\beta}{\alpha-\beta a} = k$,

$\mathrm{Tg\,NB}m = \dfrac{(k+b)(k-b) \pm 2(kb-1)+2\sqrt{(1+k^2)(1+b^2)}}{(k+b)(kb-1) \mp 2(k-b) \pm (k+b)\sqrt{(1+k^2)(1+b^2)}}$.

His vero inventis, & reperta simul semidiametro $\mathrm{B}m = \sqrt{\mathrm{PQ}^2+(\mathrm{BQ}-\mathrm{P}m)^2}$, invenitur semiaxis transversus Hyperbolæ $\mathrm{BN} = \dfrac{\mathrm{B}m}{\mathrm{Tg\,LBN}} \sqrt{\dfrac{\mathrm{Tg\,LBN}^2 - \mathrm{Tg\,NB}m^2}{1+\mathrm{Tg\,NB}m^2}}$ (*)

& semiaxis conjugatus $= \mathrm{B}m \sqrt{\dfrac{\mathrm{Tg\,LBN}^2 - \mathrm{Tg\,NB}m^2}{1+\mathrm{Tg\,NB}m^2}}$.

§. 6.

(*) Ut veritas hujus æquationis facilius perspiciatur, sequentem ejus demonstrationem hic apponere necesse duximus.

Sit ex m in BN (productam) demissa perpendicularis mn, & quoniam est semiaxis conjugatus Hyperbolæ MNS $=$ BN. Tg LBN, erit ex natura hujus Hyperbolæ, $mn^2 = \mathrm{Tg\,LBN}^2(\mathrm{B}n^2 - \mathrm{BN}^2)$; unde, ob $mn = \mathrm{B}n.\ \mathrm{Tg\,NB}m$, erit $\mathrm{B}n^2.\ \mathrm{Tg\,NB}m^2 = (\mathrm{B}n^2 - \mathrm{BN}^2)\ \mathrm{Tg\,LBN}^2$. Hinc autem eruitur $\mathrm{B}n^2 = \dfrac{\mathrm{BN}^2.\ \mathrm{Tg\,LBN}^2}{\mathrm{Tg\,LBN}^2 - \mathrm{Tg\,NB}m^2}$, adeoque

$mn^2 = \dfrac{\mathrm{BN}^2.\ \mathrm{Tg\,LBN}^2\ \mathrm{Tg\,NB}m^2}{\mathrm{Tg\,LBN}^2 - \mathrm{Tg\,NB}m^2}$, & $\mathrm{B}m =$

$= \sqrt{\mathrm{B}n^2+mn^2} = \mathrm{BN}.\ \mathrm{Tg\,LBN} \sqrt{\dfrac{1+\mathrm{Tg\,NB}m^2}{\mathrm{Tg\,LBN}^2 - \mathrm{Tg\,NB}m^2}}$,

vel $\mathrm{BN} = \dfrac{\mathrm{B}m}{\mathrm{Tg\,LBN}} \sqrt{\dfrac{\mathrm{Tg\,LBN}^2 - \mathrm{Tg\,NB}m^2}{1+\mathrm{Tg\,NB}m^2}}$.

§. 6.

