Åbo Akademi

FACULTY OF
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MASTER'S THESIS

Wilson loops in lattice gauge theories

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2022

Abstract

This thesis is based on the article "Wilson loops in finite Abelian lattice gauge theories" by M. Forsström, J. Lenells and F. Viklund [1]. In the article, lattice gauge theories on $\mathbb{Z}^4$ for a finite Abelian structure group are considered and the expectation value for the Wilson loop observable at weak coupling is computed. The purpose of this thesis is to explain this article in more detail and to give the theory necessary to understand the article.

In this thesis, we consider the lattice $\mathbb{Z}^4$, the structure group $\mathbb{Z}_n$ and a faithful and one-dimensional representation. Basic theory for groups, representations and lattices is discussed. To state the main result, several definitions, e.g. the Wilson loop observable and Wilson action, are given. The main result is given as a theorem, where we have an inequality for the limit of the expectation value of the Wilson loop observable.

The theory necessary to prove the main result is given in this thesis. Theory for discrete exterior calculus is given in the third chapter. This includes theory for k-cells and k-forms as well as definitions and applications for both the exterior derivative and the co-derivative. Furthermore, two versions of the Poincaré lemma are given and applied to problems, e.g. for writing the given measure as a measure on plaquette configurations instead of spin configurations. The Hodge dual of the lattice $\mathbb{Z}^r$ is defined and both examples and lemmas, which are important for later proofs, are given.

In the fourth chapter, vortices and oriented surfaces are defined using the theory from the previous chapter. It is important to note that these definitions might differ from other sources. Various lemmas are stated and proved. The most important result in this chapter is a proposition, in which a probability is computed, that is applied several times in the proof of the main result.

Since the limit of the expectation of the Wilson loop observable is computed, both its existence and translation invariance must be proved. A more general theorem, which proves the existence and translation invariance for a real-valued function, is given and proved with Ginibre's inequality. This theorem is then applied to the Wilson loop observable.

The last part of this thesis is the proof of the main theorem. To prove this theorem, the theory and results given in the earlier chapters are applied. The proof is divided into two parts, which are then combined to achieve the desired result.

Contents

Chapter 1

Introduction

Trying to mathematically describe the quantum field theories in the Standard Model is both a fascinating and difficult task. One method aiming to do this is lattice gauge theories. In these theories, space-time is discretised to a four-dimensional Euclidean lattice. Then, quantum field theories are approximated with methods from probability theory. Last, by taking a scaling limit, it might be possible to extend the model back to the continuous case and, thus, describe continuum quantum field theories.

Mathematical models for lattice gauge theories have been studied for a long time. They can be divided into two groups depending on whether its coupling constant is small or large. If the coupling constant is assumed to be small, we say that the gauge theory is in the weak coupling regime. The recently published articles by S. Chatterjee [2] and S. Cao [3] both compute the expectation values of a Wilson loop at weak coupling. This thesis will focus on the article by Forsström et al. [1], which extends Chatterjee's article. Since this model is in finite Abelian lattice gauge theories, it might not be relevant to the Standard Model, but it is still an interesting mathematical model to study.

The purpose of this thesis is to explain the third version of this article so that a master's student in mathematics understands it. The reader is expected to be familiar with probability theory, measure theory, complex analysis and algebra. Other necessary mathematical theory, e.g. group theory and discrete exterior calculus, will be discussed in this thesis and the reader is not required to have prior knowledge of them.

Chapter 2

The expectation value of the Wilson loop

In this chapter, the result for the expectation value of the Wilson loop observable will be given as a theorem. Before that, some theory necessary to understand this theorem is given. First, some basic theory for groups and representations is given. Second, the lattice used in the main theorem is described and terms necessary to the theorem, such as the Wilson action and the Wilson loop, are defined. Third, the main theorem is stated.

2.1 Groups

In this thesis, we will have a group $(G, +)$, which is a finite non-trivial Abelian group. Therefore, we have to define what that is, beginning with the definition of a group. A group is a set equipped with a binary operation that satisfies the group axioms in the definition below.

Definition 2.1. *(Group)* $(G, +)$ *is a group if :*
For all $a, b \in G : a + b \in G$ *(Closure)*
For all $a, b, c \in G : (a + b) + c = a + (b + c)$ *(Associativity)*
There exists an identity element $e \in G$ *for all* $a \in G$*:* $a + e = e + a = a$ *(Identity)*
There exists an inverse element $-a \in G$ *for all* $a \in G$*:* $a - a = -a + a = e$ *(Inverse).*

An Abelian group is a group, for which the elements commute.

Definition 2.2. *(Abelian group) A group is Abelian if for all* $a, b \in G : a + b = b + a$ *(Commutativity).*

Example 2.3. *Some examples of Abelian groups under ordinary addition are* $(\mathbb{Z}, +)$, $(\mathbb{R}, +)$ *and* $(\mathbb{Q}, +)$. *These groups are all infinite.*

In this thesis, we assume that the group G, which is often called structure group, is $G = (\mathbb{Z}_n, +)$, which is a finite group under addition modulo n. Here, the set $\mathbb{Z}_n$ is $\mathbb{Z}_n = \{0, 1, \ldots, n-1\}$ and for $a, b \in \mathbb{Z}_n$ we have that $a+b := a+b \bmod n$. The identity element for this group is $e = 0$.

2.2 Representations

In this section, two different kinds of representations, which we will have use of later, will be defined. Last, the representation that will be used in this thesis is given. First, we must define what a representation is. Therefore, we define group homomorphisms and the general linear group before giving the definition for a representation.

Definition 2.4. *(Group homomorphism) Let* $(G, *)$ *and* $(H, \diamond)$ *be groups. Then* $f : (G, *) \mapsto (H, \diamond)$ *is a homomorphism if for all* $a, b \in G$ *it holds that*

$$f(a * b) = f(a) \diamond f(b).$$

Definition 2.5. *(General linear group of a vector space) The general linear group of a vector space* V *is the group of linear invertible mappings from* V *to* V *and is denoted by* $GL(V)$.

It is well known that this is a group with the operation matrix multiplication.

Definition 2.6. *(Representation) A representation* ρ *of* $(G, +)$ *on a vector space* V *is a group homomorphism from* G *to* $GL(V)$.

From these definitions, it follows that the following property holds for a representation ρ of $(G, +)$:

$$\rho(g + g') = \rho(g)\rho(g') \quad \text{for } g, g' \in G. \tag{2.1}$$

We continue by discussing faithful representations.

Definition 2.7. *(Faithful representation) A representation ρ is faithful if it is injective.*

A representation ρ is injective if $\rho(g) = \rho(g')$ implies $g = g'$. Let $\mathbb{I}_V$ be the identity matrix for the vector space V. Then, from equation (2.1) follows that

$$\rho(e)\rho(g') = \rho(e + g') = \rho(g') \quad \text{for all } g' \in G.$$

Thus,

$$\rho(e) = \mathbb{I}_V.$$

In this thesis, ρ is defined as a faithful and one-dimensional representation of the group $G = \mathbb{Z}_n$. That the representation is one-dimensional means that the dimension of the vector space V is $\dim(V) = 1$. We derive the representation and begin with calculating the value of $\rho(0)$. Since 0 is the identity element for this group, we have

$$\rho(0) = \rho(e) = 1. \tag{2.2}$$

Since

$$\rho(1)^n = \rho(n \cdot 1) = \rho(e) = 1,$$

it follows that $\rho(g)$ must be the nth roots of unity (Recall that a root of unity is a complex or real solution to $x^n = 1$, where n is a positive integer and the nth root of unity is given by $e^{2k\pi i/n}$). Since ρ is the roots of unity, we obtain

$$|\rho(g)| = 1 \quad \text{for all } g \in G. \tag{2.3}$$

Last, the representation must take the same value for g and $g + n$. Hence, it follows that the representation must be given by

$$\rho(g) = e^{g \cdot 2\pi i m/n}, \quad g \in G = \mathbb{Z}_n, \tag{2.4}$$

for some $m \in \{1, \ldots, n-1\}$ relatively prime to n. For faithfulness of the representation, it is necessary that m is relatively prime to n.

We define a unitary representation and show that the representation ρ is unitary.

Definition 2.8. *(Unitary representation) A representation ρ is unitary if $\rho(g)$ is a unitary operator. The operator $\rho(g)$ is unitary if $\rho(g)^* = \rho(g)^{-1}$ for every $g \in G$, where $\rho(g)^*$ is the conjugate transpose of $\rho(g)$.*

To show that the representation ρ, defined in equation (2.4), is unitary, we calculate $\rho(g)\rho(g)^*$. The conjugate transpose of $\rho(g)$ is $\rho(g)^* = e^{-g\cdot 2\pi i m/n}$ and

$$\rho(g)\rho(g)^* = e^{g\cdot 2\pi i m/n} e^{-g\cdot 2\pi i m/n} = e^0 = 1.$$

Thus, $\rho(g)^* = \rho(g)^{-1}$ and the representation is unitary.

To summarise this section, the representation $\rho(g) = e^{g\cdot 2\pi i m/n}$ is a faithful, unitary and one-dimensional representation of $G = \mathbb{Z}_n$, $\rho(0) = 1$ and $\rho(g)$ is a root of unity. Last, the choice of m will determine the representation ρ.

2.3 The lattice $\mathbb{Z}^4$

Consider the lattice $\mathbb{Z}^4$, which is four-dimensional with a vertex at every integer coordinate. An edge is called positively oriented if the coordinate increases when traversing it. From every vertex x, exactly four positively oriented edges emerge to its nearest neighbours. These edges are denoted by dx_i, where $i = 1, 2, 3, 4$. The opposite direction, the negative orientation, of an edge is given by $-dx_i$. An oriented plaquette p is a pair of oriented edges that share a vertex and is defined as $p := dx_i \wedge dx_j$, where $\wedge$ is the wedge product. The boundary ∂p of a plaquette $p = dx_{j_1} \wedge dx_{j_2}$ is given by the edges $\partial p := \{dx_{j_1}, dx_{j_2} + (d\mathbf{e}_{j_1})_{j_2}, -(dx_{j_1} + (d\mathbf{e}_{j_2})_{j_1}), -dx_{j_2}\}$, where $d\mathbf{e}_{j_1}$ and $d\mathbf{e}_{j_1}$ are unit vectors. An illustration of an oriented plaquette and its boundary is given in Figure 2.1 below. The terms oriented edges, oriented plaquettes and boundary are defined in more detail in Section 3.1 and Section 3.3.

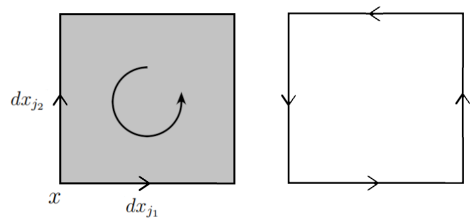

Figure 2.1: An oriented plaquette $p = dx_{j_1} \wedge dx_{j_2}$ and the edges in its boundary ∂p.

We continue by defining some subsets of the lattice. For a given $N \geq 1$, let $B_N = \mathbb{Z}^4 \cap [-N, N]^4$. Then the set E_N, whose elements are called e, is the set containing all oriented edges, whose both endpoints are in B_N. An oriented edge with the opposite direction of e is given by $-e$. An edge $e \in E_N$ is a boundary edge of B_N if there is a plaquette p with a corner that does not belong to B_N, such that $e \in \partial p$. The set of oriented plaquettes, which have all their boundary edges in E_N, are denoted by P_N. Elements in P_N are called p. Let the set $\sum_{E_N}$ be the set of functions $\sigma : E_N \to G$ for which $\sigma_e = -\sigma_{-e}$ and $\sigma_e \neq 0$ for all $e \in E_N$. The elements σ in Σ_{E_N} are called spin configurations. For $\sigma \in \Sigma_{E_N}$, $(d\sigma)_p$ is defined as

$$(d\sigma)_p := \sigma_{e_1} + \sigma_{e_2} + \sigma_{e_3} + \sigma_{e_4}, \quad p \in P_N,$$

where e_i are oriented edges in the boundary of p.

Last, we have some definitions for loops. Let $a_0, a_1, \ldots, a_n$ be vertices. A loop is then a sequence of oriented edges $e_0, e_1, \ldots, e_n$, where e_i is a vertex between a_{i-1} and a_i for $i = 1, \ldots, n$ such that $a_n = a_0$. Let γ be a loop, then the length $|\gamma|$ is defined as the number of edges in it. A loop, which is closed and oriented, is called simple if all oriented edges in γ are distinct from each other. Let e be an edge in γ. Then the edge $e' \in \gamma$, $e' \neq e$, is a corner edge if e' or $-e'$ shares a plaquette with the edge e. Edges which are not corner edges are called non-corner edges. This is illustrated in Figure 2.2, where the corner edges are given by the black edges and non-corner edges by the grey edges.

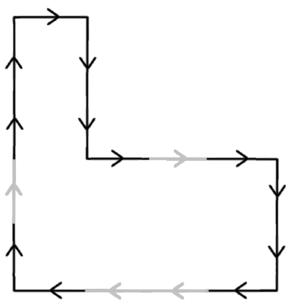

Figure 2.2: Corner and non-corner edges of a loop.

2.4 The Wilson loop observable

We begin with the definition of the Wilson action before defining the probability measure that will be used. Thereafter, the Wilson loop observable and the expectation value of it are defined. In the next definition, we have the notation $\Re(x)$, which gives the real part of the complex number x.

Definition 2.9. *The Wilson action corresponding to the structure group G, representation ρ and fixed N is given by*

$$S(\sigma) := -\sum_{p \in P_N} \Re\rho((d\sigma)_p), \quad \sigma \in \Sigma_{E_N}.$$

The probability measure μ_H is the uniform measure on the set $\sum_{E_N}$. From this measure, an associated probability measure $\mu_{\beta,N}$ on $\sum_{E_N}$ can be defined by weighting the uniform measure by $e^{-\beta S(\sigma)}$.

Definition 2.10. *For each $\beta > 0$, the probability measure $\mu_{\beta,N}$ on $\sum_{E_N}$ is*

$$\mu_{\beta,N}(\sigma) := Z_{\beta,N}^{-1} e^{-\beta S(\sigma)} \mu_H(\sigma), \quad \sigma \in \Sigma_{E_N}, \tag{2.5}$$

where $Z_{\beta,N}$ is a normalising constant.

Definition 2.11. *(The Wilson loop observable) Given a simple loop $\gamma \subseteq E_N$, the Wilson loop observable W_γ is*

$$W_\gamma := W_\gamma(\sigma) := \rho\left(\sum_{e \in \gamma} \sigma_e\right), \quad \sigma \in \Sigma_{E_N}. \tag{2.6}$$

Definition 2.12. *(Expectation of the Wilson loop observable) The expectation of the Wilson loop observable with respect to the measure $\mu_{\beta,N}$, using free boundary conditions, is*

$$\mathbb{E}_{\beta,N}[W_\gamma] = \sum_{\sigma \in \sum_{E_N}} W_\gamma(\sigma)\mu_{\beta,N}(\sigma).$$

Free boundary conditions means that they are no modifications at the boundary of B_N. Last, the definition for the limit of this expectation.

Definition 2.13. *(The limit of the expectation of the Wilson loop observable) The limit of the expectation of the Wilson loop observable $\langle W_\gamma \rangle_\beta$ is*

$$\langle W_\gamma \rangle_\beta := \lim_{N \to \infty} \mathbb{E}_{\beta,N}[W_\gamma]$$

when it exists.

In Chapter 5, we will see that the limit of the expected value of the Wilson loop observable both exists and is translation invariant.

2.5 The functions $\theta(\beta)$ and $\lambda(\beta)$

Before the main theorem is stated, two functions, $\theta(\beta)$ and $\lambda(\beta)$, must be defined. For $\beta \geq 0$, $\theta(\beta)$ is defined as

$$\theta(\beta) := \frac{\sum_{g\in G} \rho(g) e^{12\beta\Re\rho(g)}}{\sum_{g\in G} e^{12\beta\Re\rho(g)}} \tag{2.7}$$

and $\lambda(\beta)$ as

$$\lambda(\beta) := \max_{g\in G\setminus\{0\}} \frac{e^{\beta\Re(\rho(g))}}{e^{\beta\Re(\rho(0))}}. \tag{2.8}$$

We define a function $\phi_\beta(g)$ and write the functions $\lambda(\beta)$ and $\theta(\beta)$ with it. The function $\phi_\beta : G = \mathbb{Z}_n \to \mathbb{R}$ is defined by

$$\phi_\beta(g) := e^{\beta\Re\rho(g)}, \qquad g \in G. \tag{2.9}$$

Hence, the functions $\theta(\beta)$ and $\lambda(\beta)$ can be written as

$$\theta(\beta) = \frac{\sum_{g\in G} \rho(g)\phi_\beta(g)^{12}}{\sum_{g\in G} \phi_\beta(g)^{12}} \tag{2.10}$$

and

$$\lambda(\beta) = \max_{g\in G\setminus\{0\}} \frac{\phi_\beta(g)}{\phi_\beta(0)}. \tag{2.11}$$

2.6 The result for the expectation value

Theorem 2.14. *Let $n \in \mathbb{Z}$ and $n \geq 2$. Consider lattice gauge theory with the structure group $G = \mathbb{Z}_n$ and a one-dimensional and faithful representation ρ of G. Let $\gamma \in \mathbb{Z}^4$ be any simple oriented loop, $\ell = |\gamma|$ its length and ℓ_c the number of corner edges in γ. Then, the limit of the expectation value of the Wilson loop observable exists. Furthermore, for all $\beta_0 > 0$ chosen large enough, there exist constants $C' = C'(\beta_0)$ and $C'' = C''(\beta_0)$ such that for any $\beta \geq \beta_0$, the following inequality holds:*

$$|\langle W_\gamma\rangle_\beta - e^{-\ell(1-\theta(\beta))}| \leq C' \left[\frac{\sqrt{\ell_c}}{\ell} + \lambda(\beta)^2\right]^{C''}. \tag{2.12}$$

The constants C' and C'' are defined by equation (6.66) in Section 6.3.

From this inequality, the value of $\langle W_\gamma\rangle_\beta$ (the limit of the expectation of the Wilson loop) can be estimated. The function $e^{-\ell(1-\theta(\beta))}$ takes values in $]0, 1]$, since the constant $\theta(\beta)$ only takes values in $]0,1]$ (This is proved in Section 6.3.). The right-hand side is small, especially if we have a large β and a long loop with few corners. If $\ell(1-\theta(\beta))$ is very large, the exponential function will be close to zero and hence $\langle W_\gamma \rangle_\beta \approx 0$. Likewise, if $\ell(1-\theta(\beta))$ is very small, the exponential function will be close to 1 and we have $\langle W_\gamma \rangle_\beta \approx 1$.

Another approach to this is that $\ell(1-\theta(\beta))$ is large when β is chosen to be much smaller than ℓ. Then it is likely that there are a lot of plaquettes p near the loop γ for which $d\sigma(p) \neq 0$. This implies that there are so many plaquettes that the model will have independence and therefore gives the result that the expectation is close to zero. Similarly, when $\ell(1-\theta(\beta))$ is small, β is much larger than ℓ. Hence, it is unlikely that there are any plaquettes p near γ for which $d\sigma(p) \neq 0$. Therefore, the expectation will be close to one.

