DE
INVENIENDIS LINEIS CURVIS EX DATIS RADII CURVATURÆ PROPRIETATIBUS, PROBLEMATA.

Conf. Ampl. Facult. Philof. Aboënf.

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ABOÆ,

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THESES.

Thes. I.

Distinctionem inter γνώμας & νόηματα recte facientes Rhetores, merito præcipiunt, hæc & meliora in universum esse, & largius usurpari in orationibus posse.

Thes. II. Vere non minus quam pulchre cecinit HORATIUS: *Ut sylvæ foliis pronos mutantur in annos, prima cadunt: ita verborum vetus interit ætas, & juvenum ritu florent modo nata vigentque.*

Thes. III. Pueris ac adolescentibus fundamenta linguæ alicujus discendæ auctore bono explicando posituris, versio quidem atque interpretatio verborum accurata necessaria est, at hæc tamen minime sufficit vel ad linguæ usum recte addiscendum, nisi rerum quoque ab auctore propositarum diligens adjungatur explanatio.

Thes. IV. In integrandis quantitatibus differentialibus formæ irrationalis, eximium sæpissime præbent usum substitutiones quantitatum trigonometricarum.

Thes. V. Facili constructione geometrica invenitur relatio illa Fluxionum Sinus, Cosinus, Tangentis, Secantis atque Arcus circularis, qua, pro Sinu toto = 1, est $d\ Sin\ nv = ndv\ Cos\ nv$, $d\ Cos\ nv = -ndv\ Sin\ nv$, $d\ Tg\ nv = ndv\ Sec\ nv^2$, & $d\ Sec\ nv = ndv\ Tg\ nv\ Sec\ nv$.

Thes. VI. Fluxionibus autem Sinus & Cosinus cognitis, facilius earum ope, quam per reversionem Serierum, vel ope quantitatum imaginariarum repeti̇untur Series, quibus pro data longitudine Arcus, computatur valor Sinus & Cosinus correspondentis.

§. I.

Osculum appellant Mathematici Circulum illum, qui curvam quamvis Lineam in aliquo ejus puncto exacte adeo atque intime contingit, ut in puncto contactus inter hanc Curvam atque Circulum *osculum* nullus alius Circulus describi possit. Eamdem itaque habent curvaturam Circulus osculus & Curva in hocce puncto contactus. Radium Circuli osculi, qui etiam *Radius curvaturæ* dicitur, inversam sequentem rationem curvaturæ Circuli osculantis, in inversa quoque semper esse ratione curvaturæ ipsius Lineæ curvæ in puncto contactus, hinc patet. Quare ut in diversis punctis Curvæ alicujus Curvatura diversa est, sic etiam ejusdem Curvæ pro diversis punctis Radii curvaturæ diversæ sunt longitudinis. Sunt autem hi semper functiones quædam vel algebraicæ vel transscendentes Coordinatarum ipsius Curvæ, ita ut pro quavis Curva atque quovis ejus puncto determinari queat Radius curvaturæ. Ope Calculi Differentialis generalissimam exhibuerunt Mathematici formulam cujusvis Curvæ determinandi Radios Curvaturæ. Ita pro Ordinatis inter se parallelis, denotante x Abscissam, y Ordinatam, & a Sinum anguli Coordinatarum pro Sinu toto $= b$, invenerunt esse generaliter Radium

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curvaturæ $= \frac{b(dx^2 + dy^2)^{\frac{3}{2}}}{a(dyddx - dxddy)}$. Pro Ordinatis autem e puncto aliquo fixo progredientibus, denotante r radium circuli in quo computantur Abscissæ, atque facta $p = \frac{dy}{dx}$, docuerunt eum esse $= \frac{dx(r^2p^2 + y^2)^{3:2}}{dx(r^2p^2 + y^2) + r^2(pdy - ydp)}$. Sic quoque inverse, ope Calculi Integralis, ex dato Radio curvaturæ investigari potest æquatio Curvæ, quoties nimirum integratio quantitatum differentialium succedit. Problemata quædam huc pertinentia solvimus, atque tua, Lector candide, venia publicæ luci committere audemus.

§. II.