Quando eſt $b = \frac{\alpha a + \beta}{\alpha - \beta a}$, Curvam deſcriptam Parabolam eſſe invenimus (§. 4.), æquatione $(y + bx)^2 + \frac{\gamma}{\alpha - \beta a}(\overline{1 - ab}.y + \overline{a + b}.x) = o$ expreſſam. Sit AM (Fig. 3.) Axis Abſciſſarum & ei in A perpendiculariter inſiſtens AN Axis ordinatarum, ad quos Axes æquatio Parabolæ allata refertur, & erit A initium Abſciſſarum. Facta $y = o$ in Parabolæ noſtræ æquatione, ea in hanc reducitur: $b^2 x^2 + \frac{\gamma(a+b)x}{\alpha - \beta a} = o$, quæ dat $x = o$, & $x = -\frac{\gamma(a+b)}{(\alpha - \beta a)b^2}$, unde concluditur, facta $AM = -\frac{\gamma(a+b)}{(\alpha - \beta a)b^2}$, lineam AM in A & M a Parabola ſecari. Similiter faciendo $x = o$, invenitur $y^2 + \frac{\gamma(1 - ab)}{\alpha - \beta a}.y = o$, quæ æquatio indigitat, Parabolam, præterquam quod per A transeat, facta $AN = -\frac{\gamma(1 - ab)}{\alpha - \beta a}$, in N lineæ AN occurrere. Sit Parabolæ MAN Axis principalis BCS, cui occurrit Parabola in C, ita ut ſit C vertex ejus primarius. Ut hujus Axis innoteſcat poſitio atque Parameter, determinandæ ſunt quantitates conſtantes AB, AD, & DC, exſiſtentibus B & D punctis illis, ubi lineis AN & AM occurrit Axis. Fiat ergo $AB = m$, AD $= n$, BD $= p$, & CD $= q$; atque facta Abſciſſa AP $= x$, & ei normali Applicata PQ $= y$, quæ Axi CDS in R occurrat, ducatur ex Q in CDS normalis QS. Sit CS $= t$, & SQ $= u$. Ob Triang. ABD ∽ PRD ∽ RSQ eſt $n : p :: u :$ QR, unde QR $= \frac{pu}{n}$, & $n : m :: u :$ SR, unde SR $= \frac{mu}{n}$, adeoque RD $= t - \frac{mu}{n} - q$. Eſt quoque $p : m ::$ RD : PR & $p : n ::$ RD : PD, unde inveniuntur PR $= \frac{mt}{p} - \frac{m^2 u}{np} - \frac{mq}{p}$ & PD $= \frac{nt - mu - nq}{p}$. Hinc vero habetur $x =$ PD + DA $= \frac{nt - mu + n(p - q)}{p}$, & $y =$ PR + RQ $= \frac{mt + nu - mq}{p}$. His Coordinatarum x & y valoribus ſubſtitutis in æqv.

$(y + bx)^2 + \frac{\gamma}{\alpha - \beta a}(\overline{1 - ab}.y + \overline{a + b}.x) = 0$, ea in hanc transformatur: $(n - bm)^2 u^2 + 2(n - bm)(bn + m)tu + (bn + m)^2 t^2 + 2(n - bm)(bn.\overline{p - q} - mq) \Big\} u +$

$$\left.\begin{matrix} + \frac{\gamma(1 - ab)np}{\alpha - \beta a} \\ - \frac{\gamma(a + b)mp}{\alpha - \beta a} \end{matrix}\right\}$$

$$\left.\begin{array}{r} +2(bn+m)(bn.\overline{p-q}-mq) \\ +\dfrac{\gamma(1-ab)mp}{\alpha-\beta a} \\ +\dfrac{\gamma(a+b)np}{\alpha-\beta a} \end{array}\right] t \left.\begin{array}{l} +(bn.\overline{p-q}-mq)^2 \\ -\dfrac{\gamma(1-ab)mpq}{\alpha-\beta a} \\ +\dfrac{\gamma(a+b)(p-q)np}{\alpha-\beta a} \end{array}\right\} = 0.$$ Quum autem ad Axem Parabolæ MAN ejusque Applicatas orthogonales relata erit hæcce æquatio, initio Abſciſſarum in ipſo vertice ſumto, evaneſcere debent termini tu, t^2, u & quantitas ultima conſtans; quare eſſe debet $bn+m=o$, $2(n-bm)(bn.\overline{p-q}-mq) + \frac{\gamma(1-ab)np}{\alpha-\beta a} - \frac{\gamma(a+b)mp}{\alpha-\beta a} = o$, & $(bn.\overline{p-q}-mq)^2 - \frac{\gamma(1-ab)mpq}{\alpha-\beta a} + \frac{\gamma(a+b)(p-q)np}{\alpha-\beta a} = o$. Cumque præterea ſit $p^2 = m^2 + n^2$, harum quatuor æquationum ope inveniuntur: $m = \frac{\gamma}{2(\alpha-\beta a)}$, $n = -\frac{\gamma}{2(\alpha-\beta a)b}$, $p = \pm\frac{\gamma\sqrt{1+b^2}}{2(\alpha-\beta a)b}$, & $q = \pm\frac{\gamma(2a+b)}{4ab(\alpha-\beta a)\sqrt{1+b^2}}$. Hisce valoribus in Coëfficientem termini t ſubſtitutis, habetur æquatio Parabolæ MAN ad Axem BCS relata: $u^2 = \pm\frac{\gamma^3 a(1+b^2)\sqrt{1+b^2}}{4(\alpha-\beta a)^3 b^2}\,.\,t$, adeoque Parameter hujus Axis $= \pm\frac{\gamma^3 a(1+b^2)\sqrt{1+b^2}}{4(\alpha-\beta a)^3 b^2}$.