Remark 2.15. *The theorem holds when β_0 is chosen such that both $5(|G|-1)\lambda(\beta_0)^2 < 1$ and*

$$\max_{g_1,\ldots,g_6 \in G} \left[\frac{\sum_{g\in G} e^{-2\beta \sum_{k=1}^{6} \Re\rho(g+g_k)}}{\max_{g\in G} e^{-2\beta \sum_{k=1}^{6} \Re\rho(g+g_k)}} - \left| \arg\max_{g\in G} e^{-2\beta \sum_{k=1}^{6} \Re\rho(g+g_k)} \right| \right] \leq \frac{1-\cos(2\pi/n)}{8}$$

hold.

Chapter 3

Discrete exterior calculus

In this chapter, we focus on discrete exterior calculus for cell complexes of the lattice $\mathbb{Z}^r$ and its subset $B_N := [-N, N]^r \cap \mathbb{Z}^r$ when $r \geq 1$. However, the theory will later only be used for cell complexes of $\mathbb{Z}^4$. We assume that the group G is Abelian. Hence, the results can be applied to both $G = \mathbb{Z}$ and $G = \mathbb{Z}_n$. We begin with theory for k-cells and k-forms.

3.1 Oriented k-cells and k-forms

Earlier, both the oriented 1-cell and the oriented 2-cell were briefly discussed. Then they were called an oriented edge and an oriented plaquette. The 1-cell and the 2-cell will now be given new definitions and a general k-cell is defined. Consider the lattice $\mathbb{Z}^r$, which has a vertex at every point $x \in \mathbb{Z}^r$ with integer coordinates. Between every two neighbouring vertices, there exists an edge $\hat{e}$.

Definition 3.1. *(Oriented* 1*-cell or edge) The edge $\hat{e}$ between two neighbouring vertices can be divided into two oriented edges, e and $-e$, with the same endpoints. These oriented edges are called* 1*-cells.*

To define the direction of an oriented edge, unit vectors must first be defined. Let $\mathbf{e}_1 = (1, 0, 0, \ldots, 0), \mathbf{e}_2 = (0, 1, 0, \ldots, 0), \ldots, \mathbf{e}_r = (0, \ldots, 0, 1)$ and consider the edges from the origin to one of the vertices $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_r$. Edges, which have both the same length and direction as these edges, are unit vectors. These unit vectors are denoted by $d\mathbf{e}_1, \ldots, d\mathbf{e}_r$. Positive orientation is defined as the direction of the unit vectors and negative orientation as the opposite direction. Therefore, when $x \in \mathbb{Z}^r$ and $j \in \{1, 2, \ldots, r\}$, a positively oriented edge e can be written as $e = dx_j := x + d\mathbf{e}_j$. Then, the negatively oriented edge is given by $-e = -dx_j = x + \mathbf{e}_j - d\mathbf{e}_j$. Recall from Section 2.3 that in $\mathbb{Z}^4$ the set E_N was denoted to be the set of oriented edges whose both endpoints are in the set $\mathbb{Z}^4 \cap [-N, N]^4$. We expand this definition to the lattice $\mathbb{Z}^r$ and define E_N to be the set of oriented edges, whose both endpoints are in the set $B_N = \mathbb{Z}^r \cap [-N, N]^r$. Furthermore, the set of only positively oriented edges in B_N is given by the set E_N^+ and the set of only negatively oriented edges is given by E_N^-.

In the definition for oriented k-cells, the notation $e_1 \wedge e_2$ will be used. This is the wedge product between two oriented edges e_1 and e_2. The wedge product is defined to have the following properties: For two edges $e_1, e_2 \in E_N$, we have

$$e_1 \wedge e_2 = -(e_2 \wedge e_1) = (-e_2) \wedge e_1 = e_2 \wedge (-e_1)$$

and

$$e_1 \wedge e_1 = 0.$$

For two or more oriented edges $e_1, e_2, \ldots, e_k$ in E_N, the wedge product $e_1 \wedge \cdots \wedge e_k$ is zero if the edges $e_1, \ldots, e_k$ do not have a common endpoint.

Definition 3.2. *(Oriented k-cell) Let $e_1, \ldots, e_k$ be oriented edges with $e_1 \wedge \cdots \wedge e_k \neq 0$. Then $e_1 \wedge \cdots \wedge e_k$ is an oriented k-cell.*

A k-cell $e_1 \wedge \cdots \wedge e_k$ is positively oriented if there is an $x \in \mathbb{Z}^r$ and $j_1 < j_2 < \cdots < j_k$ such that $e_i = dx_{j_i}$. Correspondingly, the k-cell $-(e_1 \wedge \cdots \wedge e_k)$ is negatively oriented. Some examples of k-cells are illustrated in the figure below. Notice that a 0-cell is a vertex.

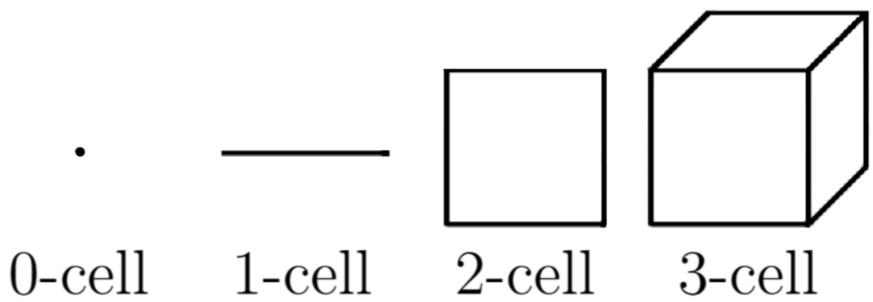

Figure 3.1: Examples of k-cells.

From now on, a k-cell will refer to an oriented k-cell.

Definition 3.3. *(Oriented plaquette) An oriented plaquette p is an oriented 2-cell.*

Since the oriented plaquette is a k-cell, the oriented plaquette $p = e_1 \wedge e_2$ is positively oriented if there is an $x \in \mathbb{Z}^r$ and $j_1 < j_2$ such that $e_1 = dx_{j_1}$ and $e_2 = dx_{j_2}$. Therefore, a positively oriented plaquette can be written as $p \coloneqq dx_{j_1} \wedge dx_{j_2}$. The set of oriented plaquettes whose all boundary edges are in E_N is given by P_N.

We continue with theory about k-forms, which are discrete differential forms.

Definition 3.4. *(k-form) A function f, which is defined on a subset of k-cells in $\mathbb{Z}^r$, for which $f(c) = -f(-c)$ is a k-form.*

Consider a k-form f, which has the value $f_{j_1,\ldots j_k}(x)$ on the k-cell $c = dx_{j_1} \wedge \cdots \wedge dx_{j_k}$. Then the k-form f is represented by

$$f(x) = \sum_{1 \leq j_1 < \cdots < j_k \leq r} f_{j_1,\ldots,j_k}(x) dx_{j_1} \wedge \cdots \wedge dx_{j_k}. \tag{3.1}$$

The function $f_{j_1,\ldots j_k}(x)$ can also be written shortly as f_c.

Example 3.5. *Two examples of 2-forms in $\mathbb{Z}^4$ are*

$$f_1(x) = f_{12}(x)dx_1 \wedge dx_2 + f_{34}(x)dx_3 \wedge dx_4$$

and

$$f_2(x) = f_{12}(x)dx_1 \wedge dx_2 + f_{13}(x)dx_1 \wedge dx_3.$$

Next, we define both non-trivial k-forms and the set support.

Definition 3.6. *(Non-trivial k-form) A k-form f is non-trivial if there exists one or more k-cells c for which $f_c \neq 0$.*

Definition 3.7. *(Support) The support of a k-form f is the set of oriented k-cells c, for which the k-form f satisfies $f(c) \neq 0$. The support of f is denoted by* $\operatorname{supp} f$.

Example 3.8. *Let A be a subset of E_N and define for each $e \in E_N$ the function f by*

$$f(e) \coloneqq \begin{cases} 1 & \text{if } e \in A, \\ -1 & \text{if } -e \in A, \\ 0 & \text{otherwise.} \end{cases}$$

Since $f(e) = -f(-e)$ *and* e *is a* 1*-cell, we have that* f *is a* 1*-form. The function only takes the value zero outside* A*, hence* $\operatorname{supp} f = A$.

The set Σ_{E_N} in $\mathbb{Z}^r$ is defined as the set of G-valued 1-forms, whose support is in E_N. Its elements are denoted by σ and called spin configurations. Recall the definition for Σ_{E_N} in $\mathbb{Z}^4$: the set of functions $\sigma : E_N \to G$ for which $\sigma_e = -\sigma_{-e}$ and $\sigma_e \neq 0$ for all $e \in E_N$. We see that this agrees with the definition for $\mathbb{Z}^r$.

Last, the restriction $(\sigma|_E)_e$ for a spin configuration $\sigma \in \Sigma_{E_N}$, a set $E \subseteq E_N$ and an edge $e \in E_N$ is defined by

$$(\sigma|_E)_e := \begin{cases} \sigma_e & \text{if } e \in E, \\ 0 & \text{otherwise.} \end{cases} \tag{3.2}$$

3.2 The exterior derivative

In this section, the exterior derivative is defined and some examples are given. When the exterior derivative operates on a k-form, the result is a $k+1$-form.

Definition 3.9. *(Exterior derivative) Given* $h : \mathbb{Z}^r \to G, x \in \mathbb{Z}^r$*, and* $i \in \{1, 2, \ldots, r\}$*, let*

$$\partial_i h(x) := h(x + \boldsymbol{e}_i) - h(x). \tag{3.3}$$

If $k \in \{0, 1, \ldots, r-1\}$ *and* f *is a* G*-valued* k*-form, the exterior derivative* d *of* f *is defined as*

$$df(x) = \sum_{1 \leq j_1 < \cdots < j_k \leq r} \sum_{i=1}^{r} \partial_i f_{j_1,\ldots,j_k}(x) dx_i \wedge (dx_{j_1} \wedge \ldots \wedge dx_{j_k}), \quad x \in \mathbb{Z}^r. \tag{3.4}$$

Next is an example for how the exterior derivative is calculated for 2-forms.

Example 3.10. *We calculate the exterior derivative of the* 2*-forms* $f_1 = f_{12}(x)dx_1 \wedge dx_2 + f_{34}(x)dx_3 \wedge dx_4$ *and* $f_2 = f_{12}(x)dx_1 \wedge dx_2 + f_{13}(x)dx_1 \wedge dx_3$*. We begin with calculating the exterior derivative for the* 2*-form* f_1*. By the definition of the exterior derivative, we have*

$$\begin{aligned} df_1 &= d(f_{12}(x)dx_1 \wedge dx_2 + f_{34}(x)dx_3 \wedge dx_4) \\ &= \partial_3 f_{12}(x)dx_1 \wedge dx_2 \wedge dx_3 + \partial_4 f_{12}(x)dx_1 \wedge dx_2 \wedge dx_4 \\ &\quad + \partial_1 f_{34}(x)dx_3 \wedge dx_4 \wedge dx_1 + \partial_2 f_{34}(x)dx_3 \wedge dx_4 \wedge dx_2. \end{aligned}$$

We calculate the exterior derivative for f_2:

$$\begin{aligned} df_2 &= d(f_{12}(x)dx_1 \wedge dx_2 + f_{13}(x)dx_1 \wedge dx_3) \\ &= \partial_3 f_{12}(x)dx_1 \wedge dx_2 \wedge dx_3 + \partial_4 f_{12}(x)dx_1 \wedge dx_2 \wedge dx_4 \\ &\quad + \partial_2 f_{13}(x)dx_1 \wedge dx_3 \wedge dx_2 + \partial_4 f_{13}(x)dx_1 \wedge dx_3 \wedge dx_4 \\ &= (\partial_3 f_{12}(x) - \partial_2 f_{13}(x))dx_1 \wedge dx_2 \wedge dx_3 + \partial_4 f_{12}(x)dx_1 \wedge dx_2 \wedge dx_4 \\ &\quad + \partial_4 f_{13}(x)dx_1 \wedge dx_3 \wedge dx_4. \end{aligned}$$

A closed k-form can now be defined.

Definition 3.11. *(Closed k-form) A k-form f is closed if $df = 0$.*

Last, we see that we earlier had an definition for the exterior derivative on a spin configuration on the lattice $\mathbb{Z}^4$.

Example 3.12. *Recall the definitions in Section 2.3. For $\sigma \in \Sigma_{E_N}$, we defined*

$$(d\sigma)_p \coloneqq \sigma_{e_1} + \sigma_{e_2} + \sigma_{e_3} + \sigma_{e_4}, \quad p \in P_N,$$

where e_i are edges in the boundary of p. This is the exterior derivative of $\sigma \in \Sigma_{E_N}$.

3.3 Boundary and co-boundary

Recall the definition for the boundary of a plaquette on $\mathbb{Z}^4$ in Section 2.3. We now expand this definition to $\mathbb{Z}^r$. When $x \in \mathbb{Z}^r$ and $j_1 < j_2$, the boundary of a plaquette $p = dx_{j_1} \wedge dx_{j_2}$ is defined as

$$\partial p \coloneqq \{dx_{j_1}, dx_{j_2} + (d\mathbf{e}_{j_1})_{j_2}, -(dx_{j_1} + (d\mathbf{e}_{j_2})_{j_1}), -dx_{j_2}\}.$$

This is illustrated in Figure 3.2 on page 17. The boundary for a plaquette can also be defined as the set of edges dx_j for which

$$(d(\mathbb{I}_{x=\hat{x}}dx_j))_p = 1,$$

where $\mathbb{I}_{x=\hat{x}}$ is the Kronecker delta function of x with mass at $\hat{x} \in \mathbb{Z}^r$. Next, we prove that these two definitions are equivalent.

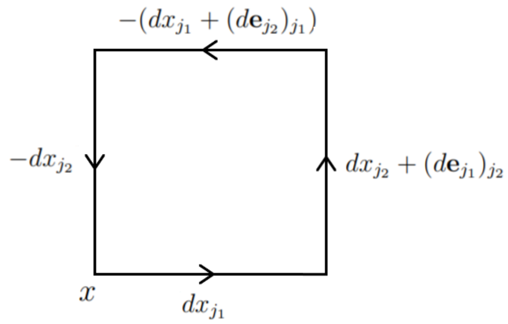

Figure 3.2: The boundary of the plaquette $p = dx_{j_1} \wedge dx_{j_2}$.

We prove this by letting the exterior derivative operate on the 1-form $\mathbb{I}_{x=\hat{x}} dx_j$, when $\hat{x} \in \mathbb{Z}^r$ and $j \in \{1, 2, \ldots, r\}$. By equations (3.4) and (3.3) for the function $h = \mathbb{I}_{x=\hat{x}}$, we have

$$\begin{aligned} d(\mathbb{I}_{x=\hat{x}} dx_j) &= \sum_{i=1}^{r} (\partial_i \mathbb{I}_{x=\hat{x}}) dx_i \wedge dx_j = \sum_{i=1}^{r} (\mathbb{I}_{x+\mathbf{e}_i=\hat{x}} - \mathbb{I}_{x=\hat{x}}) dx_i \wedge dx_j \\ &= \sum_{i=1}^{r} (\mathbb{I}_{x+\mathbf{e}_i=\hat{x}} - \mathbb{I}_{x=\hat{x}}) dx_i \wedge dx_j - \sum_{i=j+1}^{r} (\mathbb{I}_{x+\mathbf{e}_i=\hat{x}} - \mathbb{I}_{x=\hat{x}}) dx_j \wedge dx_i. \end{aligned}$$

For edges that are not in the boundary for neither the positive nor the negative direction, both $\mathbb{I}_{x+\mathbf{e}_i=\hat{x}} = 0$ and $\mathbb{I}_{x=\hat{x}} = 0$. Hence,

$$(d(\mathbb{I}_{x=\hat{x}} dx_j))_p = \begin{cases} (\mathbb{I}_{x+\mathbf{e}_i=\hat{x}} - \mathbb{I}_{x=\hat{x}}) & \text{if } e \in \partial p, \\ -(\mathbb{I}_{x+\mathbf{e}_i=\hat{x}} - \mathbb{I}_{x=\hat{x}}) & \text{if } -e \in \partial p, \\ 0 & \text{otherwise,} \end{cases}$$

where p is chosen as an oriented plaquette and $e = \hat{x} + d\mathbf{e}_j = d\hat{x}_j$ as an oriented edge. Since $\mathbb{I}_{x+\mathbf{e}_i=\hat{x}} - \mathbb{I}_{x=\hat{x}} = 1$ for edges in the boundary, we have

$$(d(\mathbb{I}_{x=\hat{x}} dx_j))_p = \begin{cases} 1 & \text{if } e \in \partial p, \\ -1 & \text{if } -e \in \partial p, \\ 0 & \text{otherwise.} \end{cases} \tag{3.5}$$

Thus, the definitions are equivalent.

From this follows that if f is a 1-form, $1 \leq j_1 < j_2 \leq r$ and the plaquette p is given by $p = dx_{j_1} \wedge dx_{j_2}$, then we have that

$$(df)_p = (df)_{j_1,j_2}(x) = \sum_{e \in \partial p} f_e. \tag{3.6}$$

Next, the definition for the boundary for general k-cells and a general version of formula (3.6) for k-forms.

Definition 3.13. *(Boundary) Let $k \in \{1, 2, \dots, r\}$ and c be a k-cell. The boundary $\partial\hat{c}$ of c is the set of $(k-1)$-cells $\hat{c} = dx_{j_1} \wedge \cdots \wedge d\hat{x}_{j_{k_1}}$ for which*

$$(d(\mathbb{I}_{x=\hat{x}} d\hat{x}_{j_1} \wedge \cdots \wedge d\hat{x}_{j_{k-1}}))_c = 1.$$

Let c_0 be a $(k+1)$-cell, then

$$(df)_{c_0} = \sum_{c \in \partial c_0} f_c. \tag{3.7}$$

The boundary is connected with the co-boundary, which is defined next.

Definition 3.14. *(Co-boundary) The co-boundary $\hat{\partial} c$ of a k-cell c is defined as the set of every $(k+1)$-cells $\hat{c}$ for which $c \in \partial\hat{c}$.*

Example 3.15. *The co-boundary $\hat{\partial} e$ of an edge e is the set of all oriented plaquettes p that contain e. An illustration is given in Figure 3.3. Furthermore, $e \in \partial p$ is equivalent to $p \in \hat{\partial} e$.*

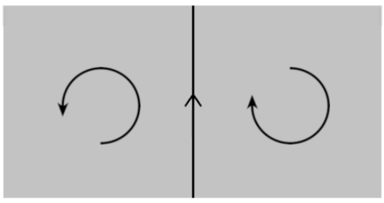

Figure 3.3: An edge and two of the plaquettes in its co-boundary.

Example 3.16. *For an edge $e \in E_N$, a plaquette p cannot contain both e and $-e$. Therefore, the intersection $\hat{\partial} e \cap \hat{\partial}(-e)$ is empty. Moreover, if $p \in \hat{\partial}(-e)$, then $-p \in \hat{\partial} e$.*

Last, boundary cells are defined and one example is given. Notice that a k-cell is in B_N if all corners of the k-cell is in B_N.

Definition 3.17. *(Boundary cell) For $k \in \{0, 1, \dots, r-1\}$, a k-cell c in B_N is a boundary cell of B_N if there exists a $(k+1)$-cell $\hat{c} \in \hat{\partial} c$ which is not in B_N.*

Example 3.18. *An edge $e \in E_N$ is a boundary edge of B_N if there exists one plaquette $p \in \hat{\partial} e$ which is outside B_N. A plaquette $p \in P_N$ is a boundary plaquette of B_N if there exists a 3-cell $c \in \hat{\partial} p$ which is outside B_N. This is illustrated in Figure 3.4, where one of the boundary edges is given by the blue edge and one of the boundary plaquettes is given by the blue plaquette.*

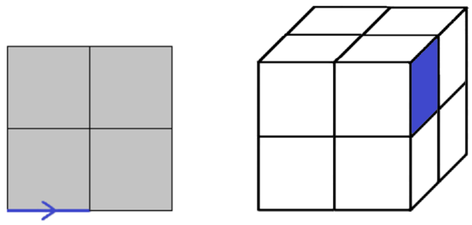

Figure 3.4: Example of a boundary edge and a boundary plaquette.