PROBLEMA. *Invenire Curvam, cujus Radius curvaturæ est* $= x + \frac{xdy^2}{dx^2}$, *denotante* x *Abscissam, &* y *Ordinatam, atque exsistentibus Ordinatis inter se parallelis & quidem Coordinatis orthogonalibus.*

Sit dx constans, atque erit pro Coordinatis orthogonalibus Radius curvaturæ $= \frac{(dx^2 + dy^2)^{3:2}}{-dxddy}$ (§. I.). Habetur itaque ex hypothesi æquatio:

$x +$

$x + \frac{xdy^2}{dx^2} = \frac{(dx^2 + dy^2)^{3:2}}{-dxddy}$, vel facta debita reductione: $xddy = -dx\sqrt{dx^2 + dy^2}$. Ut hæc integretur, fiat $dy = zdx$; quo facto erit $ddy = dxdz$. His autem valoribus in æquationem differentialem secundi ordinis: $xddy = -dx\sqrt{dx^2 + dy^2}$ substitutis, atque terminis rite translatis, prodit æquatio differentialis primi ordinis hæc: $\frac{dz}{\sqrt{1+z^2}} = -\frac{dx}{x}$. Fiat ulterius $v = z + \sqrt{1+z^2}$, adeoque $v^2 - 2vz + z^2 = 1 + z^2$, & hinc $z = \frac{v^2-1}{2v}$, $\sqrt{1+z^2} = \frac{v^2+1}{2v}$, ut etiam, sumtis utrinque Fluxionibus, $dz = \frac{(v^2+1)dv}{2v^2}$. Substitutis his valoribus, habetur $\frac{dz}{\sqrt{1+z^2}} = \frac{dv}{v} = -\frac{dx}{x}$, atque integrando hanc æquationem obtinetur $L.v = L.a - L.x = L.\frac{a}{x}$, (denotante L Logarithmum Hyperbolicum, & a quantitatem constantem), ergo etiam $v = z + \sqrt{1+z^2} = \frac{a}{x}$. Hinc autem invenitur $z = \frac{a^2-x^2}{2ax}$, adeoque hoc valore substituto erit $dy = \frac{(a^2-x^2)dx}{2ax} = \frac{adx}{2x} - \frac{xdx}{2a}$, & u-

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trinque integrando habetur æquatio Curvæ quæſitæ hæc: $y = aL.\sqrt{x} - \frac{x^2}{4a} + b$, ubi b eſt quantitas conſtans corrigens.

§. III.

PROBLEMA. *Invenire æquationem Curvæ, cujus pro Coordinatis orthogonalibus Radius curvaturæ eſt* = $\frac{(3 + x^2 - 2\sqrt{1 + x^2})^{3:2}}{-x}$, *denotante* x *Abſcisſam.*

Eſt itaque pro dx conſtante $\frac{(dx^2 + dy^2)^{3:2}}{dxddy} = \frac{(3 + x^2 - 2\sqrt{1 + x^2})^{3:2}}{x}$, ſeu $(3 + x^2 - 2\sqrt{1 + x^2})^{3:2} dxddy = x(dx^2 + dy^2)^{3:2}$, quæ æquatio, facta $dy = zdx$, adeoque $ddy = dxdz$, & ſubſtitutis hisce valoribus, abit in hanc differentialem primi ordinis $\frac{dz}{(1 + z^2)^{3:2}} = \frac{xdx}{(3 + x^2 - 2\sqrt{1 + x^2})^{3:2}}$. In membro priori hujus æquationis integrando ponatur $\sqrt{1 + z^2} = z + \sqrt{u - 1}$; quare erit $1 + z^2 = z^2 + 2z\sqrt{u - 1} + u - 1$, & hinc $z = \frac{2 - u}{2\sqrt{u - 1}}$, atque $1 + z^2 = \frac{u^2}{4(u - 1)}$, vel $(1 + z^2)^{3:2}$