§. 7.

Sit jam ſecundi ordinis Linea DNK (Fig. 1.), quæ æquatione generali: $\alpha z^2+\beta zv+\gamma v^2+\delta z+\varepsilon v+\zeta = o$

exprimitur. Substitutis in ea valoribus $z = \frac{(ax+y)(bx+y)}{(a+b)x+(1-ab)y}$, & $v = \frac{(x-ay)(bx+y)}{(a+b)x+(1-ab)y}$ (vide §. 2.), habetur pro Linea descripta MS æquatio hæcce:

$$\mathfrak{S}.\ \left.\begin{matrix}+\alpha\\-\beta a\\+\gamma a^2\end{matrix}\right\}y^4 \left.\begin{matrix}+2\alpha(a+b)\\+\beta(-2ab-a^2)\\-2\gamma a(1-ab)\end{matrix}\right\}xy^3 \left.\begin{matrix}+\alpha(a^2+4ab+b^2)\\+\beta(a+2(1-a^2)b-ab^2)\\+\gamma(1-4ab+a^2b^2)\end{matrix}\right\}x^2y^2$$

$$\left.\begin{matrix}+2\alpha ab(a+b)\\+\beta(2ab+(1-a^2)b^2)\\+2\gamma b(1-ab)\end{matrix}\right\}x^3y \left.\begin{matrix}+\alpha a^2b^2\\+\beta ab^2\\+\gamma b^2\end{matrix}\right\}x^4 \left.\begin{matrix}+\delta(1-ab)\\-\varepsilon a(1-ab)\end{matrix}\right\}y^3.$$

$$\left.\begin{matrix}+\delta(a+b)(2-ab)\\-\varepsilon(a.\overline{a+b}-\overline{1-ab}^2)\end{matrix}\right\}xy^2 \left.\begin{matrix}+\delta((a+b)^2+ab(1-ab))\\+\varepsilon(a+2b)(1-ab\end{matrix}\right\}x^2y$$

$$\left.\begin{matrix}+\delta ab(a+b)\\+\varepsilon b(a+b)\end{matrix}\right\}x^3+\zeta(1-ab)^2y^2+2\zeta(a+b)(1-ab)xy+$$

$\zeta(a+b)^2x^2 = 0$, quæ ostendit, Lineam MS quarti ordinis esse posse. Hoc casu autem, quando DNK aliqua est Sectionum Conicarum, illam semper quarti esse ordinis, minime putes. Talem enim inter se relationem habere possunt quantitates constantes, æquationem $\mathfrak{S}$ ingredientes, ut in Factores reales tertii, secundi, & etiam primi ordinis, quorum quisqve sui ordinis Lineam aliquam exprimit, resolvi queat hæc æquatio. Sic ex. gr. si manentibus reliquis, evanescat ζ, hoc est, si in Curva DNK nihil aliud mutetur, quam ut ea per punctum A ducatur, Linea descripta MS ultra gradum tertium jam non assurgit. Deficientibus enim tum in æquatione $\mathfrak{S}$ terminis y^2, xy & x^2,

re-

reliquum ejus resolvi potest in Factores duos, unum primi ordinis: $y+bx=o$, alterum vero hunc tertii ordinis:

$$\mathfrak{H}.\ \left.\begin{matrix}+\alpha\\ -\beta a\\ +\gamma a^2\end{matrix}\right\} y^3 \left.\begin{matrix}+\alpha(2a+b)\\ +\beta(1-a^2-ab)\\ -\gamma a(2-ab)\end{matrix}\right\} xy^2 \left.\begin{matrix}+\alpha a(a+2b)\\ +\beta(a+\overline{1-a^2}.b)\\ +\gamma(1-2ab)\end{matrix}\right\} x^2y+$$

$$\left.\begin{matrix}+\alpha a^2 b\\ +\beta a b\\ +\gamma b\end{matrix}\right\} x^3 \left.\begin{matrix}+\delta(1-ab)\\ -\varepsilon a(1-ab)\end{matrix}\right\} y^2 \left.\begin{matrix}+\delta(2a+b(1-a^2))\\ +\varepsilon(1-2ab-a^2)\end{matrix}\right\} xy+$$