3.4 The Poincaré Lemma

In this section, the Poincaré Lemma will be given without proof and then applied to both the set Σ_{P_N} (which will be defined later in this section) and for writing the earlier given measure as a measure on elements in the set Σ_{P_N} instead of spin configurations. Another version of this lemma, the Poincaré Lemma for the co-derivative, will be given in the next section. Before the Poincaré lemma is stated, the definition for a box and a cube is given.

Definition 3.19. *(Box and cube) A set of the form $([a_1, b_1] \times \cdots \times [a_r, b_r]) \cap \mathbb{Z}^r$ is a box if for each $j \in \{1, 2, \ldots, r\}$, $\{a_j, b_j\} \subset \mathbb{Z}$ satisfies $a_j < b_j$. A box is a cube if all intervals $[a_j, b_j], 1 \leq j \leq r$, are of the same length.*

An example of a box is the set B_N.

Lemma 3.20. *(The Poincaré lemma). Let $k \in \{0, 1, \ldots, r-1\}$ and B be a box in $\mathbb{Z}^r$. Then the exterior derivative d is a surjective mapping from the set of G-valued k-forms with support contained in B onto the set of G-valued closed $(k+1)$-forms with support contained in B. If the group G is finite and the number of closed G-valued k-forms with support contained in B is m, then the map is an*

m-to-1 correspondence. If $k \in \{0, 1, 2, \ldots, r-1\}$ and f is a closed $(k+1)$-form that vanishes on the boundary of B, then there is a k-form h that also vanish on the boundary of B and satisfies $dh = f$.

For the proof, see [2, Lemma 2.2].

3.4.1 The set Σ_{P_N}

The set $\sum_{P_N}$ is defined as the set of closed G-valued 2-forms ω with support in P_N. The 2-forms in $\sum_{P_N}$ are called plaquette configurations. Since the set Σ_{P_N} only contains closed forms, we obtain

$$d\omega = 0 \quad \text{if } \omega \in \Sigma_{P_N}.$$

Recall that the 1-forms in the set Σ_{E_N} do not have to be closed, while the set Σ_{P_N} is defined to only contain closed 2-forms. The restriction for a plaquette configuration $\omega \in \Sigma_{P_N}$ is defined similar to how a restriction was defined for a spin configuration $\sigma \in \Sigma_{E_N}$ in equation (3.2). For $\omega \in \Sigma_{P_N}$, a set $P \subseteq P_N$ and a plaquette $p \in P_N$, the restriction $(\omega|_P)_p$ is defined as

$$(\omega|_P)_p := \begin{cases} \omega_p & \text{if } p \in P, \\ 0 & \text{otherwise.} \end{cases} \tag{3.8}$$

Next is an example of a restriction for a 2-form in P_N.

Example 3.21. *For $\omega \in \Sigma_{P_N}$, consider the restriction $\omega|_{\text{supp }\omega}$. By definition, each ω has support in P_N. Hence, $\omega|_{\text{supp }\omega} = \omega$.*

We apply the Poincaré lemma to the k-form $\omega \in \Sigma_{P_N}$ and the box B_N. Recall, that the set of G-valued 1-forms with support contained in P_N is the set Σ_{E_N} and that the 1-forms in Σ_{E_N} are called σ. Furthermore, we have that both σ and ω vanishes on the boundary of P_N. By the Poincaré Lemma, we have that the exterior derivative from the set Σ_{E_N} onto the set Σ_{P_N} is a surjective map. Since G is finite, there exists an m for which the map is a m-to-1 correspondence. Furthermore, since ω is closed, there is a σ such that $d\sigma = \omega$. As a result of these three conclusions, we have that $\omega \in \Sigma_{P_N}$ if and only if there exists a $\sigma \in \Sigma_{E_N}$ for which $d\sigma = \omega$.

3.4.2 The measure

Earlier, the measure $\mu_{\beta,N}$ was defined in Definition 2.5 as a measure on spin configurations:

$$\mu_{\beta,N}(\sigma) = Z_{\beta,N}^{-1} e^{\beta \sum_{p \in P_N} \Re \rho((d\sigma)_p)} \mu_H(\sigma), \quad \sigma \in \Sigma_{E_N}.$$

We will later use this measure on plaquette configurations instead of spin configurations. Therefore, we have to map spin configurations to plaquette configurations. A problem is that every plaquette configuration does not arise from a spin configuration. Nonetheless, the Poincaré Lemma says that $\omega \in \Sigma_{P_N}$ if and only if there exists a $\sigma \in \Sigma_{E_N}$ such that $d\sigma = \omega$. Furthermore, the map from spin configurations $\sigma \in \Sigma_{E_N}$ to plaquette configurations $\omega \in \Sigma_{P_N}$ is a many-to-1 correspondence. Since $|\{\sigma \in \Sigma_{E_N} : d\sigma = \omega\}|$ does not depend on the choice of ω, we have

$$\sum_{\sigma \in \Sigma_{E_N} : d\sigma = \omega} e^{\beta \sum_{p \in P_N} \Re \rho((d\sigma)_p)} = |\{\sigma \in \Sigma_{E_N} : d\sigma = \omega\}| e^{\beta \sum_{p \in P_N} \Re \rho(\omega_p)}.$$

Thus, a measure for plaquette configurations can be created:

$$\begin{aligned} \mu_{\beta,N}(\{\sigma \in \Sigma_{E_N} : d\sigma = \omega\}) &= \frac{\sum_{\sigma \in \Sigma_{E_N} : d\sigma = \omega} e^{\beta \sum_{p \in P_N} \Re \rho((d\sigma)_p)}}{\sum_{\sigma \in \Sigma_{E_N}} e^{\beta \sum_{p \in P_N} \Re \rho((d\sigma)_p)}} \\ &= \frac{e^{\beta \sum_{p \in P_N} \Re \rho(\omega_p)}}{\sum_{\omega' \in \Sigma_{P_N}} e^{\beta \sum_{p \in P_N} \Re \rho(\omega'_p)}} \\ &= \frac{\prod_{p \in P_N} \phi_\beta(\omega_p)}{\sum_{\omega' \in \Sigma_{P_N}} \prod_{p \in P_N} \phi_\beta(\omega'_p)} = \mu_{\beta,N}(\{\omega\}). \end{aligned} \tag{3.9}$$

3.5 The co-derivative and its Poincaré Lemma

The definition of the co-derivative is similar to the definition of the exterior derivative. They both operate on a k-form, but when the co-derivative operates the result is a $k-1$-form instead of a $k+1$-form.

Definition 3.22. *(Co-derivative) Given $h : \mathbb{Z}^r \to G, x \in \mathbb{Z}^r$, and an $i \in \{1, 2, \ldots, r\}$, let*

$$\bar{\partial}_i h(x) \coloneqq h(x) - h(x - \mathbf{e}_i). \tag{3.10}$$

If $k \in \{1, 2, \ldots, r-1\}$ and f is a G-valued k-form, the co-derivative δ of f is defined by

$$\delta f(x) := \sum_{1 \le j_1 < \cdots < j_k \le r} \sum_{i=1}^{k} (-1)^i \bar{\partial}_{j_i} f_{j_1,\ldots,j_k}(x) dx_{j_1} \wedge \cdots \wedge dx_{j_{i-1}} \wedge dx_{j_{i+1}} \wedge \ldots \wedge dx_{j_k} \tag{3.11}$$

for $x \in \mathbb{Z}^r$.

Example 3.23. *We calculate the co-derivative of the 2-forms $f_1 = f_{12}(x)dx_1 \wedge dx_2 + f_{34}(x)dx_3 \wedge dx_4$ and $f_2 = f_{12}(x)dx_1 \wedge dx_2 + f_{13}(x)dx_1 \wedge dx_3$.*

By definition, we have that the co-derivative of f_1 is

$$\delta f_1(x) = -\bar{\partial}_1 f_{12} dx_2 + \bar{\partial}_2 f_{12} dx_1 - \bar{\partial}_3 f_{34} dx_4 + \bar{\partial}_4 f_{34} dx_3$$

and the co-derivative of f_2 is

$$\delta f_2(x) = -\bar{\partial}_1 f_{12} dx_2 + \bar{\partial}_2 f_{12} dx_1 - \bar{\partial}_1 f_{13} dx_3 - \bar{\partial}_3 f_{13} dx_1.$$

Similarly to the calculations of $(d(\mathbb{I}_{x=\hat{x}} dx_j))_p$, we calculate the co-derivative of the 2-form $f := \mathbb{I}_{x=\hat{x}} dx_{i_1} \wedge dx_{i_2}$ where $x \in \mathbb{Z}^r$. Let $\hat{x} \in \mathbb{Z}^r$, $1 \le i_1 < i_2 \le r$ and $p_0 = d\hat{x}_{i_1} \wedge \hat{x}_{i_2} \in P_N$. For $j, j_1, j_2 \in \{1, 2, \ldots, r\}$, we have by equation (3.10) that

$$\bar{\partial}_j f_{j_1,j_2}(x) = \bar{\partial}_j \mathbb{I}_{x=\hat{x}, j_1=i_1, j_2=i_2} = \mathbb{I}_{x=\hat{x}, j_1=i_1, j_2=i_2} - \mathbb{I}_{x=\hat{x}-\mathbf{e}_j, j_1=i_1, j_2=i_2}.$$

Letting the co-derivative operate on f, we obtain

$$\begin{aligned}
\delta f(x) &= \sum_{1 \le j_1 < \cdots < j_2 \le r} \sum_{i=1}^{2} (-1)^i \bar{\partial}_i f_{j_1,j_2}(x) dx_{j_1} \wedge \cdots \wedge dx_{j_{i-1}} \wedge dx_{j_{i+1}} \wedge \ldots \wedge dx_{j_2} \\
&= (-1)^1 \bar{\partial}_i \mathbb{I}_{x=\hat{x}} dx_{i_2} + (-1)^2 \bar{\partial}_i \mathbb{I}_{x=\hat{x}} dx_{i_1} = -\bar{\partial}_i \mathbb{I}_{x=\hat{x}} dx_{i_2} + \bar{\partial}_i \mathbb{I}_{x=\hat{x}} dx_{i_1} \\
&= -\mathbb{I}_{x=\hat{x}} dx_{i_2} + \mathbb{I}_{x-\mathbf{e}_{i_1}=\hat{x}} dx_{i_2} + \mathbb{I}_{x=\hat{x}} dx_{i_1} - \mathbb{I}_{x-\mathbf{e}_{i_2}=\hat{x}} dx_{i_1}.
\end{aligned}$$

Thus,

$$(\delta f)_e = \begin{cases} 1 & \text{if } e \in \partial p_0, \\ -1 & \text{if } -e \in \partial p_0, \\ 0 & \text{otherwise.} \end{cases}$$

As a result, when $1 \le j \le n$ and f is a 2-form, we obtain for an edge $e = dx_j$ that

$$(\delta f)_e = (\delta f)_j(x) = \sum_{p \in \hat{\partial} c} f_p.$$

This result can be extended to general k-forms. For a k-form f and a $(k-1)$-cell c_0, we have

$$(\delta f)_{c_0} = \sum_{c \in \hat{\partial} c_0} f_c. \tag{3.12}$$

Last, the version of the Poincaré lemma for the co-derivative is stated.

Lemma 3.24. *(The Poincaré lemma for the co-derivative) Let $k \in \{1, 2, \ldots, r-1\}$ and f be a G-valued k-form on $\mathbb{Z}^r$ which is zero outside a finite region and satisfies $\delta f = 0$. Then there is a $(k+1)$-form h such that $f = \delta h$. Moreover, if f is equal to zero outside a box B, then there is a choice of h that is equal to zero outside B.*

For the proof, see [2, Lemma 2.7]. Notice that if a k-form is zero outside a finite region, the support of the k-form is contained in this finite region. This lemma will be applied when theory for oriented surfaces is discussed.

3.6 The Hodge dual

We call the lattice $\mathbb{Z}^r$ for the primal lattice and create a copy of it, which does not have its vertices at the same points as the primal lattice $\mathbb{Z}^r$. This copy, whose vertices are at the centres of the r-cells of the primal lattice, is called the dual lattice and denoted $*\mathbb{Z}^r$. Therefore, there is a bijection between the r-cells in the primal lattice and the 0-cells, i.e. vertices, in the dual lattice. Furthermore, we have a bijection between the set of k-cells of the primal lattice $\mathbb{Z}^r$ and the set of $(r-k)$-cells of the dual lattice $*\mathbb{Z}^r$, which we will define. For a cell c, the Hodge dual of the cell is denoted by $*c$. The operator $*$ is called the Hodge star operator and is additive.

We aim to define the bijection between a k-cell and its Hodge dual. First, some theory for permutations of sets is given. The sign of a permutation p is given by $\operatorname{sgn}(p)$ and takes the value 1 if the permutation is even and -1 if the permutation is odd. The sign can be calculated with the formula $\operatorname{sgn}(p) = (-1)^m$, where m is the number of transpositions used to obtain the rearrangement from the given set.

We begin with a vertex x and a r-cell $dx_1 \wedge \cdots \wedge dx_r$ in the primal lattice $\mathbb{Z}^r$. The point at the centre of this r-cell is in the dual lattice given by $y := *(dx_1 \wedge \cdots \wedge dx_r)$. The negatively oriented edges emerging from y are denoted by $dy_1 =$

$y - d\mathbf{e}_1, \ldots, dy_r = y - d\mathbf{e}_r$. Let $k \in \{0, 1, \ldots, r\}$ and $1 \le i_1 < \ldots < i_k \le r$ be given. Since $x \in \mathbb{Z}^r$, the k-cell $c = dx_{i_1} \wedge \ldots \wedge dx_{i_k}$ is also in the primal lattice $\mathbb{Z}^r$. Let $j_1, \ldots, j_{r-k}$ be an enumeration of the set $\{1, 2, \ldots, r\} \setminus \{i_1, \ldots, i_k\}$. Consider the permutation that maps $(1, 2, \ldots, r)$ to $(i_1, \ldots, i_k, j_1, \ldots, j_{r-k})$. Then the Hodge dual of the k-cell c is defined as

$$* (dx_{i_1} \wedge \cdots \wedge dx_{i_k}) \coloneqq \operatorname{sgn}(i_1, \ldots, i_k, j_1, \ldots, j_{r-k}) dy_{j_1} \wedge \cdots \wedge dy_{j_{r-k}}. \tag{3.13}$$

Likewise, we can define the Hodge dual of a $r - k$-cell in the dual lattice by

$$\begin{aligned} *(dy_{j_1} \wedge \cdots \wedge dy_{j_{r-k}}) &\coloneqq \operatorname{sgn}(j_1, \ldots, j_{r-k}, i_1, \ldots, i_k) dx_{i_1} \wedge \cdots \wedge dx_{i_k} \\ &= (-1)^{k(r-k)} \operatorname{sgn}(i_1, \ldots, i_k, j_1, \ldots, j_{r-k}) dx_{i_1} \wedge \cdots \wedge dx_{i_k}. \end{aligned} \tag{3.14}$$

We continue with some examples.

Example 3.25. *Let the primal lattice be $\mathbb{Z}^3$. For a 3-cell in the primal lattice, we have that $r - k = 3 - 3 = 0$. Hence, the Hodge dual of a 3-cell is a 0-cell at the centre of the 3-cell. For a 2-cell in the primal lattice, the Hodge dual is a 1-cell in the dual lattice. This is illustrated in the figure below. Furthermore, the Hodge dual of a 1-cell is a 2-cell and the Hodge dual of a 0-cell is a 3-cell.*

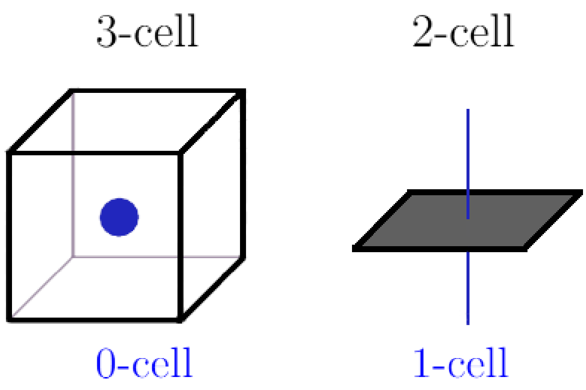

Figure 3.5: A 3-cell and a 2-cell in the primal lattice (the black cells) and their Hodge duals in the dual lattice (the blue cells).

Example 3.26. *We calculate the Hodge dual of the 2-cells $dx_1 \wedge dx_2$ and $dx_2 \wedge dx_4$ in $\mathbb{Z}^4$. In the first case, the permutation of $\{1, 2, 3, 4\}$ is even. The permutation* *of* $\{2,4,1,3\}$ *is odd, since the set* $\{1,2,3,4\}$ *is obtained from* $\{2,4,1,3\}$ *with three transpositions. Thus,*

$$*\,(dx_1 \wedge dx_2) = \text{sgn}(1,2,3,4)(dy_3 \wedge dy_4) = dy_3 \wedge dy_4,$$
$$*\,(dx_2 \wedge dx_4) = \text{sgn}(2,4,1,3)(dy_1 \wedge dy_3) = -dy_1 \wedge dy_3.$$

Example 3.27. *We calculate the Hodge dual of the* 3*-cell* $dx_1 \wedge dx_3 \wedge dx_5$ *in* $\mathbb{Z}^5$*. The permutation of* $\{1,3,5,2,4\}$ *is odd, since the set* $\{1,2,3,4,5\}$ *is obtained from* $\{1,3,5,2,4\}$ *with three transpositions. Thus,*

$$*(dx_1 \wedge dx_3 \wedge dx_5) = \text{sgn}(1,3,5,2,4)(dy_2 \wedge dy_4) = -dy_2 \wedge dy_4.$$

Next is the definition for the Hodge dual of a box.

Definition 3.28. *(Hodge dual* $*B$ *of a box* B*) For a box* $B \in \mathbb{Z}^r$*, the corresponding box in the dual lattice* $*\mathbb{Z}^r$ *is*

$$*B := \{y \in *\mathbb{Z}^r : \exists x \in B \text{ such that } y \text{ is a corner in } *\,x\}.$$

Example 3.29. *A cube* $B \in \mathbb{Z}^2$ *is represented by the black cube in Figure 3.6, where the vertices in the primal lattice are given by the grey points and the vertices in the dual lattice by the blue crosses. Then* $*B \in *\mathbb{Z}^2$ *is given by the blue cube and* $**B \in \mathbb{Z}^2$ *is given by the red cube. Assume that* B *is of width* b*. Then we see from the figure that* $*B$ *is of width* $b+1$ *and* $**B$ *of width* $b+2$*.*

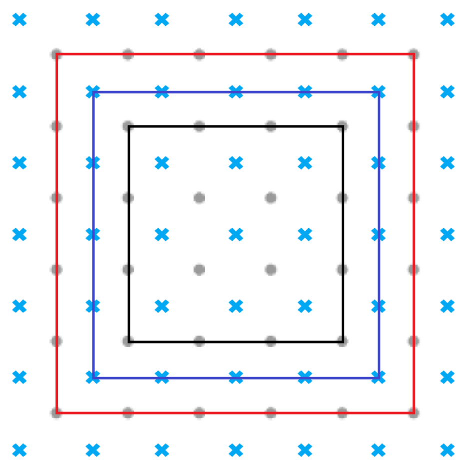

Figure 3.6: The Hodge dual of a cube in $\mathbb{Z}^2$.