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$= \frac{u^3}{8(u-1)^{3:2}}$, ut etiam ſumtis fluxionibus, $dz = \frac{-udu}{4(u-1)^{3:2}}$. Cum autem hæ fiunt ſubſtitutiones, prodit $\frac{dz}{(1+z^2)^{3:2}} = -\frac{2du}{u^2}$, adeoque erit $\int \frac{dz}{(1+z^2)^{3:2}} = \frac{2}{u} + C$, hoc eſt, reſtituto valore ipſius $u = 2(\sqrt{1+z^2} - z)\sqrt{1+z^2}$, erit $\int \frac{dz}{(1+z^2)^{3:2}} = \frac{1}{(\sqrt{1+z^2}-z)\sqrt{1+z^2}} + C$. Sed pro caſu $z = 0$ fit $\int \frac{dz}{(1+z^2)^{3:2}} = 0$, atque $\frac{1}{(\sqrt{1+z^2}-z)\sqrt{1+z^2}} = 1$; quare erit $0 = 1 + C$, vel $C = -1$, adeoque hac facta correctione, $\int \frac{dz}{(1+z^2)^{3:2}} = \frac{1}{(\sqrt{1+z^2}-z)\sqrt{1+z^2}} - 1 = \frac{z}{\sqrt{1+z^2}}$. Ut quoque inveniatur Integrale $\int \frac{xdx}{(3+x^2-2\sqrt{1+x^2})^{3:2}}$, obſervandum eſt, eſſe $3 + x^2 - 2\sqrt{1+x^2} = 1 + (1-\sqrt{1+x^2})^2$. Fiat ergo $1 - \sqrt{1+x^2} = v$, & erit $xdx = (v-1)\,dv$, adeoque $\int \frac{xdx}{(3+x^2-2\sqrt{1+x^2})^{3:2}} = \int \frac{(v-1)dv}{(1+v^2)^{3:2}} = \int \frac{vdv}{(1+v^2)^{3:2}}$ —

$-\int \frac{dv}{(1+v^2)^{3:2}}$. Est autem $\int \frac{v\,dv}{(1+v^2)^{3:2}} = \frac{-1}{\sqrt{1+v^2}}$, & $\int \frac{dv}{\sqrt{1+v^2}} = \frac{v}{\sqrt{1+v^2}}$, adeoque $\int \frac{x\,dx}{(3+x^2-2\sqrt{1+x^2})^{3:2}} = -\frac{1+v}{\sqrt{1+v^2}} = \frac{\sqrt{1+x^2}-2}{\sqrt{(1+(1-\sqrt{1+x^2})^2)}}$, neglecta quantitate constante corrigente. Est itaque $\frac{z}{\sqrt{1+z^2}} = \frac{\sqrt{1+x^2}-2}{\sqrt{(1+(1-\sqrt{1+x^2})^2)}}$. Hinc autem eruitur $z = \frac{\sqrt{1+x^2}-2}{\sqrt{(2(\sqrt{1+x^2}-1))}}$, adeoque $dy = \frac{(\sqrt{1+x^2}-2)\,dx}{\sqrt{(2(\sqrt{1+x^2}-1))}}$. Ut jam integretur membrum posterius hujus æquationis, sumatur ϕ angulus talis, ut pro Sinu toto $= 1$, sit $2Tg\,\phi^2 = \sqrt{1+x^2}-1$, quo facto erit $\sqrt{1+x^2} - 2 = 2Tg\,\phi^2 - 1$, & $x = 2\,Tg\,\phi\,Sec\,\phi$, adeoque, ob $d(Tg\,\phi) = Sec\,\phi^2 d\phi$, & $d(Sec\,\phi) = Tg\,\phi\,Sec\,\phi d\phi$, erit $dx = 2\,Sec\,\phi^3\,d\phi + 2\,Tg\,\phi^2\,Sec\,\phi\,d\phi$, quare post institutam substitutionem obtinetur $\frac{(\sqrt{1+x^2}-2)\,dx}{\sqrt{(2(\sqrt{1+x^2}-1))}} = 2\,Tg\,\phi\,Sec\,\phi^3\,d\phi + 2\,Tg\,\phi^3\,Sec\,\phi\,d\phi - \frac{Sec\,\phi^3\,d\phi}{Tg\,\phi} - Tg\,\phi\,Sec\,\phi d\phi = 2Sin\,\phi\,Cos\,\phi^{-4}d\phi + 2Sin\,\phi^3\,Cos\,\phi^{-4}d\phi - Sin\,\phi^{-1}\,Cos\,\phi^{-2}d\phi - Sin\,\phi\,Cos\,\phi^{-2}d\phi$. Est autem