$\left.\begin{matrix}+\delta a(a+b)\\ +\varepsilon(a+b)\end{matrix}\right\} x^2=o$. Si quando est $\zeta=o$, simul accidat, ut sit $\delta=o=\varepsilon$, in æquatione $\mathfrak{H}$ evanescunt termini y^2, xy & x^2, cum reliquum ejus resolvitur in Factores $y+bx=o$, & $\left.\begin{matrix}+\alpha\\ -\beta a\\ +\gamma a^2\end{matrix}\right\} y^2 \left.\begin{matrix}+2\alpha a\\ +\beta(1-a^2)\\ -2\gamma a\end{matrix}\right\} xy \left.\begin{matrix}+\alpha a^2\\ +\beta a^2\\ +\gamma\end{matrix}\right\} x^2=o$,

quarum hic vel præbet æquationem pro Ellipsi, quando est $(2\alpha a+\beta.\overline{1-a^2}-2\gamma a)^2<4(\alpha-\beta a+\gamma a^2)(\alpha a^2+\beta a+\gamma)$; vel systema indigitat duarum rectarum, quarum æquationes sunt: $y+\left[\frac{2\alpha a+\beta.\overline{1-a^2}-2\gamma a}{2(\alpha-\beta a+\gamma a^2)}\right.$

$\left.\pm\frac{\sqrt{(2\alpha a+\beta.\overline{1-a^2}-2\gamma a)^2-4(\alpha-\beta a+\gamma a^2)(\alpha a^2+\beta a+\gamma)}}{2(\alpha-\beta a+\gamma a^2)}\right].x$

$=o$, quando est $(2\alpha a+\beta.\overline{1-a^2}-2\gamma a)^2>4(\alpha-\beta a+\gamma a^2)(\alpha a^2+\beta a+\gamma)$; vel unam tantum repræsentat rectam, æquatione: $y+\frac{2\alpha a+\beta.\overline{1-a^2}-2\gamma a}{2(\alpha-\beta a+\gamma a^2)}.x=o$ definitam, quando nimirum est $(2\alpha a+\beta.\overline{1-a^2}-2\gamma a)^2=4(\alpha-\beta a+\gamma a^2)(\alpha a^2+\beta a+\gamma)$.

Sic

Sic quoque exempli loco hic afferamus casum illum, quo est $\gamma = o$ & $b = o$. Tum enim æqv. $\mathfrak{S}$ resolvitur in hos reales Factores: $y + ax = o$, & $\mathfrak{J}$). $(\alpha - \beta a)y^3 + (\alpha a + \beta)xy^2 + (\delta - \varepsilon a)y^2 + (\delta a + \varepsilon)xy + \zeta y + \zeta ax = o$, quorum hic pro $\zeta = o$ habet Factores $y = o$, & $(\alpha - \beta a)y^2 + (\alpha a + \beta)xy + (\delta - \varepsilon a)y + (\delta a + \varepsilon)x = o$, qui est æquatio ad Hyperbolam.

§. 8.

Æquationis $\mathfrak{S}$, quæ ostendit, Lineam descriptam MS, exsistente DNK aliqua Sectionum Conicarum, quarti esse posse ordinis, membrum supremum:

$$\left.\begin{matrix} +\alpha \\ -\beta a \\ +\gamma a^2 \end{matrix}\right\} y^4 \left.\begin{matrix} +2\alpha(a+b) \\ +\beta(1-2ab-a^2) \\ -2\gamma a(1-ab) \end{matrix}\right\} xy^3 \left.\begin{matrix} +\alpha(a^2+4ab+b^2) \\ +\beta(a+2(1-a^2)b-ab^2) \\ +\gamma(1-4ab+a^2b^2) \end{matrix}\right\} x^2y^2$$

$$\left.\begin{matrix} +2\alpha ab(a+b) \\ +\beta(2ab+(1-a^2)b^2) \\ +2\gamma b(1-ab) \end{matrix}\right\} x^3y \left.\begin{matrix} +\alpha a^2b^2 \\ +\beta ab^2 \\ +\gamma b^2 \end{matrix}\right\} x^4$$

in hos quatuor resolvi potest Factores simplices: $y + bx$, $y + bx$, & $y + \left(\frac{2\alpha a + \beta(1-a^2) - 2\gamma a}{2(\alpha - \beta a + \gamma a^2)} \pm \right.$

$$\left. \pm \sqrt{\frac{(2\alpha a + \beta(1-a^2) - 2\gamma a)^2}{4(\alpha - \beta a + \gamma a^2)^2} - \frac{\alpha a^2 + \beta a + \gamma}{\alpha - \beta a + \gamma a^2}}\right)x,$$

quorum duos priores semper esse reales atque inter se æquales apparet. Hos vero quatuor considerando Factores, sequentes quinque observantur casus.