Example 3.30. *The box B is defined as $B := ([a_1, b_1] \times \ldots \times [a_r, b_r]) \cap \mathbb{Z}^r$. Then, the Hodge dual is*

$$*B = \left(\left[a_1 - \frac{1}{2}, b_1 + \frac{1}{2} \right] \times \ldots \times \left[a_r - \frac{1}{2}, b_r + \frac{1}{2} \right] \right) \cap *\mathbb{Z}^r.$$

Taking the Hodge dual again, we obtain

$$**B = ([a_1 - 1, b_1 + 1] \times \ldots \times [a_r - 1, b_r + 1]) \cap \mathbb{Z}^r.$$

Thus, the width of a side in $*B$ is 1 larger than its corresponding side in B and the width of a side in $**B$ is 2 larger than its corresponding side in B. Consequently, $B \subsetneq **B$. For example, if B is a cube of width b, then $**B$ is a cube of width $b + 2$ containing B.

Next, one lemma for the Hodge dual of a box is given without proof.

Lemma 3.31. *Let B be a box in $\mathbb{Z}^r$. Then a k-cell c is outside B if and only if $*c$ is either outside $*B$ or in the boundary of $*B$. If c is a k-cell outside B that contains a $(k-1)$-cell of B, then $*c$ belongs to the boundary of $*B$.*

For the proof, see [2, Lemma 2.4]. We define the Hodge dual for k-forms.

Definition 3.32. *(Hodge dual of a G-valued k-form) Given a G-valued k-form f on $\mathbb{Z}^r$, the Hodge dual $*f$ of f is an $(r-k)$-form defined as*

$$* f(y) := \sum_{1 \le i_1 < \cdots < i_k \le r} f_{i_1,\ldots,i_k}(x) \operatorname{sgn}(i_1, \ldots, i_k, j_1, \ldots, j_{r-k}) dy_{j_1} \wedge \cdots \wedge d_{y_{j_{r-k}}}, \tag{3.15}$$

where $y = *(dx_1 \wedge \cdots \wedge dx_r)$ and the sequence $j_1, \ldots, j_{r-k}$ depends on $i_1, \ldots, i_k$.

From this definition and (3.13) follows that

$$*f(*c) = f(c)$$

for a k-cell $c = dx_1 \wedge \cdots \wedge dx_r$.

Example 3.33. *We calculate the Hodge dual of the 2-form $f = f_{12}(x)dx_1 \wedge dx_2 + f_{34}(x)dx_3 \wedge dx_4$ in $\mathbb{Z}^4$. By Definition 3.32, we have*

$$\begin{aligned} *f(y) &= f_{12}(x) \operatorname{sgn}(1,2,3,4) dy_3 \wedge dy_4 + \operatorname{sgn}(3,4,1,2) f_{34}(x) dy_1 \wedge dy_2 \\ &= f_{12}(x) dy_3 \wedge dy_4 + f_{13}(x) dy_2 \wedge dy_4. \end{aligned}$$

For the Hodge dual of a k-form holds the following formula:

$$* (*f) = (-1)^{k(r-k)} f. \tag{3.16}$$

We prove this by calculating $*(*f)$:

$$\begin{aligned}
*(*f) &\overset{(3.15)}{=} * \left(\sum_{1 \le i_1 < \cdots < i_k \le r} f_{i_1,\ldots,i_k}(x) \operatorname{sgn}(i_1, \ldots, i_k, j_1, \ldots, j_{r-k}) dy_{j_1} \wedge \cdots \wedge d_{y_{j_{r-k}}} \right) \\
&= \sum_{1 \le i_1 < \cdots < i_k \le r} f_{i_1,\ldots,i_k}(x) \operatorname{sgn}(i_1, \ldots, i_k, j_1, \ldots, j_{r-k}) * (dy_{j_1} \wedge \cdots \wedge d_{y_{j_{r-k}}}) \\
&\overset{(3.14)}{=} \sum_{1 \le i_1 < \cdots < i_k \le r} f_{i_1,\ldots,i_k}(x) \operatorname{sgn}(i_1, \ldots, i_k, j_1, \ldots, j_{r-k}) (-1)^{k(r-k)} \\
&\quad \cdot \operatorname{sgn}(i_1, \ldots, i_k, j_1, \ldots, j_{r-k}) dx_{i_1} \wedge \cdots \wedge dx_{i_k} \\
&= \sum_{1 \le i_1 < \cdots < i_k \le r} f_{i_1,\ldots,i_k}(x) (-1)^{k(r-k)} dx_{i_1} \wedge \cdots \wedge dx_{i_k} \\
&= (-1)^{k(r-k)} \sum_{1 \le i_1 < \cdots < i_k \le r} f_{i_1,\ldots,i_k}(x) dx_{i_1} \wedge \cdots \wedge dx_{i_k} \\
&\overset{(3.1)}{=} (-1)^{k(r-k)} f.
\end{aligned}$$

The exterior derivative has different definitions on the primal and on the dual lattice.

Definition 3.34. *(Exterior derivative on the dual cell lattice) The exterior derivative d on the dual cell lattice is defined by*

$$df(y) = \sum_{1 \le j_1 < \cdots < j_k \le r} \sum_{i=1}^{r} \bar{\partial}_i f_{j_1,\ldots,j_k}(y) dy_i \wedge (dy_{j_1} \wedge \ldots \wedge dy_{j_k}), \quad y \in *\mathbb{Z}^r.$$

Note that $\bar{\partial}_i$ is used instead of ∂_i, although the exterior derivative is denoted by the same symbol d in both cases. Last, a lemma is given.

Lemma 3.35. *For any G-valued k-form f on $\mathbb{Z}^r$ and any $x \in \mathbb{Z}^r$,*

$$\delta f(x) = (-1)^{r(k+1)+1} * (d(*f(y))),$$

where $y = *(dx_1 \wedge \cdots \wedge dx_r)$ is the centre of the r-cell $(dx_1 \wedge \cdots \wedge dx_r)$.

For the proof, see [2, Lemma 2.3].

Chapter 4

Vortices and oriented surfaces

In this chapter, vortices and oriented surfaces are discussed. Lemmas crucial for the proof of the main theorem are stated and proved. From now on, the lattice is always $\mathbb{Z}^4$, the group G is a finite Abelian group and the representation ρ of G is fixed.

4.1 Vortices

We begin with defining vortices before discussing both decompositions and distributions of vortices. Some theory for minimal vortices will also be given. First, the definition for an irreducible plaquette configuration is given. This definition is necessary for the definition of a vortex.

Definition 4.1. *(Irreducible plaquette configuration) A plaquette configuration $\omega \in \Sigma_{P_N}$ is irreducible if there does not exist a non-empty set $P \subsetneq \operatorname{supp} \omega$ such that $\omega|_P \in \Sigma_{P_N}$.*

Recall from the definition of Σ_{P_N} that $d\omega = 0$ for $\omega \in \Sigma_{P_N}$. Therefore, in the above definition $\omega|_P \in \Sigma_{P_N}$ is equivalent to $d(\omega|_P) = 0$.

Definition 4.2. *(Vortex) Let $\sigma \in \Sigma_{E_N}$. A non-trivial and irreducible plaquette configuration $\nu \in \Sigma_{P_N}$ is a vortex in σ if $(d\sigma)|_{supp\, \nu} = \nu$.*

4.1.1 Vortex decompositions

The goal of this section is to prove that if a spin configuration $\sigma \in \Sigma_{E_N}$, then $d\sigma$ can be written as a sum of vortices with disjoint supports in σ. To prove this claim, we need the results from the following two lemmas.

Lemma 4.3. *Let* $\omega \in \Sigma_{P_N}$ *and let* $\nu \in \Sigma_{P_N}$ *be such that* $\omega|_{supp\,\nu} = \nu$. *Then* $\omega|_{P_N \setminus supp\,\nu} \in \Sigma_{P_N}$.

Proof. We prove this by showing that $d(\omega|_{P_N \setminus \text{supp } \nu}) = 0$. Since $\omega \in \Sigma_{P_N}$ and $\nu \in \Sigma_{P_N}$, we have that $d\omega = 0$ and $d\nu = 0$. Since $\omega \in \Sigma_{P_N} \subseteq P_N$ and supp ν, it follows that $\omega|_{P_N \setminus supp\,\nu} = \omega|_{P_N} - \omega|_{\text{supp } \nu} = \omega - \nu$. Thus,

$$d(\omega|_{P_N \setminus \text{supp } \nu}) = d(\omega - \nu) = d\omega - d\nu = 0 - 0 = 0$$

and $\omega|_{P_N \setminus \text{supp } \nu} \in \Sigma_{P_N}$. □

Lemma 4.4. *Let* $\omega \in \Sigma_{P_N}$. *Then either* ω *is irreducible or there exist non-trivial* $\nu^{(1)}, \nu^{(2)} \in \Sigma_{P_N}$ *with disjoint supports contained in* supp ω, *such that* $\omega = \nu^{(1)} + \nu^{(2)}$.

Proof. We assume that ω is reducible and then prove that there exists non-trivial $\nu^{(1)}, \nu^{(2)} \in \Sigma_{P_N}$ with disjoint supports contained in supp ω such that $\omega = \nu^{(1)} + \nu^{(2)}$. By Definition 4.1, there exists at least one non-empty set $P \subsetneq$ supp ω such that $\nu^{(1)} := \omega|_P \in \Sigma_{P_N}$. Consequently, $P =$ supp $\nu^{(1)}$ and

$$\omega|_{\text{supp } \nu^{(1)}} = \nu^{(1)}. \tag{4.1}$$

Therefore, $\nu^{(1)}$ fulfils the assumptions for Lemma 4.3 and

$$\nu^{(2)} := \omega|_{P_N \setminus supp\,\nu^{(1)}} \in \Sigma_{P_N}. \tag{4.2}$$

Since supp $\nu^{(1)}$ is non-empty, $\nu^{(1)}$ contains at least one 2-cell for which $\nu^{(1)} \neq 0$. Similarly, $\nu^{(2)}$ contains at least one 2-cell for which $\nu^{(2)} \neq 0$. Thus, they are both non-trivial. Since supp $\nu^{(2)} \subseteq P_N \setminus$ supp $\nu^{(1)}$, the intersection of supp $\nu^{(1)}$ and supp $\nu^{(2)}$ must be empty. Hence, they have disjoint supports. Last, we conclude that

$$\nu^{(1)} + \nu^{(2)} \overset{(4.1),(4.2)}{=} \omega|_{\text{supp } \nu^{(1)}} + \omega|_{P_N \setminus \text{supp } \nu^{(1)}} = \omega|_{P_N} = \omega.$$

□

We now have everything necessary to prove and state the main result of this section.

Lemma 4.5. *Let $\sigma \in \Sigma_{E_N}$. Then $d\sigma$ can be written as a sum of vortices in σ with disjoint supports.*

Proof. Since $\sigma \in \Sigma_{E_N}$ and $d\sigma|_{\text{supp } d\sigma} = d\sigma$, it follows from Definition 4.2 that there are two cases to consider. First, if $d\omega$ is irreducible, then $d\sigma$ is a vortex. Second, we consider the case when $d\omega$ is not irreducible. Since $d\sigma \in \Sigma_{P_N}$, by Lemma 4.4 there exists non-trivial $\nu^{(1)}, \nu^{(2)} \in \Sigma_{P_N}$ with disjoint supports in supp $d\sigma$ for which $d\sigma = \nu^{(1)} + \nu^{(2)}$.

If $\nu^{(1)}$ and $\nu^{(2)}$ are both irreducible, we have a sum of two non-trivial vortices with disjoint supports. Otherwise, by Lemma 4.4, $\nu^{(1)}$ and/or $\nu^{(2)}$ can be decomposed further. This decomposition of 2-forms is repeated until we only have irreducible and non-trivial plaquette configurations $\nu^{(j)} \in \Sigma_{P_N}$. Since this decomposition cannot consist of more than $|P_N| < \infty$ parts, there must exist an $m \leq |P_N| < \infty$ for which $d\sigma$ can be written as a finite sum of m vortices with disjoint supports, i.e. $d\sigma = \sum_{j=1}^{m} \nu^{(j)}$. □

4.1.2 Minimal vortices

In this section, we define minimal vortices and prove that a minimal vortex ν, whose support does not contain any boundary plaquettes of P_N, can be written using an edge in E_N and an element $g \in G \setminus \{0\}$. To achieve this result, two lemmas are necessary. We begin with the definition for a minimal vortex.

Definition 4.6. *(Minimal vortex) Let $\sigma \in \Sigma_{E_N}$. A vortex ν in σ is minimal if $|\text{supp } \nu| = 12$.*

The following lemma says that the support of a minimal vortex, which contains no boundary plaquettes of P_N, can be written as a union of the co-boundary of a positively oriented edge e_0 and the co-boundary of the negatively oriented edge $-e_0$.

Lemma 4.7. *Let $\sigma \in \Sigma_{E_N}$ and let ν in σ be a vortex for which* supp ν *contains no boundary plaquettes of P_N. Then $|\text{supp } \nu| \geq 12$. If $|\text{supp } \nu| = 12$, then there exists an edge $e_0 \in E_N$ such that* supp $\nu = \hat{\partial} e_0 \cup \hat{\partial}(-e_0)$.

For a proof of this lemma, see [3, Lemma 3.4.6]. Before stating the next lemma about minimal vortices, the Bianchi Lemma is given.

Lemma 4.8. *(Bianchi Lemma) If $\omega \in \sum_{P_N}$ and c is an oriented 3-cell in B_N, then*

$$\sum_{p \in \partial c} \omega_p = 0. \tag{4.3}$$

Proof. Let c be an oriented 3-cell in B_N. Since $\omega \in \Sigma_{P_N}$, we have that

$$\sum_{p \in \partial c} \omega_p \overset{(3.7)}{=} (d\omega)_c = 0.$$

$\square$

The following lemma is the main result of this section.

Lemma 4.9. *Let $\sigma \in \Sigma_{E_N}$ and let ν be a minimal vortex in σ. If the support of ν contains no boundary plaquettes of P_N, then there exists an edge $dx_j \in E_N$ and $g \in G \setminus \{0\}$ such that for all $p \in P_N$, we have*

$$\nu_p = (d(g dx_j))_p.$$

Proof. By definition, $|\operatorname{supp}\ \nu| = 12$. Since the support of ν contains no boundary plaquettes of P_N, it follows from Lemma 4.7 that there exists an edge $e_0 = dx_j \in E_N$ such that $\operatorname{supp}\ \nu = \hat{\partial} e_0 \cup \hat{\partial}(-e_0)$.

A 3-cell c can not contain more than two plaquettes in $\hat{\partial} e_0 \cup \hat{\partial}(-e_0)$. Hence, we have two plaquettes p_1 and p_2 such that $p_1 \in \hat{\partial} e_0$ and $p_2 \in \hat{\partial}(-e_0)$. By Lemma 4.8,

$$\sum_{p \in \partial c} \nu_p = \nu_{p_1} + \nu_{p_2} = 0.$$

Since the plaquettes are in the support of ν, we have $\nu_{p_1} \neq 0 \neq \nu_{p_2}$. Hence, $\nu_{p_1} = g$ and $\nu_{p_2} = -g$ for a $g \in G \setminus \{0\}$. To conclude,

$$\nu_p = \begin{cases} g & \text{if } p \in \hat{\partial} e_0, \\ -g & \text{if } p \in \hat{\partial}(-e_0). \end{cases}$$

Since $p \in \hat{\partial}(-e_0)$ is equivalent to $-e_0 \in \partial p$, we see from equation (3.5) that $(d(g dx_j))_p$ can be written as

$$(d(g dx_j))_p = \begin{cases} g & \text{if } p \in \hat{\partial} e_0, \\ -g & \text{if } p \in \hat{\partial}(-e_0). \end{cases}$$

Thus, $\nu_p = (d(g dx_j))_p$. $\square$

This lemma will be applied in the proof of Proposition 6.1.

4.1.3 Distribution of vortices

In this section, the focus is on distributions of vortices. With the help of two lemmas, we will state and prove Proposition 4.12 that is of great importance in Chapter 6. We begin with the first lemma, which gives us an upper bound for a useful probability.

Lemma 4.10. *Let* $\nu \in \Sigma_{P_N}$. *Then*

$$\mu_{\beta,N}(\{\sigma \in \Sigma_{E_N} : (d\sigma)|_{\text{supp }\nu} = \nu\}) \leq \lambda(\beta)^{|\text{supp }\nu|}.$$

Proof. Let $P := \text{supp }\nu$ and define the sets $\mathcal{E}_P^\nu := \{\omega \in \Sigma_{P_N} : \omega|_P = \nu\}$ and $\mathcal{E}_P^0 := \{\omega \in \Sigma_{P_N} : \omega|_P = 0\}$. By the Poincare Lemma, there exists a $\sigma \in \Sigma_{E_N}$ such that $d\sigma = \omega$ if and only if $\omega \in \Sigma_{P_N}$. Therefore, by equation (3.9) we have that $\mu_{\beta,N}(\mathcal{E}_P^\nu) = \mu_{\beta,N}(\{\sigma \in \Sigma_{E_N} : (d\sigma)|_{\text{supp }\nu} = \nu\})$. We calculate an upper bound for the probability. Since $\mathcal{E}_P^0 \subseteq \Sigma_{P_N}$ and ϕ_β is positive, we have

$$\begin{aligned}\mu_{\beta,N}(\{\sigma \in \Sigma_{E_N} : (d\sigma)|_{\text{supp }\nu} = \nu\}) = \mu_{\beta,N}(\mathcal{E}_P^\nu) &= \frac{\sum_{\omega\in\mathcal{E}_P^\nu}\prod_{p\in P_N}\phi_\beta(\omega_p)}{\sum_{\omega\in\Sigma_{P_N}}\prod_{p\in P_N}\phi_\beta(\omega_p)} \\ &\leq \frac{\sum_{\omega\in\mathcal{E}_P^\nu}\prod_{p\in P_N}\phi_\beta(\omega_p)}{\sum_{\omega\in\mathcal{E}_P^0}\prod_{p\in P_N}\phi_\beta(\omega_p)}.\end{aligned} \tag{4.4}$$

We aim to compare the denominator to a sum over $\omega \in \mathcal{E}_P^\nu$. From the definitions of the sets $\mathcal{E}_P^\nu$ and $\mathcal{E}_P^0$ follows that

$$\sum_{\omega\in\mathcal{E}_P^0}\prod_{p\in P_N}\phi_\beta(\omega_p) = \sum_{\omega\in\mathcal{E}_P^\nu}\prod_{p\in P_N}\phi_\beta((\omega-\nu)_p).$$