$\int Sin$

$\int Sin\,\varphi\,Cos\,\varphi^{-4}d\varphi = \frac{1}{3}Cos\,\varphi^{-3} = \frac{1}{3}(1 + Tg\,\varphi^2)^{3:2}$; $\int Sin\,\varphi^3\,Cos\,\varphi^{-4}d\varphi = \frac{1}{3}Sin\,\varphi^2\,Cos\,\varphi^{-3} - \frac{2}{3}\int Sin\,\varphi\,Cos\,\varphi^{-2}d\varphi = \frac{1}{3}Sin\,\varphi^2\,Cos\,\varphi^{-3} - \frac{2}{3}Cos\,\varphi^{-1} = \frac{1}{3}(Tg\,\varphi^2 - 2)\sqrt{1 + Tg\,\varphi^2}$; $\int Sin\,\varphi^{-1}\,Cos\,\varphi^{-2}d\varphi = Cos\,\varphi^{-1} + \int Sin\,\varphi^{-1}d\varphi = \sqrt{1 + Tg\,\varphi^2} + L.\,Tg\,\frac{1}{2}\varphi$; atque $\int Sin\,\varphi\,Cos\,\varphi^{-2}d\varphi = Sec\,\varphi = \sqrt{1 + Tg\,\varphi^2}$ (°). Hisque omnibus valoribus collectis & reductis, obtinetur $\int \frac{\sqrt{(1+x^2} - 2)\,dx}{\sqrt{(2(\sqrt{1+x^2} - 1))}} = -\frac{4}{3}(2 - Tg\,\varphi^2) - L.\,Tg\,\frac{1}{2}\varphi + C = C - \frac{4}{6}(5 - \sqrt{1 + x^2}) - L.\,\frac{\sqrt{(1 + \sqrt{(1+x^2)}) - \sqrt{2}}}{\sqrt{(\sqrt{1+x^2} - 1)}} = y$, æquatio quæsita Curvæ.

§. IV.

Problema. *Invenire æquationem Curvæ, cujus pro Coordinatis orthogonalibus Radius curvaturæ est* $= \frac{dy\,d^3y\,(dx^2 + dy^2)^{3:2}}{dx\,ddy^3}$.

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(°) Cfr. Dissert. *de Integratione Fluxionum formæ Sin* z^m *Cos* $z^n dz$, a Joh. H. Lindquist, Præside M. J. Walle-

Fluat Abſciſſa x uniformiter, ut ſit $ddx = 0$, & erit ex hypotheſi $\frac{dyd^3y(dx^2+dy^2)^{3:2}}{dxddy^3} = \frac{(dx^2+dy^2)^{3:2}}{-dxddy}$, unde facta debita reductione, habetur hæc æquatio differentialis tertii ordinis: $dyd^3y + ddy^2 = 0$. Ut hæc integretur, fiat $ddy = zdy^2$, unde erit $d^3y = 2z^2dy^3 + dzdy^2$. His autem in æquatione $dyd^3y + ddy^2 = 0$ adhibitis ſubſtitutionibus, ea ad hanc transformatur primi ordinis: $3dy + \frac{dz}{z^2} = 0$, quæ integrata dat $3y + A = 1:z$, (denotante A quantitatem conſtantem arbitrariam). Hinc erit $z = \frac{1}{A+3y}$, adeoque $\frac{ddy}{dy} = \frac{dy}{A+3y}$, unde iterum integrando eruitur $L\frac{Cdy}{dx} = \frac{1}{3}L(A+3y)$, (ſi L denotat Logarithmum Hyperbolicum, & C quantitatem aliquam conſtantem), vel, transeundo a Logarithmis ad quantitates abſolutas, $dx = \frac{Cdy}{(A+3y)^{1:3}}$. Ut autem hæc æquatio integretur, fiat $Tg\,\varphi^2 = \frac{3y}{A}$, (poſito Sinu toto $= 1$), adeoque $dy = \frac{2}{3}A\,Tg\,\varphi\,Sec\,\varphi^2\,d\varphi$, & $(A+3y)^{1:3} =$ $A^{1:3}$

NIO, Aboæ 1768 edit., & *Acta Holmienſ.* pro Ann. 1758, pag. 194.