Cas. I. Si est $(2\alpha a + \beta(1-a^2) - 2\gamma a)^2 < 4(\alpha - \beta a + \gamma a^2)(\alpha a^2 + \beta a + \gamma)$, duo Factores sunt reales atque

atque inter se æquales, reliquis existentibus imaginariis.

CAS. II. Si est $(2\alpha a + \beta(1-a^2)-2\gamma a)^2 = 4(\alpha-\beta a + \gamma a^2)(\alpha a^2 + \beta a + \gamma)$, non vero $b = \frac{2\alpha a + \beta(1-a^2)-2\gamma a}{2(\alpha-\beta a+\gamma a^2)}$, præter duos Factores semper æquales reliqui etiam inter se sunt æquales.

CAS. III. Si vero, quando est $(2\alpha a + \beta(1-a^2) -2\gamma a)^2 = 4(\alpha-\beta a+\gamma a^2)(\alpha a^2+\beta a+\gamma)$, simul sit $b = \frac{2\alpha a+\beta(1-a^2)-2\gamma a}{2(\alpha-\beta a+\gamma a^2)}$, omnes quatuor Factores sunt reales & inter se æquales.

CAS. IV. Si est $(2\alpha a+\beta(1-a^2)-2\gamma a)^2 > 4(\alpha-\beta a +\gamma a^2)(\alpha a^2+\beta a+\gamma)$, non autem $b = \frac{2\alpha a+\beta(1-a^2)-2\gamma a}{2(\alpha-\beta a+\gamma a^2)}$ $\pm \sqrt{\frac{(2\alpha a+\beta(1-a^2)-2\gamma a)^2}{4(\alpha-\beta a+\gamma a^2)^2} - \frac{\alpha a^2+\beta a+\gamma}{\alpha-\beta a+\gamma a^2}}$, omnes Factores sunt reales, duo inter se æquales, reliqui vero inæquales.

CAS. V. Si vero, quando est $(2\alpha a+\beta(1-a^2) -2\gamma a)^2 > 4(\alpha-\beta a+\gamma a^2)(\alpha a^2+\beta a+\gamma)$, simul sit $b = \frac{2\alpha a+\beta(1-a^2)-2\gamma a}{2(\alpha-\beta a+\gamma a^2)}$ $\pm \sqrt{\frac{(2\alpha a+\beta(1-a^2)-2\gamma a)^2}{4(\alpha-\beta a+\gamma a^2)^2} - \frac{\alpha a^2+\beta a+\gamma}{\alpha-\beta a+\gamma a^2}}$, erunt omnes Factores reales, eorumque tres inter se æquales.

Hi caſus enumerati, etiamſi non omnes quidem comprehendant Lineas quarti ordinis, deſunt enim caſus, ubi nulli Factorum membri ſupremi æquationis ſunt reales, ut etiam quando duo Factores ſunt reales & inæquales inter ſe, nec non quando omnes quatuor Factores ſunt reales & inæquales; magnum tamen omnino præbent numerum generum harum Linearum, ut videre licet in Cel. EULERI *Introd. in Analyſ. infinitor. Tom. II, Cap. XI.*

Linearum tertii ordinis, quæ deſcribi noſtra methodo poſſunt, quando eſt Linea DNK ſecundi ordinis, illæ ſaltem hic conſiderari debuiſſent, quas præbent æquationes 𝔥 & 𝔍 (§. 7.). Id autem facere cum non permittat anguſtia harum pagellarum, nominaſſe tantum juvabit, Lineas illas tertii ordinis omnium ſex decem ſpecierum Eulerianarum ſuppeditare æquationes has 𝔥 & 𝔍.

§. 9.