Since the mapping $\omega \to \omega - \nu$ from $\mathcal{E}_P^\nu$ to $\mathcal{E}_P^0$ is bijective, this gives

$$\begin{aligned}\frac{\sum_{\omega\in\mathcal{E}_P^\nu}\prod_{p\in P_N}\phi_\beta(\omega_p)}{\sum_{\omega\in\mathcal{E}_P^0}\prod_{p\in P_N}\phi_\beta(\omega_p)} &= \frac{\sum_{\omega\in\mathcal{E}_P^\nu}\prod_{p\in P_N}\phi_\beta(\omega_p)}{\sum_{\omega\in\mathcal{E}_P^\nu}\prod_{p\in P_N}\phi_\beta((\omega-\nu)_p)} \\ &= \frac{\sum_{\omega\in\mathcal{E}_P^\nu}\prod_{p\in P}\phi_\beta(\omega_p)\prod_{p\in P_N\setminus P}\phi_\beta(\omega_p)}{\sum_{\omega\in\mathcal{E}_P^\nu}\prod_{p\in P}\phi_\beta((\omega-\nu)_p)\prod_{p\in P_N\setminus P}\phi_\beta((\omega-\nu)_p)}.\end{aligned} \tag{4.5}$$

To simplify it further, notice that $\nu_p = 0$ when $p \in P_N \setminus P$. Therefore,

$$\phi_\beta((\omega-\nu)_p) = \phi_\beta(\omega_p) \quad \text{for } p \in P_N \setminus P. \tag{4.6}$$

For a plaquette $p \in P$ and a plaquette configuration $\omega \in \mathcal{E}_P^\nu$, we have by definition that $\omega_p = \nu_p$. Thus,

$$(\omega - \nu)_p = \omega_p - \nu_p = 0 \quad \text{for } p \in P \text{ and } \omega \in \mathcal{E}_P^\nu. \tag{4.7}$$

Combining the previous conclusions, we obtain

$$\begin{aligned}
\mu_{\beta,N}(\mathcal{E}_P^\nu) &\overset{(4.4),(4.5)}{\leq} \frac{\sum_{\omega\in\mathcal{E}_P^\nu} \prod_{p\in P} \phi_\beta(\omega_p) \prod_{p\in P_N\smallsetminus P} \phi_\beta(\omega_p)}{\sum_{\omega\in\mathcal{E}_P^\nu} \prod_{p\in P} \phi_\beta((\omega-\nu)_p) \prod_{p\in P_N\smallsetminus P} \phi_\beta((\omega-\nu)_p)} \\
&\overset{(4.6),(4.7)}{=} \frac{\sum_{\omega\in\mathcal{E}_P^\nu} \prod_{p\in P} \phi_\beta(\omega_p) \prod_{p\in P_N\smallsetminus P} \phi_\beta(\omega_p)}{\sum_{\omega\in\mathcal{E}_P^\nu} \prod_{p\in P} \phi_\beta(0) \prod_{p\in P_N\smallsetminus P} \phi_\beta(\omega_p)} \\
&= \frac{\prod_{p\in P} \phi_\beta(\nu_p) \sum_{\omega\in\mathcal{E}_P^\nu} \prod_{p\in P_N\smallsetminus P} \phi_\beta(\omega_p)}{\prod_{p\in P} \phi_\beta(0) \sum_{\omega\in\mathcal{E}_P^\nu} \prod_{p\in P_N\smallsetminus P} \phi_\beta(\omega_p)} \\
&= \frac{\prod_{p\in P} \phi_\beta(\nu_p)}{\prod_{p\in P} \phi_\beta(0)} = \prod_{p\in \operatorname{supp}\,\nu} \frac{\phi_\beta(\nu_p)}{\phi_\beta(0)} \\
&\overset{(2.11)}{\leq} \prod_{p\in \operatorname{supp}\,\nu} \lambda(\beta) = \lambda(\beta)^{|\operatorname{supp}\,\nu|}.
\end{aligned}$$

Thus, $\mu_{\beta,N}(\{\sigma \in \Sigma_{E_N} : (d\sigma)|_{\operatorname{supp}\,\nu} = \nu\}) \leq \lambda(\beta)^{|\operatorname{supp}\,\nu|}$. □

The following lemma is about how many irreducible plaquette configurations there can exist.

Lemma 4.11. *For each $p_0 \in P_N$ and each $m \geq 6$, there are at most $5^{m-1}(|G| - 1)^m$ irreducible $\nu \in \Sigma_{P_N}$ with $p_0 \in \operatorname{supp}\,\nu$ and $|\operatorname{supp}\,\nu| \geq 2m$.*

Proof. We construct an injective map from the set of irreducible 2-forms in Σ_{P_N} with $p_0 \in \operatorname{supp}\,\nu$ and $|\operatorname{supp}\,\nu| \geq 2m$ to the set of sequences $\nu^{(1)}, \nu^{(2)}, \ldots \nu^{(m)}$ of G-valued 2-forms on P_N.

Let $\nu \in \Sigma_{P_N}$ be irreducible, $p_0 \in \operatorname{supp}\,\nu$ and $|\operatorname{supp}\,\nu| = 2m$. Define $\nu^{(0)} :=$ $0 \in \Sigma_{P_N}$, $p_1 := p_0$ and for $p \in P_N$:

$$\nu_p^{(1)} := \begin{cases} \nu_{p_1} & \text{if } p = p_1, \\ -\nu_{p_1} & \text{if } p = -p_1, \\ 0 & \text{else.} \end{cases}$$

Let $k \in \{1, 2, \ldots, m\}$ and assume that the 2-forms $\nu^{(1)}, \nu^{(2)}, \ldots, \nu^{(k)}$ are given such that for each $j \in \{1, 2, \ldots, k\}$, the two following statements hold:

(a) supp $\nu^{(j)} \setminus$supp $\nu^{(j-1)} = \{p_j, -p_j\}$ for some $p_j \in P_N$. This implies that $|\text{supp } \nu^{(j)}| = 2j$.
(b) $\nu|_{\text{supp } \nu^{(j)}} = \nu^{(j)}$.

We check that (a) and (b) hold if $k = 1$. Since the sequence contains only one element, we only have to check the conditions for $\nu^{(1)}$. (a) supp $\nu^{(1)} \setminus$supp $\nu^{(0)} = \{p_1, -p_1\} \setminus \{\varnothing\} = \{p_1, -p_1\}$ and (b) $\nu|_{\text{supp } \nu^{(1)}} = \nu^{(1)}$.

We prove that $\nu^{(m)} = \nu$ by considering two different cases: $d\nu^{(k)} = 0$ and $d\nu^{(k)} \not\equiv 0$. First, we assume that there exists an sequence satisfying (a) and (b) for which $d\nu^{(k)} = 0$. From (b) follows that $\nu|_{\text{supp } \nu^{(k)}} = \nu^{(k)} \in \Sigma_{P_N}$. Since ν is irreducible, there exists no non-empty set $P \subsetneq$ supp ν such that $\nu|_P \in \Sigma_{P_N}$. Since it followed from (a) that supp $\nu^{(k)}$ is non-empty, we have $\nu^{(k)} = \nu$. Since $|\text{supp } \nu^{(k)}| = 2k$, we must have

$$2k = |\text{supp } \nu^{(k)}| = |\text{supp } \nu| = 2m.$$

Thus, $k = m$ and $\nu^{(m)} = \nu$.

Consider the second case, where we assume that the sequence is such that $d\nu^{(k)} \not\equiv 0$. Since we had $k = m$ in the first case, we now have that $k < m$. Therefore, we expand the sequence by defining $\nu^{(k+1)}$ such that (a) and (b) hold. Since $d\nu^{(k)} \not\equiv 0$, there exists an oriented 3-cell $c \in B_N$ such that $(d\nu^{(k)})_c \neq 0$. We assume that both the 3-cells in the set B_N and the arbitrary total ordering of the plaquettes are given. Denote the first 3-cell for which $(d\nu^{(k)})_c \neq 0$ by c_{k+1}. Furthermore, $(d\nu)_{c_{k+1}} = 0$. Hence, there exists at least one plaquette $p \in \partial_{c_{k+1}} \setminus$supp $\nu^{(k)}$. We denote the first plaquette for which $p \in \partial_{c_{k+1}} \setminus$supp $\nu^{(k)}$ by p_{k+1} and let

$$\nu_p^{(k+1)} := \begin{cases} \nu_{p_{(k+1)}} & \text{if } p = p_{k+1}, \\ -\nu_{p_{(k+1)}} & \text{if } p = -p_{k+1}, \\ \nu_p^{(k)} & \text{otherwise.} \end{cases}$$

We saw earlier that (a) and (b) holds for $k = 1$. We assume that $\nu^{(1)}, \ldots, \nu^{(k)}$ satisfies both (a) and (b), then they also hold for $\nu^{(k+1)}$. Therefore, (a) and (b) hold for all $\nu^{(k)}$, where $k \in \{1, 2, \ldots, m\}$. Since it follows from a) that $|\text{supp } \nu^{(m)}| = 2m$, from (b) that $\nu|_{\text{supp } \nu^{(m)}} = \nu^{(m)}$ and we assumed that $|\text{supp } \nu| = 2m$, we have that $\nu^{(m)} = \nu$. Therefore, the injective map is constructed.

Since this mapping is injective, we can derive the upper bound for the set of sequences instead of the irreducible vortices with $p_0 \in$ supp ν and $|\text{supp } \nu| \geq$

$2m$. Therefore, we calculate how many such sequences $(\nu^{(1)}, \nu^{(2)}, \ldots, \nu^{(m)})$ there are. Since $\nu_{p_k} \in G \setminus \{0\}$, there is for each $k \in \{1, 2, \ldots, m\}$ $|G| - 1$ choices for ν_{p_k}. For each $k \in \{1, 2, \ldots, m-1\}$, there is a given 3-cell c_{k+1} for which we have at most 5 choices for p_{k+1} for each $k \in \{1, 2, \ldots, m-1\}$. Combining this, we have at most $5^{m-1}(|G|-1)^m$ sequences $(\nu^{(1)}, \nu^{(2)}, \ldots, \nu^{(m)})$. Hence, there exists at most $5^{m-1}(|G|-1)^m$ irreducible vortices $p_0 \in \operatorname{supp} \nu$ and $|\operatorname{supp} \nu| \geq 2m$. □

Recall the definition of $\lambda(\beta)$. Since $\rho(0) = 1$ and the largest value $\Re\rho(g) = \cos(\frac{g \cdot 2\pi m}{n})$ can take for $g \neq 0$ is $\cos(2\pi/n)$, it follows that $\lambda(\beta)$ can be written as

$$\begin{aligned}\lambda(\beta) &\coloneqq \max_{g \in G\setminus\{0\}} \frac{e^{\beta\Re(\rho(g))}}{e^{\beta\Re((\rho(0)))}} = \max_{g \in G\setminus\{0\}} e^{\beta\Re(\rho(g)-\rho(0))} \\ &= \max_{g \in G\setminus\{0\}} e^{\beta\Re(\rho(g)-1)} = e^{\beta(\cos(2\pi/n)-1)}.\end{aligned}$$

Hence,

$$\lim_{\beta\to\infty} \lambda(\beta) = \lim_{\beta\to\infty} e^{\beta(\cos(2\pi/n)-1)} = 0.$$

Thus, there always exists a β_0 for which $5(|G|-1)\lambda(\beta)^2 < 1$ for $\beta > \beta_0$. Finally, we have everything necessary to state and prove the proposition.

Proposition 4.12. *Fix any $\beta_0 > 0$ such that $5(|G|-1)\lambda(\beta)^2 < 1$ for all $\beta > \beta_0$. Fix $p_0 \in P_N$ and $M \geq 6$. Then*

$$\begin{aligned}&\mu_{\beta,N}(\{\sigma \in \Sigma_{E_N} : \exists \text{ a vortex } \nu \text{ in } \sigma \text{ with } p_0 \in \operatorname{supp} \nu \text{ and } |\operatorname{supp} \nu| \geq 2M\}) \\ &\qquad \leq C_0^{(M)} \lambda(\beta)^{2M}\end{aligned}$$

for all $\beta > \beta_0$, where

$$C_0^{(M)} \coloneqq \frac{5^M(|G|-1)^M}{1 - 5(|G|-1)\lambda(\beta)^2}. \tag{4.8}$$

Proof. Let $m \in \mathbb{Z}_+$ and $p_0 \in P_N$. For $\nu \in \Sigma_{P_N}$, we have

$$\mu_{\beta,N}(\{\sigma \in \Sigma_{E_N} : (d\sigma)|_{\operatorname{supp} \nu} = \nu\}) \overset{(4.10)}{\leq} (\lambda(\beta))^{|\operatorname{supp} \nu|} = \lambda(\beta)^{2m}.$$

From Lemma 4.11 follows that there are at most $5^{m-1}(|G|-1)^m$ irreducible plaquette configurations $\nu \in \Sigma_{P_N}$, such that $p_0 \in \operatorname{supp} \nu$ and $|\operatorname{supp} \nu| = 2m$. Therefore,

$$\begin{aligned}&\mu_{\beta,N}(\{\sigma \in \Sigma_{E_N} : \exists \text{ a vortex } \nu \text{ in } \sigma \text{ with } p_0 \in \operatorname{supp} \nu \text{ and } |\operatorname{supp} \nu| = 2m\}) \\ &\leq 5^{m-1}(|G|-1)^m \lambda(\beta)^{2m} \leq (5(|G|-1))^m \lambda(\beta)^{2m}.\end{aligned}$$

We sum over every $m \geq M$:

$$\begin{aligned}
&\mu_{\beta,N}(\{\sigma \in \Sigma_{E_N} : \exists \text{ a vortex } \nu \text{ in } \sigma \text{ with } p_0 \in \text{supp } \nu \text{ and } |\text{supp } \nu| \geq 2M) \\
&= \sum_{m=M}^{\infty} \mu_{\beta,N}(\{\sigma \in \Sigma_{E_N} : \exists \text{ a vortex } \nu \text{ in } \sigma \text{ with } p_0 \in \text{supp } \nu \text{ and} \\
&\quad |\text{supp } \nu| = 2m\}) \\
&\leq \sum_{m=M}^{\infty} (5(|G|-1))^m \lambda(\beta)^{2m} = \frac{(5(|G|-1))^M \lambda(\beta)^{2M}}{1 - 5(|G|-1)\lambda(\beta)^2} = C_0^{(M)} \lambda(\beta)^{2M}.
\end{aligned}$$

The second to last step follows from the sum being a geometric sum, which converges when $(5(|G|-1))\lambda(\beta)^2 < 1$. □

4.2 Oriented surfaces

This section focuses on oriented surfaces. Their relation to simple loops is given by the second lemma. First, the definitions for an oriented surface and its boundary are given.

Definition 4.13. *(Oriented surface) A $\mathbb{Z}$-valued 2-form q on P_N is an oriented surface if we have for every $e \in E_N$ that*

$$(\delta q)_e = \sum_{p \in \hat{\partial} e} q_p \in \{-1, 0, 1\}. \tag{4.9}$$

If a 2-form q is such that $|q_p| = 1$ for all $p \in P_N$, then the oriented surface is a collection of oriented plaquettes, where every plaquette shares at least one edge with another plaquette in the collection. An example of an oriented surface and its boundary is illustrated in Figure 4.1.

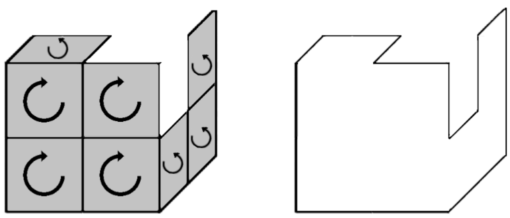

Figure 4.1: Example of an oriented surface and its boundary when $|q_p| = 1$.

Definition 4.14. *(The boundary of an oriented surface) Let q be an oriented surface. Then*

$$B_q := \{e \in E_N : \sum_{p \in \hat{\partial} e} q_p = 1\}$$

is the boundary of q.

Before stating the first lemma, we define for an oriented surface q that

$$q_p^+ := \begin{cases} \max(q_p, 0) & \text{for } p \in P_N, \\ 0 & \text{for } p \notin P_N. \end{cases} \tag{4.10}$$

Lemma 4.15. *Let q be an oriented surface with boundary B_q. Then,*

$$\sum_{p \in P_N} q_p^+ (d\sigma)_p = \sum_{e \in B_q} \sigma_e, \quad \textit{for all } \sigma \in \Sigma_{E_N}.$$

Note that this lemma is a discrete version of Stokes theorem.

Proof. Let $\sigma \in \Sigma_{E_N}$ and $p \in P_N$. Recall that the set E_N^+ only contains the positively oriented edges in the set E_N and that the set E_N^- only contains the negatively oriented edges. Furthermore,

$$\sigma_e = -\sigma_{-e}. \tag{4.11}$$

From equation (3.7) follows that $(d\sigma)_p = \sum_{e \in \partial p} \sigma_e$. Therefore,

$$\begin{aligned}
\sum_{p \in P_N} q_p^+ (d\sigma)_p &= \sum_{p \in P_N} q_p^+ \sum_{e \in \partial p} \sigma_e \\
&= \sum_{e \in E_N} \sigma_e \sum_{p \in (\hat{\partial} e) \cap P_N} q_p^+ \\
&\overset{(4.10)}{=} \sum_{e \in E_N} \sigma_e \sum_{p \in \hat{\partial} e} q_p^+ \\
&= \sum_{e \in E_N^+} \sigma_e \sum_{p \in \hat{\partial} e} q_p^+ + \sum_{e \in E_N^-} \sigma_e \sum_{p \in \hat{\partial} e} q_p^+ \\
&\overset{(4.11)}{=} \sum_{e \in E_N^+} \sigma_e \sum_{p \in \hat{\partial} e} q_p^+ - \sum_{e \in E_N^+} \sigma_e \sum_{p \in \hat{\partial}(-e)} q_p^+ \\
&= \sum_{e \in E_N^+} \sigma_e \left(\sum_{p \in \hat{\partial} e} q_p^+ - \sum_{p \in \hat{\partial}(-e)} q_p^+ \right).
\end{aligned}$$

Recall from Example 3.16 that $\hat{\partial} e \cap \hat{\partial}(-e) = \varnothing$. Moreover, if $p \in \hat{\partial}(-e)$, then $-p \in \hat{\partial} e$. Since q is a 2-form, we have for $p \in \hat{\partial} e$ that $q_{-p} = -q_p$. Therefore,

$$q_p^+ - q_{-p}^+ = \max(q_p, 0) - \max(q_{-p}, 0) = \max(q_p, 0) - \max(-q_p, 0) = q_p \tag{4.12}$$

and

$$\begin{aligned}\sum_{e \in E_N^+} \sigma_e \left(\sum_{p \in \hat{\partial} e} q_p^+ - \sum_{p \in \hat{\partial}(-e)} q_p^+ \right) &= \sum_{e \in E_N^+} \sigma_e \left(\sum_{p \in \hat{\partial} e} q_p^+ - \sum_{p \in \hat{\partial} e} q_{-p}^+ \right) \\ &\overset{(4.12)}{=} \sum_{e \in E_N^+} \sigma_e \sum_{p \in \hat{\partial} e} q_p.\end{aligned}$$

Furthermore,

$$\sigma_e \sum_{p \in \hat{\partial} e} q_p = -\sigma_{-e} \sum_{p \in \hat{\partial} e} (-q_{-p}) = \sigma_{-e} \sum_{p \in \hat{\partial} e} q_{-p} = \sigma_{-e} \sum_{-p \in \hat{\partial} e} q_p = \sigma_{-e} \sum_{p \in \hat{\partial}(-e)} q_p. \tag{4.13}$$

Therefore,

$$\begin{aligned}\sum_{e \in E_N^+} \sigma_e \sum_{p \in \hat{\partial} e} q_p &\overset{(4.13)}{=} \sum_{e \in E_N^+} \left(\frac{1}{2} \left(\sigma_e \sum_{p \in \hat{\partial} e} q_p + \sigma_{-e} \sum_{p \in \hat{\partial}(-e)} q_p \right) \right) = \frac{1}{2} \sum_{e \in E_N} \sigma_e \sum_{p \in \hat{\partial} e} q_p \\ &\overset{(4.9)}{=} \frac{1}{2} \sum_{\substack{e \in E_N: \\ \sum_{p \in \hat{\partial} e} q_p = -1}} \sigma_e \sum_{p \in \hat{\partial} e} q_p + \frac{1}{2} \sum_{\substack{e \in E_N: \\ \sum_{p \in \hat{\partial} e} q_p = 0}} \sigma_e \sum_{p \in \hat{\partial} e} q_p \\ &\quad + \frac{1}{2} \sum_{\substack{e \in E_N: \\ \sum_{p \in \hat{\partial} e} q_p = 1}} \sigma_e \sum_{p \in \hat{\partial} e} q_p \\ &= \frac{1}{2} \sum_{\substack{e \in E_N: \\ \sum_{p \in \hat{\partial} e} q_p = -1}} \sigma_e \sum_{p \in \hat{\partial} e} q_p + \frac{1}{2} \sum_{\substack{e \in E_N: \\ \sum_{p \in \hat{\partial} e} q_p = 1}} \sigma_e \sum_{p \in \hat{\partial} e} q_p \\ &= \frac{1}{2} \sum_{\substack{e \in E_N: \\ \sum_{p \in \hat{\partial} e} q_p = 1}} \sigma_e \sum_{p \in \hat{\partial} e} q_p + \frac{1}{2} \sum_{\substack{e \in E_N: \\ \sum_{p \in \hat{\partial} e} q_p = 1}} \sigma_e \sum_{p \in \hat{\partial} e} q_p \\ &= \sum_{e \in B_q} \sigma_e \sum_{p \in \hat{\partial} e} q_p = \sum_{e \in B_q} \sigma_e.\end{aligned}$$

Thus,

$$\sum_{p \in P_N} q_p^+ (d\sigma)_p = \sum_{e \in B_q} \sigma_e.$$

□

This lemma is applied in the final proof of this section. The next lemma says that a simple loop is the boundary of an oriented surface. This is illustrated in Figure 4.2.