$A^{1:3}\, Sec\, \varphi^{2:3}$, atque institutis his substitutionibus erit $\frac{dy}{(A+3y)^{1:3}} = \frac{2}{3} A^{2:3}\, Sin\, \varphi\, Cos\, \varphi^{-7:3} d\varphi$. Est itaque $\int \frac{C dy}{(A+3y)^{1:3}} = \frac{A^{2:3} C}{2 Cos\, \varphi^{4:3}} = \frac{1}{2} C (A + 3y)^{2:3}$, adeoque si D est quantitas constans, $(x + D)^3 = \frac{1}{8} C^3 (A + 3y)^2$, quæ est æquatio quæsita Curvæ, hæcque Curva esse videtur algebraica tertii ordinis.

§. V.

PROBLEMA. *Invenire Curvam, cujus Radius curvaturæ est* $= - \frac{dy (dx^2 + dy^2)^{3:2}}{dx d^3 y}$, *quando sunt Coordinatæ orthogonales.*

Comparando valorem Radii curvaturæ datum $- \frac{dy (dx^2 + dy^2)^{3:2}}{dx d^3 y}$ cum generali illo, pro x uniformiter fluente, $\frac{(dx^2 + dy^2)^{3:2}}{- dx ddy}$, obtinetur post debitam reductionem hæc æquatio differentialis tertii ordinis: $d^3 y - dy ddy = 0$, quæ, facta $ddy = z dy^2$, adeoque $d^3 y = 2z^2 dy^3 + dz dy^2$, atque substitutis his valoribus ipsarum ddy & $d^3 y$, migrat in hanc primi ordinis: $dy = \frac{dz}{z(1-2z)} = \frac{dz}{z} + \frac{2dz}{1-2z}$, ex qua integrando obtinetur $y = L\,\frac{Az}{1-2z}$ (si L denotat Logarithmum hyperbolicum, & A constantem arbitrariam). Hinc autem, transeundo a Logarithmis ad quantitates absolutas, existente $L.N = 1$, obtinetur $N^y = \frac{Az}{1-2z}$, adeoque $z = \frac{N^y}{A+2N^y}$. Hoc valore ipsius z in æquatione $ddy = zdy^2$ adhibito, obtinetur hæc æquatio $\frac{ddy}{dy} = \frac{N^y dy}{A+2N^y}$, unde integrando invenitur $L\,\frac{Bdy}{dx} = \frac{1}{2}L(A + 2N^y)$, adeoque $dx = \frac{Bdy}{\sqrt{A+2N^y}}$, & $x + D = \int \frac{Bdy}{\sqrt{A+2N^y}}$, existentibus B & D quantitatibus constantibus, Integralium corrigendorum gratia additis. Ut autem inveniatur integrale $\int \frac{Bdy}{\sqrt{A+2N^y}}$, fiat, pro Sinu toto $= 1$, $Tg\,\phi^2 = \frac{2N^y}{A}$, quo facto erit $\sqrt{A + 2N^y} = A^{1:2}\,Sec\,\phi$, & sumtis fluxionibus $dy = 2Sin\,\phi^{-1}\,Cos\,\phi^{-1}d\phi$, adeoque $\int \frac{Bdy}{\sqrt{A+2N^y}} = \frac{2B}{A^{1:2}}\int Sin\,\phi^{-1}d\phi = \frac{2B}{\sqrt{A}}L.Tg\,\frac{1}{2}\phi$ $= \frac{2B}{\sqrt{A}}L\,\frac{\sqrt{(A+N^y)} - \sqrt{A}}{\sqrt{(2N^y)}} = x + D$, quæ est æquatio ad Curvam quæsitam.

§. V.

§. VI.