Sicut Lineas primi, ſecundi, tertii & quarti ordinis methodo §. 2. allata deſcribi poſſe oſtendimus, ita ſimiliter, adhibitis in deſcribendo Lineis altiorum ordinum, altiorum quoque ordinum Curvæ delineatæ habentur, quod ex iis, quæ ſeqvuntur, generaliter apparet. Sit nimirum Linea algebraica DNK (Fig. 1.) ordinis n, atque erit ejus æquatio generalis: $\alpha z^{n} + \beta v z^{n-1} + \gamma v^{2} z^{n-2} + \dots + \zeta v^{n-1} z + \eta v^{n} + \theta z^{n-1} + \kappa v z^{n-2} + \dots \mu z + \nu v + \varrho = 0$. Quando in ea ea ſubſtituuntur valores $z = \frac{(ax+y)(bx+y)}{(a+b)x+(1-ab)y}$ & $v = \frac{(x-ay)(bx+y)}{(a+b)x+(1-ab)y}$, æquatio pro Curva deſcripta erit hæcce:

$$\alpha(ax+y)^n(bx+y)^n + \beta(x-ay)(ax+y)^{n-1}(bx+y)^n + \gamma(x-ay)^2(ax+y)^{n-2}(bx+y)^n + \cdots + \zeta(x-ay)^{n-1}(ax+y)(bx+y)^n + \eta(x-ay)^n(bx+y)^n + \theta(ax+y)^{n-1}(bx+y)^{n-1}(\overline{a+b}.x+\overline{1-ab}.y) + \cdots + \mu(ax+y)(bx+y)(\overline{a+b}.x+\overline{1-ab}.y)^{n-1} + \nu(x-ay)(bx+y)(\overline{a+b}.x+\overline{1-ab}.y)^{n-1} + \varrho(\overline{a+b}.x+\overline{1-ab}.y)^n = 0,$$

quæ ordinis $2n$ exſiſtens, indigitat Lineam MS altioris eſſe non poſſe ordinis, quam exprimit numerus $2n$. Semper vero, quando DNK eſt ordinis n, ad ordinem numero $2n$ expreſſum non aſſurgere Lineam MS, etiam ex ſequentibus videri poteſt exemplis. Deficiant in æquatione Lineæ DNK reliquis omnibus terminis præter αz^n, $\beta v z^{n-1}$ & ηv^n, ita ut illa ſit: $\alpha z^n + \beta v z^{n-1} + \eta v^n = 0$, & erit æquatio Lineam MS exprimens hæc: $\alpha(ax+y)^n + \beta(ax+y)^{n-1}(x-ay) + \eta(x-ay)^n = 0$, ordinis nimirum n. Sit $b=0$, exponens n termini z major exponente termini v (quod de ſequentibus etiam æquationibus valeat), & æquatio pro DNK hæcce: $\alpha z^n + \nu v + \varrho = 0$; atque erit æquatio pro MS talis: $\alpha y^n(ax+y) + \nu y(x-ay) + \varrho(ax+y) = 0$, ordinis nimirum $n+1$. Manente

nente $b = o$, fit Lineæ DNK æquatio: $\alpha z^n + \lambda v^2 + \varrho = o$, & habetur Lineæ MS æquatio: $\alpha y^n (ax+y)^2 + \lambda y^2 (x-ay)^2 + \varrho (ax+y)^2 = o$, nempe ordinis $n+2$. Similiter exſiſtente $b = o$, & æqv. $\alpha z^n + \kappa v^3 + \varrho = o$ pro Linea DNK; erit æquatio Lineæ MS hæcce: $\alpha y^n (ax+y)^3 + \kappa y^3 (x-ay)^3 + \varrho (ax+y)^3 = o$, ordinis numero $n+3$ expresſi. Hinc apparet, ſi ſic in tali æquatione Lineæ DNK ordinis n termini v exponens unitate ſemper augeatur, usque quo ad numerum n perveniatur, numerum quoque, qui ordinem indigitat Lineæ MS, usque ad $2n$ ſuccesſive creſcere, atque adeo ad ordinem quemvis, qui aliquo numerorum $n, n+1, n+2, n+3, \text{-----} 2n$ exprimitur, pertinere poſſe Lineam MS. Hinc autem patet, Lineas cujuscunque ordinis algebraicas noſtra methodo deſcribi poſſe, & quidem adhibendam eſſe Lineam ordinis non minoris, quam qui numero n exprimitur, ſi deſcribenda eſt Linea ordinis $2n$, & Lineam ordinis non minoris quam $n+1$, ſi habeatur deſcripta Linea ordinis $2n+1$. Simul quoque apparet, ſi omnium ordinum Lineæ algebraicæ deſcribendæ ſunt, non omnium ordinum Lineis in deſcribendo opus eſſe, ſed illis tantum, quarum ordinem exprimunt numeri n, $2n+1$, $4n+3$, $8n+7$, $16n+15$, $\text{----} 2^m n + 2^m - 1$, exſiſtentibus tam m quam [n] numeris integris.—