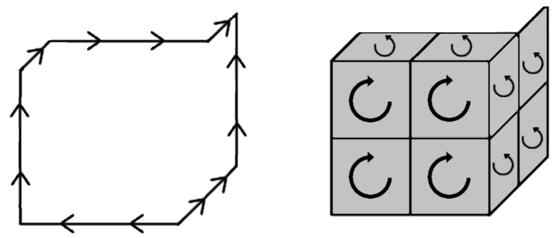

Figure 4.2: A simple loop γ and an oriented surface, whose boundary is γ.

Lemma 4.16. *Let γ in B_N be a simple loop contained in the cube $B \subseteq B_N$. Then there exists an oriented surface q, whose support is in B, such that γ is the boundary of q.*

Proof. Let γ be a given simple loop contained in the cube B. For each $e \in E_N$, define the 1-form f^γ as

$$f_e^\gamma := \begin{cases} 1 & \text{if } e \in \gamma, \\ -1 & \text{if } -e \in \gamma, \\ 0 & \text{otherwise.} \end{cases}$$

By definition, the cube B contains the support of f^γ. For the co-derivative of f^γ, we have that $\delta f_e^\gamma = 0$. Hence, the assumptions of Lemma 3.24 are fulfilled for f^γ. Therefore, there exists a 2-form q^γ on P_N, with support only in B, for which

$$f^\gamma = \delta q^\gamma. \tag{4.14}$$

From equation (3.12) follows that

$$(\delta q^\gamma)_e = \sum_{p \in \hat{\partial} e} q_p^\gamma \quad \text{for } e \in E_N. \tag{4.15}$$

By combining these conclusions, we obtain

$$\sum_{p \in \hat{\partial} e} q_p^\gamma \overset{(4.15)}{=} (\delta q^\gamma)_e \overset{(4.14)}{=} f_e^\gamma \in \{-1, 0, 1\}.$$

Thus, q^γ is an oriented surface with support contained in B and γ is the boundary of q^γ. $\square$

This lemma is applied in the proof of Proposition 6.1. Next, we define internal plaquettes and edges.

Definition 4.17. *(Internal plaquette) Let q be an oriented surface and let $\hat{p} \in \operatorname{supp} q$. A plaquette $\hat{p}$ is an internal plaquette of q if*

$$\sum_{p \in \hat{\partial} e} q_p = 0 \quad \text{for each } e \in \partial \hat{p}.$$

We derive another definition of internal plaquettes. If p is not an internal plaquette of the oriented surface q, then $\sum_{p\in\hat{\partial}e} q_p \in \{-1, 1\}$. For edges in the boundary of q, we have $\sum_{p\in\hat{\partial}e} q_p = 1$. Since $q_{-p} = -q_p$, we obtain

$$\sum_{p\in\hat{\partial}e} q_p = -\sum_{p\in\hat{\partial}e} q_{-p} = -\sum_{-p\in\hat{\partial}e} q_p$$

and

$$\sum_{p\in\hat{\partial}e} q_p = -1 \Leftrightarrow \sum_{-p\in\hat{\partial}e} q_p = 1.$$

Recall that $p \in \hat{\partial}e$ is equivalent to $e \in \partial p$. Therefore, a plaquette $p \in \operatorname{supp} q$ is internal if

$$(\partial p \cup \partial(-p)) \cap B_q = \varnothing. \tag{4.16}$$

Definition 4.18. *(Internal edge) An edge $e \in E_N$ is an internal edge of an oriented surface q if there exists a plaquette $p \in \operatorname{supp} q$, such that $e \in \partial p$ and neither e nor $-e$ is in the boundary of q.*

Last, we have the final lemma of this chapter.

Lemma 4.19. *Let $\sigma \in \Sigma_{E_N}$ and let $\nu \in \Sigma_{P_N}$ be a vortex in σ. Let q be an oriented surface. If there exists a box B, which contains the support of ν, for which the intersection $(**B) \cap \operatorname{supp} q$ only contains internal plaquettes of q, then*

$$\sum_{p\in P_N} q_p^+ \nu_p = 0. \tag{4.17}$$

Proof. Since ν is a 2-form and the lattice is $\mathbb{Z}^4$, it follows from equation (3.16) that

$$* (*\nu) = (-1)^{2(4-2)}\nu = \nu. \tag{4.18}$$

Since $\nu \in \Sigma_{P_N}$, we have $d\nu = 0$. By Lemma 3.35, the co-derivative of $*\nu$ is

$$\delta(*\nu) = (-1)^{4(2+1)+1} * (d(*(*\nu))) \overset{(4.18)}{=} - * d\nu = -1 \cdot 0 = 0.$$

Assume that the box B contains supp ν. By Lemma 3.31, we have that the 2-form $*\nu$ has no support outside $*B$. Thus, the assumptions of Lemma 3.24 are fulfilled for $*\nu$. Therefore, there exists a 3-form g without support outside $*B$, such that $*\nu = -\delta g$. By Lemma 3.35, we have

$$\delta g = (-1)^{4(3+1)+1} * (d(*g)) = (-1) * d(*g). \tag{4.19}$$

Since $*g$ is a 1-form, $d(*g)$ is a 2-form and

$$* (*(d(*g))) \overset{(3.16)}{=} -1^{2(4-2)}(d(*g)) = d(*g). \tag{4.20}$$

Thus,

$$\nu \overset{(4.18)}{=} *(*\nu) = *(-\delta g) \overset{(4.19)}{=} *(*d(*g)) \overset{(4.20}{=} d(*g). \tag{4.21}$$

Since the support of q is finite, we have that $\sum_{p\in P_N} q_p^+ \nu_p$ is well defined. Let B_q be the boundary of q. From Lemma 4.15 follows that

$$\sum_{p\in P_N} q_p^+ \nu_p \overset{(4.21)}{=} \sum_{p\in P_N} q_p^+ (d(*g))_p = \sum_{e\in B_q} (*g)_e.$$

A k-form f_0 is elementary if there exists a k-cell c for which supp $f_0 = \{c, -c\}$. Since g only has support inside $*B$, the support is finite. Therefore, we can write the 3-form g as a finite sum of elementary 3-forms g_0, whose support is contained in $*B$. Since

$$\sum_{e\in B_q} (*g)_e = \sum_{e\in B_q} \sum_{g_0} (*g_0)_e,$$

we only have to prove that

$$\sum_{e\in B_q} (*g_0)_e = 0$$

for each g_0.

Take any elementary 3-form g_0 whose support is contained in $*B$. Then, the Hodge dual of g_0 is an elementary 1-form $*g_0$. Since g_0 is elementary, there exists a 3-cell c_0 in $*B$, such that supp $g_0 = \{c_0, -c_0\}$. Likewise, we have supp $(*g_0) = \{e_0, -e_0\}$, where $e_0 = *c_0$. Since $e_0 = *c_0$, we have $\hat{\partial} e_0 = \hat{\partial}(*c_0) = *(\partial c_0)$. Since ∂c_0 is in $*B$, it follows that $*(\partial c_0) = \hat{\partial} e_0$ is in $**B$. Therefore, $(\hat{\partial} e_0 \cap \text{supp } q) \subseteq (**B \cap \text{supp } q)$. Since we assumed that $(**B) \cap \text{supp } q$ only contains internal plaquettes of q, the plaquettes $p \in \hat{\partial} e_0 \cap \text{supp } q$ must be internal plaquettes of q. Recall that a plaquette $p \in \hat{\partial} e_0 \cap \text{supp } q$ is an internal plaquette if $(\partial p \cup \partial(-p)) \cap B_q = \varnothing$. Since $\pm e_0 \in \partial p \cup \partial(-p)$, it follows that $\pm e_0 \notin B_q$. Therefore, B_q only contains elements for which $*g_0$ is zero. Thus,

$$\sum_{e \in B_q} (*g_0)_e = 0$$

and

$$0 = \sum_{e \in B_q} (*g)_e = \sum_{p \in P_N} q_p^+ \nu_p.$$

□

Chapter 5

The existence and translation invariance of the limit

This chapter focuses on the limit of the expectation of the Wilson loop observable W_γ. This limit is also called the infinite volume limit. Both the existence and the translation invariance of this limit will be proved. For these proofs, Ginibre's inequality is needed. Before stating the inequality, we define both cones and convex cones.

Definition 5.1. *(Cone and convex cone) A set $A \subset V$, where V is a vector space, is a cone if for every $x \in A$ and positive $\alpha \in \mathbb{R}$ we have $\alpha x \in A$. The cone is convex if for positive $\alpha, \beta \in \mathbb{R}$ and $x, y \in A$ we have $\alpha x + \beta y \in A$ [6].*

We will use the notation $\mathrm{Cone}(A)$ for a convex cone generated by the set A. This means that the cone is the intersection of all convex cones containing A.

Lemma 5.2. *(Ginibre's inequality). Let K be a compact metric space and let μ be a probability measure on K. Let $C(K)$ be the algebra of complex-valued continuous functions on K. Let S be a subset of $C(K)$ which is both invariant under complex conjugation and such that for any $f_1, \dots f_m \in S$ and any choice of signs $s_1, \dots, s_m \in \{-1, 1\}$, the inequality*

$$\int\int \mu(x)\mu(y) \prod_{i=1}^{m} (f_i(x) + s_i f_i(y)) \geq 0 \tag{5.1}$$

holds. Let h be a real-valued function and let $f \in C(K)$. Define

$$\langle f \rangle_h = \int f(x) e^{-h(x)} \mu(x) \Big/ \int e^{-h(x)} \mu(x).$$

Let Cone(S) *be the convex cone generated by the set* S. *Then, for any* $f, g, -h$ *in* Cone(S), $\langle f\rangle_h$, $\langle g\rangle_h$ *and* $\langle fg\rangle_h$ *are real and*

$$\langle fg\rangle_h \geq \langle f\rangle_h \langle g\rangle_h. \tag{5.2}$$

For the proof, see [7].

In the following theorem, there is a function from the set of G-valued 1-forms on the set of all oriented edges in $\mathbb{Z}^4$. Therefore, we introduce the two following notations: E_∞ is the set of oriented edges in $\mathbb{Z}^4$ and Σ_{E_∞} is the set of G-valued 1-forms on E_∞. In the theorem, f only depends on the spins of the edges in E_M and we have that $f(\sigma) = f(\sigma|_{E_M})$ for $\sigma \in \Sigma_{E_\infty}$. To simplify, when $N \geq M$ we write $f(\sigma)$ instead of $f(\sigma|_{E_M})$ and f for the natural restriction of f to Σ_{E_N}. A translation τ of the lattice changes where the origin of the lattice is. That the limit is translation invariant means that it does not depend on where in the lattice the Wilson loop observable is.

Theorem 5.3. *Let* $G = \mathbb{Z}_n$ *and* $f : \Sigma_{E_\infty} \to \mathbb{R}$ *be a real-valued function, which only depends on the spins of edges in* E_M *for some integer* $M \geq 1$. *Let* $\beta \geq 0$. *Then,*
(i) the limit $\lim_{N\to\infty} \mathbb{E}_{\beta,N}[f(\sigma)]$ *exists.*
(ii) for any translation τ *of* $\mathbb{Z}^4$, *we have*

$$\lim_{N\to\infty} \mathbb{E}_{\beta,N}[f \circ \tau(\sigma)] = \lim_{N\to\infty} \mathbb{E}_{\beta,N}[f(\sigma)].$$

Proof. (i) To prove the existence of the limit, we write f as a finite sum of functions g_j, for which the limit of the expectation value exists. We will show that $\langle g_j\rangle_h$ (h is defined later in the proof) is both increasing and bounded, which implies that the limit for f exists.

First, we prove that f can be written as a finite sum and that we can use Ginibre's inequality on $\langle g_j\rangle_h$. For this, we need some statements from Example 4 in [7]. These statements hold when $N \geq 1$ for the group $\Gamma^{(N)} = (\sum_{E_N}, +)$. First, the group homomorphisms $\Gamma^{(N)} \mapsto \mathbb{C} \setminus \{0\}$ are given by the functions $\sigma \mapsto \prod_{e\in E_N} e^{2\pi i \sigma_e \sigma'_e/n}$, where σ' is fixed and $\sigma, \sigma' \in \sum_{E_N}$. We notice that $\sigma \mapsto \rho((d\sigma)_p)$, $\sigma \in E_N$ is one of these group homomorphisms. Second, let $S^{(N)}$ be the set of real parts of the group homomorphisms $\Gamma^{(N)} \mapsto \mathbb{C} \setminus \{0\}$. Denote the convex cone of this set by $\mathrm{Cone}(S^{(N)})$, then

$$\Re(\rho((d\sigma)_p)) \in S^{(N)} \subseteq \mathrm{Cone}(S^{(N)}).$$

Third, inequality (5.1) holds for the functions $f_1, \ldots, f_n \in S^{(N)}$ and the signs $s_1, \ldots, s_m \in \{-1, 1\}$ when μ is the uniform measure on $\sum_{E_N}$.

We show that f can be written as a finite sum of functions $g \in S^{(M)}$. Since $M \leq N$, the set Σ_{E_M} is finite. Thus, the group homomorphisms $\Gamma^{(M)} \mapsto \mathbb{C} \setminus \{0\}$ span the set of all real-valued functions on the set Σ_{E_M}. Therefore, there exists scalars $a_1, a_2, \ldots, a_m \in \mathbb{R}$ and functions $g_1, g_2, \ldots, g_m \in S^{(M)}$ such that the function f can be written as $f = a_1 g_1 + \ldots + a_m g_m$. Since the functions $g_1, \ldots, g_m \in S^{(M)}$, they are also in $\mathrm{Cone}(S^{(M)})$. For $M' \geq M$ and $j \in \{1, \ldots, m\}$, we simplify and write g_j for the natural extension of g_j from Σ_{E_M} to the larger set $\Sigma_{E_{M'}}$.

We define the function $-h$ and show that $-h \in \mathrm{Cone}(S^{(N')})$. Fix $N' \geq N \geq M$ and let $\beta, \beta' \geq 0$. Then, for $\sigma \in \Sigma_{E_{N'}}$ define

$$-h(\sigma) := -h(\sigma)_{N,N',\beta,\beta'} := \beta \sum_{p \in P_N} \Re(\rho((d\sigma)_p)) + \beta' \sum_{p \in P_{N'} \setminus P_N} \Re\left(\rho((d\sigma)_p)\right).$$

By definition, $S^{(N')}$ contains the function $\sigma \mapsto \Re(\rho((d\sigma)_p))$, $\sigma \in \Sigma_{E_{N'}}$. Since both β and β' are non-negative, it follows from the definition of a convex cone that $-h(\sigma) \in \mathrm{Cone}(S^{(N')})$.

We check the assumptions for Ginibre's inequality. Since the metric space $K = \sum_{E_N}$ is finite and equipped with discrete topology, it is compact. The set $S = S^{(N')}$ contains only continuous functions and since it consists of the real parts of the group homomorphisms, it is invariant under complex conjugation. Let the probability measure be the uniform measure μ, thus, the third statement from the example fulfils inequality (5.1). Since Ginibre's inequality will be applied to $\langle g_j \sum_{p \in P_{N'} \setminus P_N} \Re(\rho((d\sigma_p))) \rangle_h$, the functions $g_j(\sigma)$, $\sum_{p \in P_{N'} \setminus P_N} \Re(\rho((d\sigma_p)))$ and $-h$ must be in $\mathrm{Cone}(S^{(N')})$. We proved earlier that both g_j and $-h$ are in $\mathrm{Cone}(S^{(N')})$. Since $\sum_{p \in P_{N'} \setminus P_N} \Re(\rho((d\sigma_p)))$ is a sum of elements in $S^{(N')}$, it follows that it is in $\mathrm{Cone}(S^{(N')})$. Therefore, all assumptions for Ginibre's inequality are fulfilled.

We show that the derivative of $\langle g_j \rangle_h$ is non-negative in β' for $j \in \{1, 2, \ldots, m\}$. Ginibre's inequality will be applied in the calculations:

$$\frac{d}{d\beta'} \langle g_j \rangle_h = \frac{d}{d\beta'} \left(\frac{\int g_j(\sigma) e^{-h(\sigma)} \mu(\sigma)}{\int e^{-h(\sigma)} \mu(\sigma)} \right)$$

$$= \frac{\frac{d}{d\beta'}\left(\int g_j(\sigma)e^{-h(\sigma)}\mu(\sigma)\right)\cdot\int e^{-h(\sigma)}\mu(\sigma) - \int g_j(\sigma)e^{-h(\sigma)}\mu(\sigma)\cdot\frac{d}{d\beta'}\int e^{-h(\sigma)}\mu(\sigma)}{(\int e^{-h(\sigma)}\mu(\sigma))^2}$$
$$= \frac{\sum_{p\in P_{N'}\setminus P_N}\Re(\rho((d\sigma)_p))\int g_j(\sigma)e^{-h(\sigma)}\mu(\sigma)}{\int e^{-h(\sigma)}\mu(\sigma)}$$
$$- \langle g_j\rangle_h\frac{\sum_{p\in P_{N'}\setminus P_N}\Re(\rho((d\sigma)_p))\int e^{-h(\sigma)}\mu(\sigma)}{\int e^{-h(\sigma)}\mu(\sigma)}$$
$$= \Big\langle g_j(\sigma)\sum_{p\in P_{N'}\setminus P_N}\Re(\rho((d\sigma)_p))\Big\rangle_h - \langle g_j(\sigma)\rangle_h\Big\langle\sum_{p\in P_{N'}\setminus P_N}\Re(\rho(d\sigma(p))\Big\rangle_h$$
$$\overset{(5.2)}{\geq}\langle g_j(\sigma)\rangle_h\Big\langle\sum_{p\in P_{N'}\setminus P_N}\Re(\rho((d\sigma)_p))\Big\rangle_h - \langle g_j(\sigma)\rangle_h\Big\langle\sum_{p\in P_{N'}\setminus P_N}\Re(\rho((d\sigma)_p)\Big\rangle_h$$
$$= 0.$$

Thus, we have proved that $\langle g_j\rangle_{h_{N,N',\beta,\beta'}}$ is increasing in β'. Moreover, g_j depends on a finite set of edges. Hence, $\|g_j\|_\infty < \infty$ and $\langle g_j\rangle_{h_{N,N',\beta,0}}$ is bounded. The expectation value of g_j is

$$\mathbb{E}_{\beta,N}[g_j(\sigma)] = \langle g_j\rangle_{h_{N,N',\beta,0}},$$

which was proved to be both bounded and monotone in β. Therefore, the limit of $\mathbb{E}_{\beta,N}[g_j(\sigma)]$ exists.