PROBLEMA. *Invenire æquationem Curvæ, cujus, pro Coordinatis orthogonalibus, Radius curvaturæ est* $= \frac{9dy^2(dyd^3y - 2ddy^2)(dx^2 + dy^2)^{3:2}}{dxddy(8ddy^2 - 3dyd^3y)\sqrt{4ddy^2 + 9dy^4}}$.

Si comparantur valor datus Radii curvaturæ & generalis hic $\frac{(dx^2 + dy^2)^{3:2}}{-dxddy}$, obtinetur poſt debitam reductionem æquatio hæc: $(8ddy^2 - 3dyd^3y)\sqrt{(4ddy^2 + 9dy^4)} = 9(2ddy^2 - dyd^3y)dy^2$. Poſita $ddy = zdy^2$, erit $d^3y = dzdy^2 + 2z^2dy^3$, quibus valoribus in æquatione ſuperiori ſubſtitutis, prodit poſt debitam ſubſtitutionem, $dy = \frac{3dz}{2z^2} - \frac{9dz}{2z^2\sqrt{(4z^2+9)}}$, unde integrando obtinetur $y + a = -\frac{3}{2z} - \frac{9}{2}\int\frac{dz}{z^2\sqrt{4z^2+9}}$, exiſtente a conſtante. Ut ad formam commodiorem transmutetur integrale $\int\frac{dz}{z^2\sqrt{4z^2+9}}$, fiat $Tg\,\varphi = \frac{2}{3}z$, unde ſumtis fluxionibus erit $dz = \frac{3}{2}Sec\,\varphi^2 d\varphi$. Eſt quoque $\sqrt{(4z^2+9)} = 3Sec\,\varphi$, adeoque his adhibitis ſubſtitutionibus habetur $\int\frac{dz}{z^2\sqrt{4z^2+9}} = \frac{2}{3}\int Sin\,\varphi^{-2}\,Cos\,\varphi d\varphi$

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$= -\frac{2}{3}\, Sin\, \varphi^{-1}$. Cum autem ſit $Sin\, \varphi = \frac{2z}{\sqrt{4z^2 + 9}}$, adeoque $\int \frac{dz}{z^2 \sqrt{4z^2+9}} = -\frac{\sqrt{4z^2+9}}{9z}$, erit $a + y = -\frac{3}{2z} + \frac{\sqrt{(4z^2+9)}}{2z}$; unde invenitur $z = \frac{3(a+y)}{1-(a+y)^2}$. Hunc ipſius z valorem ſubſtituendo in æqu. $ddy = zdy^2$, obtinetur $\frac{ddy}{dy} = \frac{3(a+y)\,dy}{1-(a+y)^2}$, unde integrando obtinetur $L\frac{bdy}{dx} = -\frac{3}{2}\, L\,(1-(a+y)^2) = L\frac{1}{(1-(a+y)^2)^{3:2}}$, (ubi b deſignat quantitatem conſtantem). Hinc autem erit $b\,(1-(a+y)^2)^{3:2}dy = dx$. Ut hujus æquationis membrum prius ad integrationem commodius reddatur, fiat $Coſ\,\psi = a + y$, ut ſit $dy = -Sin\,\psi d\psi$, & $(1-(a+y)^2)^{3:2} = Sin\,\psi^3$, adeoque ſubſtitutis his valoribus, $(1-(a+y)^2)^{3:2}\, dy = -Sin\,\psi^4\, d\psi$. Eſt autem $\int Sin\,\psi^4\, d\psi = -\frac{1}{4} Sin\,\psi^3\, Coſ\,\psi - \frac{3}{8} Sin\,\psi\, Coſ\,\psi + \frac{3}{8}\psi$, adeoque, reſtitutis valoribus $Sin\,\psi = \sqrt{(1-(a+y)^2)}$ & $Coſ\,\psi = a + y$, erit, (ſi c denotat quantitatem conſtantem), $c + x = \frac{1}{4} b\,(a+y)\,(1-(a+y)^2)^{3:2} + \frac{3}{8} b\,(a+y)\,\sqrt{(1-(a+y)^2)} - \frac{3}{8} b\; Arc.\, Coſ\,(a+y)$, quæ eſt æquatio quæſita ad Curvam.