We have left to prove that $\lim_{N\to\infty}\mathbb{E}_{\beta,N}[f(\sigma)]$ exists:

$$\sum_{j=1}^m a_j\lim_{N\to\infty}\mathbb{E}_{\beta,N}[g_j(\sigma)] = \lim_{N\to\infty}\sum_{j=1}^m a_j\mu_{\beta,N}[g_j(\sigma)] = \lim_{N\to\infty}\mathbb{E}_{\beta,N}\left[\sum_{j=1}^m a_jg_j(\sigma)\right]$$
$$= \lim_{N\to\infty}\mathbb{E}_{\beta,N}[f(\sigma)].$$

Thus, the limit exists.

ii) We prove that the limit is translation invariant. Let τ be a translation of $\mathbb{Z}^4$ and choose $N' \geq N \geq M$ large enough for $B_{N'}$ to contain $B_N, \tau B_N$ and $\tau^{-1}B_N$. An illustration of this translation is given in Figure 5.1.

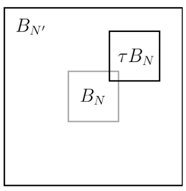

Figure 5.1: The sets B_N, $B_{N'}$ and $\tau B_{N'}$.

Define the functions $h_N := h_{N,N',\beta,0}$ and $h_{N'} := h_{N,N',\beta,\beta}$. For a translation τ of h_N, we have

$$h_N \circ \tau = \beta \sum_{p \in P_N} \Re((\rho((d\sigma)_{\tau(p)})) = \beta \sum_{p \in \tau(P_N)} \Re(\rho((d\sigma)_p)).$$

Since both τB_N and $\tau^{-1} B_N$ are in $B_{N'}$ and $\langle g_j \rangle_h$ is increasing in N (This can be proved with a similar argument to that in part i).), it follows that

$$\langle g_j \rangle_{h_N \circ \tau} \leq \langle g_j \rangle_{h_{N'}} \tag{5.3}$$

and

$$\langle g_j \rangle_{h_N \circ \tau^{-1}} \leq \langle g_j \rangle_{h_{N'}}. \tag{5.4}$$

We derive two inequalities for the expectation value of g_j and $g_j \circ \tau$:

$$\begin{aligned}
\mathbb{E}_{\beta,N}[g_j(\sigma)] &= \langle g_j \rangle_{h_N} \\
&= \langle g_j \circ \tau \circ \tau^{-1} \rangle_{h_N} \\
&= \langle g_j \circ \tau \rangle_{h_N \circ \tau} \\
&\overset{(5.3)}{\leq} \langle g_j \circ \tau \rangle_{h_{N'}} = \mathbb{E}_{\beta,N'}[g_j \circ \tau(\sigma)]
\end{aligned}$$

and

$$\mathbb{E}_{\beta,N}[g_j \circ \tau(\sigma)] = \langle g_j \circ \tau \rangle_{h_N} = \langle g_j \rangle_{h_N \circ \tau^{-1}} \overset{(5.4)}{\leq} \langle g_j \rangle_{h_{N'}} = \mathbb{E}_{\beta,N'}[g_j(\sigma)].$$

Take the limits, first for N' and then for N on both inequalities. We obtain

$$\lim_{N \to \infty} \mathbb{E}_{\beta,N}[g_j(\sigma)] \leq \lim_{N' \to \infty} \mathbb{E}_{\beta,N'}[g_j \circ \tau(\sigma)],$$

and

$$\lim_{N \to \infty} \mathbb{E}_{\beta,N}[g_j \circ \tau(\sigma)] \leq \lim_{N' \to \infty} \mathbb{E}_{\beta,N'}[g_j(\sigma)].$$

Combining these inequalities, we obtain

$$\lim_{N \to \infty} \mathbb{E}_{\beta,N}[g_j \circ \tau(\sigma)] \leq \lim_{N' \to \infty} \mathbb{E}_{\beta,N'}[g_j(\sigma)] \leq \lim_{N' \to \infty} \mathbb{E}_{\beta,N'}[g_j \circ \tau(\sigma)]$$

and as a result,

$$\lim_{N \to \infty} \mathbb{E}_{\beta,N}[g_j \circ \tau(\sigma)] = \lim_{N' \to \infty} \mathbb{E}_{\beta,N'}[g_j(\sigma)].$$

Thus, the limit of the expectation of g_j is translation invariant, but we wish to prove this property for the expectation of f. Similarly to the end of the proof in part (i), we obtain

$$\begin{aligned}\lim_{N\to\infty} \mathbb{E}_{\beta,N}[f\circ\tau(\sigma)] &= \lim_{N\to\infty} \mathbb{E}_{\beta,N}[\sum_{j=1}^{m} a_j g_j \circ \tau(\sigma)] = \sum_{j=1}^{m} a_j \lim_{N\to\infty} \mathbb{E}_{\beta,N}[g_j\circ\tau(\sigma)] \\ &= \sum_{j=1}^{m} a_j \lim_{N\to\infty} \mathbb{E}_{\beta,N}[g_j] = \lim_{N\to\infty} \mathbb{E}_{\beta,N}[\sum_{j=1}^{m} a_j g_j] \\ &= \lim_{N\to\infty} \mathbb{E}_{\beta,N}[f(\sigma)].\end{aligned}$$

Therefore, the limit of the expectation value of f is translation invariant. $\square$

We check the assumptions for the Wilson loop observable

$$W_\gamma = \rho\left(\sum_{e\in\gamma}\sigma_e\right), \ \sigma\in\Sigma_{E_N}.$$

Since the Wilson loop observable maps Σ_{E_∞} to $\mathbb{R}$, has real values and only depends on the spins of edges in E_M, the function f in Theorem 5.3 can be chosen as W_γ. Thus, the limit

$$\lim_{N\to\infty} \mathbb{E}_{\beta,N}[W_\gamma] = \langle W_\gamma\rangle_\beta$$

both exists and is translation invariant.

Chapter 6

Proof of the main theorem

The proof is divided into two cases; one when $\ell\lambda(\beta)^{12}$ is small and one when $\ell\lambda(\beta)^{12}$ is large. In the final part of the proof, these two parts are combined. We begin with the case when $\ell\lambda(\beta)^{12}$ is small.

6.1 The case when $\ell\lambda(\beta)^{12}$ is small

First, the proposition for the case when $\ell\lambda(\beta)^{12}$ is small is stated, then some lemmas necessary for the proof of the proposition are given and proved. Last, the proposition is proved.

Proposition 6.1. *Consider lattice gauge theory with the structure group $G = \mathbb{Z}_n$ and a one-dimensional faithful representation ρ of G. Let γ be a simple oriented loop in $\mathbb{Z}^4$, $\ell = |\gamma|$ the length of it and ℓ_c the number of corner edges in γ. Let N be large enough so that $\gamma \subseteq E_N$ and such that there exists a cube B of width $|\gamma|$ containing γ inside B_N. Let $\beta_0 > 0$ be such that $5(|G|-1)\lambda(\beta)^2 < 1$ for all $\beta > \beta_0$. Then for all $\beta > \beta_0$, we have*

$$|\mathbb{E}_{\beta,N}[W_\gamma] - e^{-\ell(1-\theta(\beta))}| \leq C_A e^{2C^*\ell\lambda(\beta)^{12}} \left(\sqrt{\frac{\ell_c}{\ell}} + \lambda(\beta)^2 \right).$$

Here $\theta(\beta)$ is defined by equation (2.7) and C_A by equation (6.41). Furthermore, C_A only depend on β_0, G and ρ.

To prove this proposition, we show that it is very likely that only minimal vortices, which are centred at the edges in γ, have an impact on a Wilson loop (A minimal vortex is centred at dx_j if it can be written as $d(gdx_j)$ for a $g \in G \setminus \{0\}$

and a $dx_j \in E_N$.). A number of lemmas will be needed. We begin with defining the constant C^* and recalling the definitions for $\theta(\beta)$ and $\lambda(\beta)$. Let $\beta_0 > 0$ be given and define

$$C^* := \sup_{\beta>\beta_0} \left[(1-\theta(\beta))\lambda(\beta)^{-12}\right]. \tag{6.1}$$

Recall that

$$\theta(\beta) = \frac{\sum_{g\in G} \rho(g)\phi_\beta(g)^{12}}{\sum_{g\in G} \phi_\beta(g)^{12}}$$

and

$$\lambda(\beta) = \max_{g\in G\setminus\{0\}} \frac{\phi_\beta(g)}{\phi_\beta(0)},$$

where

$$\phi_\beta(g) := e^{\beta\Re\rho(g)}, \qquad g \in G.$$

6.1.1 Technical lemmas

Lemma 6.2. *The function $\theta(\beta)$ is a real-valued function and can be written as*

$$\theta(\beta) = \frac{\sum_{g\in G} \Re(\rho(g))\phi_\beta(g)^{12}}{\sum_{g\in G} \phi_\beta(g)^{12}}. \tag{6.2}$$

.

Proof. Since $\rho(g)^{-1} = \rho(g)^*$, we have

$$\Re(\rho(-g)) = \Re(\rho(g)^{-1}) = \Re(\rho(g)^*) = \Re(\rho(g)).$$

Therefore,

$$\phi_\beta(-g) = e^{\beta\Re(\rho(-g))} = e^{\beta\Re(\rho(g))} = \phi_\beta(g) \quad \text{for } g \in G.$$

Furthermore, since $g \to -g$ is a bijection, we have that

$$\begin{aligned}
\theta(\beta) &= \frac{1}{2}\frac{\sum_{g\in G} \rho(g)\phi_\beta(g)^{12}}{\sum_{g\in G} \phi_\beta(g)^{12}} + \frac{1}{2}\frac{\sum_{g\in G} \rho(-g)\phi_\beta(-g)^{12}}{\sum_{g\in G} \phi_\beta(-g)^{12}} \\
&= \frac{1}{2}\frac{\sum_{g\in G} \rho(g)\phi_\beta(g)^{12}}{\sum_{g\in G} \phi_\beta(g)^{12}} + \frac{1}{2}\frac{\sum_{g\in G} \rho(-g)\phi_\beta(g)^{12}}{\sum_{g\in G} \phi_\beta(g)^{12}} \\
&= \frac{\sum_{g\in G} (\rho(g)+\rho(-g))\phi_\beta(g)^{12}}{\sum_{g\in G} \phi_\beta(g)^{12}} \\
&= \frac{\sum_{g\in G} \Re(\rho(g))\phi_\beta(g)^{12}}{\sum_{g\in G} \phi_\beta(g)^{12}} \in \mathbb{R}.
\end{aligned}$$

□

Lemma 6.3. *Let $\beta \geq 0$ be such that $5(|G|-1)\lambda(\beta)^2 < 1$. Then*

$$0 < 1 - \frac{2}{5^6(|G|-1)^5} < 1 - C^*\lambda(\beta)^{12} \leq \theta(\beta) \leq 1. \tag{6.3}$$

Proof. We calculate the upper bound for $1 - C^*\lambda(\beta)^{12}$:

$$\begin{aligned} 1 - C^*\lambda(\beta)^{12} &= 1 - \sup_{\beta > \beta_0} \left[(1-\theta(\beta))\lambda(\beta)^{-12}\right] \lambda(\beta)^{12} \\ &\leq 1 - \left((1-\theta(\beta))\lambda(\beta)^{-12}\right) \lambda(\beta)^{12} \\ &= \theta(\beta). \end{aligned} \tag{6.4}$$

Since Lemma 6.2 gives that $\theta(\beta)$ is real, we have

$$\theta(\beta) \leq |\theta(\beta)| = \left| \frac{\sum_{g\in G} \rho(g)\phi_\beta(g)^{12}}{\sum_{g\in G} \phi_\beta(g)^{12}} \right| \leq \frac{\sum_{g\in G} |\rho(g)|\phi_\beta(g)^{12}}{\sum_{g\in G} \phi_\beta(g)^{12}} \overset{(2.3)}{=} \frac{\sum_{g\in G} \phi_\beta(g)^{12}}{\sum_{g\in G} \phi_\beta(g)^{12}} = 1. \tag{6.5}$$

We calculate an upper bound for C^*. Notice that $\Re\rho(g) \in [-1, 1[$ for $g \in G \setminus \{0\}$ and $\phi_\beta > 0$. Hence,

$$\begin{aligned} C^* &\overset{(6.1),(6.2)}{=} \sup_{\beta > \beta_0} \left[\left(1 - \frac{\sum_{g\in G} \Re(\rho(g))\phi_\beta(g)^{12}}{\sum_{g\in G} \phi_\beta(g)^{12}} \right) \lambda(\beta)^{-12} \right] \\ &\overset{(2.2)}{=} \sup_{\beta > \beta_0} \left[\frac{\sum_{g\in G\setminus\{0\}} (1 - \Re\rho(g))\phi_\beta(g)^{12}}{\sum_{g\in G} \phi_\beta(g)^{12}} \lambda(\beta)^{-12} \right] \\ &\leq \sup_{\beta > \beta_0} \left[\frac{\sum_{g\in G\setminus\{0\}} 2\phi_\beta(g)^{12}}{\sum_{g\in G} \phi_\beta(g)^{12}} \lambda(\beta)^{-12} \right] \\ &= \sup_{\beta > \beta_0} \left[\frac{\sum_{g\in G\setminus\{0\}} 2\phi_\beta(g)^{12}}{\phi_\beta(0)^{12} + \sum_{g\in G\setminus\{0\}} \phi_\beta(g)^{12}} \lambda(\beta)^{-12} \right] \\ &\leq \sup_{\beta > \beta_0} \left[2 \sum_{g\in G\setminus\{0\}} \frac{\phi_\beta(g)^{12}}{\phi_\beta(0)^{12}} \lambda(\beta)^{-12} \right] \\ &= \sup_{\beta > \beta_0} \left[2 \sum_{g\in G\setminus\{0\}} \lambda(\beta)^{12}\lambda(\beta)^{-12} \right] \\ &= 2 \sum_{g\in G\setminus\{0\}} 1 = 2(|G|-1). \end{aligned}$$

Therefore,

$$\begin{aligned} C^*\lambda(\beta)^{12} \leq 2(|G|-1)\lambda(\beta)^{12} &= \frac{2}{5^6(|G|-1)^5} \left(5(|G|-1)\lambda(\beta)^2\right)^6 \\ &< \frac{2}{5^6(|G|-1)^5}. \end{aligned} \tag{6.6}$$

Thus,

$$0 < 1 - \frac{2}{5^6(|G|-1)^5} \overset{(6.6)}{<} 1 - C^*\lambda(\beta)^{12} \overset{(6.4)}{\leq} \theta(\beta) \overset{(6.5)}{\leq} 1.$$

□

Notice that $C^* > 0$ by equations (6.4) and (6.5). The next lemma gives two inequalities for the function θ.

Lemma 6.4. *Let $\beta > \beta_0$. For $0 \leq j \leq \ell$, we have*

$$\theta^{-j} \leq 2e^{C^*\ell\lambda(\beta)^{12}}. \tag{6.7}$$

For $1 \leq j \leq \ell$, we have

$$\theta^{-j} - 1 \leq 2jC^*\lambda(\beta)^{12}e^{C^*\ell\lambda(\beta)^{12}}. \tag{6.8}$$

For the proof of this lemma, the two following inequalities are needed.
First, for $x \in [0, \frac{1}{2}]$, we have

$$(1-x)^{-1} \leq 2e^x, \tag{6.9}$$

and second, Bernoulli's inequality: For $x \geq -1$ and $n \geq 1$, we have

$$(1+x)^n \geq 1 + nx. \tag{6.10}$$

Proof of Lemma 6.4. First, notice that

$$0 < C^*\lambda(\beta)^{12} \overset{(6.6)}{<} \frac{2}{5^6(|G|-1)^5} \leq \frac{2}{5^6} < \frac{1}{2}.$$

Therefore, inequality (6.9) can be applied on $(1 - C^*\lambda(\beta)^{12})^{-1}$. For $0 \leq j \leq \ell$, we have

$$\theta^{-j} \leq \theta^{-\ell} \overset{(6.3)}{\leq} \left((1 - C^*\lambda(\beta)^{12})^{-1}\right)^{\ell} \overset{(6.9)}{\leq} 2e^{C^*\ell\lambda(\beta)^{12}}. \tag{6.11}$$

For $1 \leq j \leq \ell$, we have by inequality (6.10) that

$$\theta^j = (1 + (\theta - 1))^j \geq 1 + j(\theta - 1).$$

Thus,

$$1 - \theta^j \leq j(1 - \theta)$$

and

$$\theta^{-j} - 1 = \theta^{-j}(1 - \theta^j) \leq j(1-\theta)\theta^{-j} \overset{(6.3),(6.11)}{\leq} 2jC^*\lambda(\beta)^{12}e^{C^*\ell\lambda(\beta)^{12}}. \tag{6.12}$$

□

Lemma 6.5. *For $\ell \geq 1$ and $\theta = \theta(\beta)$, we have*

$$|\theta^{\ell} - e^{-\ell(1-\theta)}| \leq \frac{e^{2C^{*}\ell\lambda(\beta)^{12}}}{2\ell}. \tag{6.13}$$

We state two inequalities necessary for the proof of the lemma. First, for $\ell \geq 1$, $|x| \leq 1$ and $|y| \leq 1$, we have

$$|x^{\ell} - y^{\ell}| \leq \ell|x - y|. \tag{6.14}$$

Indeed, by the intermediate value theorem, there exists a $\xi \in [-1, 1]$ such that

$$|x^{\ell} - y^{\ell}| = |\ell\xi^{\ell-1}||x - y| \leq \ell|x - y|.$$

Second, for $x > 0$, we have

$$1 + x \leq e^{x}. \tag{6.15}$$

This follows from the Taylor series for the exponential function:

$$e^{x} = \sum_{k=0}^{\infty} \frac{x^{n}}{n!},$$

which is also otherwise useful in the proof.

Proof of Lemma 6.5. Since $0 \leq 1 - \theta < 1$, it follows from the Taylor series that

$$e^{-(1-\theta)} \leq 1 + (-(1-\theta)) + \frac{(-(1-\theta))^{2}}{2} = \theta + \frac{(1-\theta)^{2}}{2}. \tag{6.16}$$

From equation (6.15) follows that

$$1 + (\theta - 1) \leq e^{(\theta-1)} \Leftrightarrow \theta \leq e^{-(1-\theta)}. \tag{6.17}$$

Thus,

$$\begin{aligned} |\theta^{\ell} - e^{-\ell(1-\theta)}| &\overset{(6.14)}{\leq} \ell|\theta - e^{-(1-\theta)}| \overset{(6.17)}{=} \ell(\theta - e^{-(1-\theta)}) \\ &\overset{(6.16)}{\leq} \frac{\ell(2\theta - 2\theta - (1-\theta)^{2})}{2} = \frac{\ell(1-\theta)^{2}}{2} \\ &\overset{(6.3)}{\leq} \frac{(\ell C^{*}\lambda(\beta)^{12})^{2}}{2\ell} \\ &\overset{(6.15)}{\leq} \frac{e^{2C^{*}\ell\lambda(\beta)^{12}}}{2\ell}. \end{aligned} \tag{6.18}$$

□

6.1.2 Probabilistic bounds on vortex sizes

We want to calculate the probabilities for the events A_1, A_2 and A_3, which are soon defined. Before that, some necessary definitions and notations are given. Let γ be a simple loop, then the cube B is defined as a fixed cube that contains γ, is inside B_N and is of width $\ell = |\gamma|$. Let q be an fixed oriented surface such that its support is contained in B and such that γ is the boundary of q (we will later prove that q exists). Denote the support of this oriented surface by $Q \coloneqq \text{supp } q$. Then, define the set Q' as the set of plaquettes $p \in Q$ for which the oriented loop γ does not intersect any cube of width $b+2$ containing p, where b is chosen as the smallest number for which the support of any irreducible $\omega \in \sum_{P_N}$ with $|\text{supp } \nu| \leq 48$ is inside the cube. Last, the set

$$\gamma_c \coloneqq \{e \in \gamma : |\partial(\hat{\partial} e) \cap (\gamma \cup -\gamma)| \geq 2\}$$

is the set of corner edges in γ. Recall that the number of corner edges in γ is given by ℓ_c.

We define the events:

$$A_1 \coloneqq \{\text{There exists no vortex } \nu \text{ in } \sigma \text{ with } |\text{supp } \nu| \geq 50 \text{ whose support intersects } Q\}$$

$$A_2 \coloneqq \{\text{There exists no vortex } \nu \text{ in } \sigma \text{ with } |\text{supp } \nu| \geq 14 \text{ and } \text{supp } \nu \cap (Q \setminus Q') \neq \varnothing\}$$

$$A_3 \coloneqq \{\text{There exists no vortex } \nu \text{ in } \sigma : \text{supp } \nu = \hat{\partial} e \cup \hat{\partial}(-e) \text{ for some } e \in \gamma_c\}.$$

Notice that the vortex in the event A_3 is minimal by Lemma 4.7. The probabilities of these events are calculated in the following three lemmas.

Lemma 6.6. *Let* $\beta_0 > 0$ *be such that* $5(|G|-1)\lambda(\beta)^2 < 1$ *for all* $\beta > \beta_0$. *Then*

$$\mu_{\beta,N}(A_1) \geq 1 - C_0^{(25)} \ell^4 \lambda(\beta)^{50}. \tag{6.19}$$

Proof. Fix $p_0 \in P_N$ and define the event

$$E_1(p_0) \coloneqq \{\text{There exists at least one vortex } \nu \text{ in } \sigma \text{ with } p_0 \in \text{supp } \nu \text{ and } |\text{supp } \nu| \geq 50\}.$$

The assumptions for Proposition 4.12 are fulfilled for E_1. Therefore,

$$\mu_{\beta,N}(E_1(p_0)) \leq C_0^{(25)}\lambda(\beta)^{50}, \quad \text{for any } p_0 \in P_N. \tag{6.20}$$

Since B is of width ℓ and supp q is contained in B, there is at most ℓ^4 choices for $p_0 \in$ supp ν for which supp ν intersects Q. Hence, by a union bound and inequality (6.20), we have

$$\mu_{\beta,N}(A_1^c) \leq \ell^4 C_0^{(25)}\lambda(\beta)^{50}.$$

Thus,

$$\mu_{\beta,N}(A_1) = 1 - \mu_{\beta,N}(A_1^c) \geq 1 - C_0^{(25)}\ell^4\lambda(\beta)^{50}.$$

□

Lemma 6.7. *Let $\beta_0 > 0$ be such that $5(|G|-1)\lambda(\beta)^2 < 1$ for all $\beta > \beta_0$. Then*

$$\mu_{\beta,N}(A_2) \geq 1 - C_1 C_0^{(7)}\ell\lambda(\beta)^{14}.$$

Proof. Fix $p_0 \in P_N$. The event E_2 is defined by

$$E_2(p_0) := \{\sigma \in \Sigma_{E_N} : \exists \text{ a vortex } \nu \text{ in } \sigma \text{ with } p_0 \in \text{supp } \nu \text{ and } |\text{supp } \nu| \geq 14\}.$$

By Proposition 4.12, we have

$$\mu_{\beta,N}(E_2(p_0)) \leq C_0^{(7)}\lambda(\beta)^{14}.$$

Consider the plaquettes $p \in Q \setminus Q'$. By definition, any cube of width $b+2$ that contains one of these plaquettes intersects γ. Hence

$$|Q \setminus Q'| \leq C_1|\gamma| = C_1\ell, \tag{6.21}$$

where the constant C_1 depends on the width of the box.

By a union bound,

$$\mu_{\beta,N}(A_2^c) \leq C_1\ell\mu_{\beta,N}(E_2(p_0)).$$

Thus,

$$\mu_{\beta,N}(A_2) = 1 - \mu_{\beta,N}(A_2^c) \geq 1 - C_1\ell\mu_{\beta,N}(E_2(p_0)) \geq 1 - C_1C_0^{(7)}\ell\lambda(\beta)^{14}.$$

□

Lemma 6.8. *Let* $\beta_0 > 0$ *be such that* $5(|G| - 1)\lambda(\beta)^2 < 1$ *for all* $\beta > \beta_0$*. Then*

$$\mu_{\beta,N}(A_3) \geq 1 - C^{(6)}\ell_c\lambda(\beta)^{12}. \tag{6.22}$$

Proof. By Proposition 4.12 for a minimal vortex and the fact that there are at most ℓ_c corner edges for which supp $\nu = \hat{\partial}e \cup \hat{\partial}(-e)$, we have that

$$\mu_{\beta,N}(A_3^c) \leq \ell_c C^{(6)}\lambda(\beta)^{12}$$

and

$$\mu_{\beta,N}(A_3) \geq 1 - C^{(6)}\ell_c\lambda(\beta)^{12}.$$

□

6.1.3 The main argument

We continue with some more definitions and notations for sets and conditional probability. Define the set of edges

$$\gamma_1 \coloneqq \gamma \setminus \gamma_c$$

and the random set of edges

$$\gamma' \coloneqq \{e \in \gamma_1 : \exists p, p' \in \hat{\partial}e \text{ with } (d\sigma)_p \neq (d\sigma)_{p'}\}.$$

Notice that γ_1 is the set of the non-corner edges of γ (Recall that both corner and non-corner edges are illustrated in Figure 2.2.).

We investigate how $d\sigma$ can be written on two different plaquettes $p, p' \in \hat{\partial}e$:

$$(d\sigma)_p = \sum_{e' \in \partial p} \sigma_{e'} = \sigma_e + \sum_{e' \in \partial p \smallsetminus \{e\}} \sigma_{e'}$$

and

$$(d\sigma)_{p'} = \sum_{e' \in \partial p'} \sigma_{e'} = \sigma_e + \sum_{e' \in \partial p' \smallsetminus \{e\}} \sigma_{e'}.$$

Therefore, the event $(d\sigma)_p \neq (d\sigma)_{p'}$ does not depend on the value of σ_e. Hence, if we know the spins of the edges which are not in $\pm\gamma_1$, then the edges in γ' are determined. Furthermore, a plaquette cannot contain two non-corner edges. Therefore, when we condition on $(\sigma_e)_{e \notin \pm\gamma_1}$, the spins $(\sigma_e)_{e \in \gamma_1}$ are independent. This is concluded in the following lemma.

Lemma 6.9. *The random set γ' is determined by the spins $(\sigma_e)_{e\notin\pm\gamma_1}$ and conditioning on the latter, $(\sigma_e)_{e\in\gamma_1}$ are independent.*

Lastly, we have one more inequality for an expectation value, the expected value of the number of edges in the set γ'.

Lemma 6.10. *Let $\beta_0 > 0$ be such that $5(|G|-1)\lambda(\beta)^2 < 1$ for all $\beta > \beta_0$. Then*

$$\mathbb{E}_{\beta,N}[|\gamma'|] \leq C_0^{(6)}\ell\lambda(\beta)^{12}. \tag{6.23}$$

Proof. If $(d\sigma)_p \neq 0$, then the plaquette is in the support of a vortex ν. Since the size of the vortex is 12 or larger, we have that $|\text{supp } \nu| \geq 12$ if supp ν contains $(d\sigma)_p \neq 0$. Then, from Proposition 4.12 follows for a fixed p_0 that

$$\mu_{\beta,N}\{\text{A plaquette } p \text{ is such that } (d\sigma)_p \neq 0\} \leq C_0^{(6)}\lambda(\beta)^{12}.$$

Since $|\gamma| = \ell$ and $\gamma' \subseteq \gamma$, there are at most ℓ plaquettes in γ' for which $(d\sigma)_p \neq 0$. Thus,

$$\mathbb{E}_{\beta,N}[|\gamma'|] \leq C_0^{(6)}\ell\lambda(\beta)^{12}.$$

□

We prove the proposition. Six inequalities and equations for expectations will be calculated and then combined to obtain the desired inequality.

Proof of Proposition 6.1. Since γ is a simple loop and $B \subseteq B_N$, we have by Lemma 4.16 there exists an oriented surface q such that its support is contained in B and γ is the boundary of q. Recall that the support of this oriented surface is Q. Let $\sigma \sim \mu_{\beta,N}$.

Since $\sigma \in E_N$, it follows from Lemma 4.5 that we can write $d\sigma$ as a sum of vortices $\nu_1, \nu_2, \ldots$ with disjoint supports. Let the decomposition be fixed and define the set V as the set of vortices in this decomposition for which the support intersects Q. For the oriented surface q with boundary γ, we have by Lemma 4.15 that

$$\sum_{e\in\gamma}\sigma_e = \sum_{p\in P_N} q_p^+(d\sigma)_p \quad \text{for } \sigma \in \Sigma_{E_N}.$$

Therefore, the Wilson loop observable W_γ can be written as

$$W_\gamma = \rho\left(\sum_{e\in\gamma}\sigma_e\right) = \rho\left(\sum_{p\in P_N} q_p^+(d\sigma)_p\right) = \rho\left(\sum_{v\in V}\sum_{p\in\text{supp }\nu} q_p^+(d\sigma)_p\right).$$

Note that since $|\rho(g)| = 1$ for $g \in G$, we have $|W_\gamma| = 1$.

The first expectation ($\mathbb{E}_{\beta,N}[|W_\gamma - W_\gamma^0|]$)

We define the subset V_0 of V as

$$V_0 \coloneqq \{\nu \in V : |\operatorname{supp}\ \nu| \leq 48\}$$

and W_γ^0 as

$$W_\gamma^0 \coloneqq \rho\left(\sum_{\nu \in V_0} \sum_{p \in \operatorname{supp}\ \nu} q_p^+(d\sigma)_p\right).$$

We estimate the expectation value of $|W_\gamma - W_\gamma^0|$. First, consider the case when the event A_1 occurs. Then there is no vortex with $|\operatorname{supp}\ \nu| \geq 50$, thus, $V_0 = V$, $W_\gamma = W_\gamma^0$ and $|W_\gamma - W_\gamma^0| = 0$. Second, consider the case when the event A_1 does not occur. Then,

$$|W_\gamma - W_\gamma^0| \leq |W_\gamma| + |W_\gamma^0| = 1 + 1 = 2.$$

Thus, by Lemma 6.6, the expectation value is

$$|\mathbb{E}_{\beta,N}[|W_\gamma - W_\gamma^0|] \leq 0 \cdot \mu_{\beta,N}(A_1) + 2 \cdot \mu_{\beta,N}(A_1^c) \leq 2C_0^{(25)}\ell^4\lambda(\beta)^{50}. \tag{6.24}$$

The second expectation ($\mathbb{E}_{\beta,N}[|W_\gamma^0 - W_\gamma^3|]$)

Consider a cube B of width b, where b is chosen as the smallest number for which the support of any irreducible $\omega \in \sum_{P_N}$ with $|\operatorname{supp}\ \nu| \leq 48$ is inside the cube. Let

$$V_1 \coloneqq \{\nu \in V_0 : \operatorname{supp}\ \nu \cap Q' \neq \varnothing\}.$$

Fix $\nu \in V_1$. Since B is a cube of width b, we have from Example 3.30 that the cube $**B$ is of width $b+2$. By definition, for plaquettes $p \in Q'$ the intersection between a cube of width $b+2$ containing p and γ is empty. Hence, the intersection $(**B) \cap Q$ only consists of internal plaquettes of q. Recall that γ is the boundary of q. Thus, from Lemma 4.19 follows that

$$\sum_{p \in \operatorname{supp}\ \nu} q_p^+(d\sigma)_p = 0 \quad \text{for } v \in V_1.$$

We define the set

$$V_2 \coloneqq V_0 \setminus V_1$$

and

$$W_\gamma^0 = \rho\left(\sum_{\nu \in V_0} \sum_{p \in \operatorname{supp}\ \nu} q_p^+(d\sigma)_p\right) = \rho\left(\sum_{\nu \in V_2} \sum_{p \in \operatorname{supp}\ \nu} q_p^+(d\sigma)_p\right).$$

We define a subset V_3 of V_2, which only contains minimal vortices. Let

$$V_3 := \{\nu \in V_2 : |\operatorname{supp}\ \nu| = 12\} = \{\nu \in V : |\operatorname{supp}\ \nu| = 12 \text{ and } \operatorname{supp}\ \nu \cap Q' = \varnothing\}$$

and

$$W_\gamma^3 := \rho\left(\sum_{\nu\in V_3}\sum_{p\in\operatorname{supp}\ \nu} q_p^+(d\sigma)_p\right).$$

We calculate the expectation value of $|W_\gamma^0 - W_\gamma^3|$. Consider the event A_2. If A_2 occurs, then there is no vortex ν with $|\operatorname{supp}\ \nu| \geq 14$. Thus, $V_3 = V_2$ and $W_\gamma^3 = W_\gamma^0$. Similarly as earlier, if A_2 does not occur, then $|W_\gamma^0 - W_\gamma^3| \leq 2$. By Lemma 6.7, the expectation value is

$$\mathbb{E}_{\beta,N}[|W_\gamma^0 - W_\gamma^3|] \leq 0 \cdot \mu_{\beta,N}(A_2) + 2 \cdot \mu_{\beta,N}(A_2^c) \leq 2C_1C_0^{(7)}\ell\lambda(\beta)^{14}. \tag{6.25}$$

The third expectation ($\mathbb{E}_{\beta,N}[|W_\gamma^3 - W_\gamma^4|]$)

Let

$$V_4 := \{\nu \in V_3 : \exists e \in \gamma \text{ such that } \operatorname{supp}\ \nu = \hat{\partial}e \cup \hat{\partial}(-e)\}$$

and

$$W_\gamma^4 := \rho\left(\sum_{\nu\in V_4}\sum_{p\in\operatorname{supp}\ \nu} q_p^+(d\sigma)_p\right).$$

Consider $\nu \in V_3 \setminus V_4$. Since ν is minimal, by Lemma 4.7 there exists an edge $e \in E_N$ such that $\operatorname{supp}\ \nu = \hat{\partial}e \cup \hat{\partial}(-e)$. Furthermore, since $\nu \notin V_4$, the edge e cannot be in the boundary of q. Thus, e is an internal edge of q. By Lemma 4.16,

$$\sum_{p\in\operatorname{supp}\ \nu} q_p^+(d\sigma)_p = 0 \quad \text{for } v \in V_3 \setminus V_4.$$

Therefore,

$$W_\gamma^3 = \rho\left(\sum_{\nu\in V_3}\sum_{p\in\operatorname{supp}\ \nu} q_p^+(d\sigma)_p\right) = \rho\left(\sum_{\nu\in V_4}\sum_{p\in\operatorname{supp}\ \nu} q_p^+(d\sigma)_p\right) = W_\gamma^4.$$

Since $W_\gamma^3 = W_\gamma^4$, the expectation of $|W_\gamma^3 - W_\gamma^4|$ is

$$\mathbb{E}_{\beta,N}[|W_\gamma^3 - W_\gamma^4|] = \mathbb{E}_{\beta,N}[0] = 0. \tag{6.26}$$

The fourth expectation ($\mathbb{E}_{\beta,N}[|W_\gamma^4 - W_\gamma^5|]$)

Let

$$V_5 := \{\nu \in V_4 : \exists e \in \gamma_1 \text{ such that } \operatorname{supp}\ \nu = \hat{\partial}e \cup \hat{\partial}(-e)\}$$

and

$$W_\gamma^5 := \rho\left(\sum_{\nu\in V_5}\sum_{p\in \mathrm{supp}\ \nu} q_p^+(d\sigma)_p\right).$$

If the event A_3 occurs, then $V_4 = V_5$ and $W_\gamma^4 = W_\gamma^5$. If A_3 does not occur, then $|W_\gamma^4 - W_\gamma^5| \leq 2$. Thus, by Lemma 6.8, the expectation value of $|W_\gamma^4 - W_\gamma^5|$ is

$$\mathbb{E}_{\beta,N}[|W_\gamma^4 - W_\gamma^5|] \leq 0\cdot\mu_{\beta,N}(A_3) + 2\mu_{\beta,N}(A_3^c) \leq 2C^{(6)}\ell_c\lambda(\beta)^{12}. \tag{6.27}$$

The fifth expectation ($\mathbb{E}_{\beta,N}[|W_\gamma^5 - W_\gamma^6|]$)

We first define a set of edges:

$$E_5 := \{e\in\gamma : \exists \nu\in V_5 \text{ such that } \mathrm{supp}\ \nu = \hat{\partial} e\cup\hat{\partial}(-e)\}.$$

Then W_γ^5 can be written as

$$\begin{aligned} W_\gamma^5 &= \rho\left(\sum_{\nu\in V_5}\sum_{p\in \mathrm{supp}\ \nu} q_p^+(d\sigma)_p\right) = \rho\left(\sum_{e\in E_5}\sum_{p\in\pm\hat{\partial} e} q_p^+(d\sigma)_p\right) \\ &= \rho\left(\sum_{e\in E_5}\sum_{p\in\hat{\partial} e} q_p(d\sigma)_p\right). \end{aligned}$$

Define

$$E_6 := \{e\in\gamma_1 : (d\sigma)_p = (d\sigma)_{p'} \text{ for all } p,p'\in\hat{\partial}(-e)\}$$

and

$$W_\gamma^6 := \rho\left(\sum_{e\in E_6}\sum_{p\in\hat{\partial} e} q_p^+(d\sigma)_p\right).$$

Notice that the set E_6 can be written as

$$E_6 = \gamma_1\setminus\gamma'. \tag{6.28}$$

Consider two disjoint edges $e, e'\in\gamma_1$, then $\hat{\partial} e\cup\hat{\partial}(-e)$ and $\hat{\partial} e'\cup\hat{\partial}(-e')$ are disjoint. Since any vortex $\nu\in V_5$ is minimal, we have by Lemma 4.9 for a vortex $\nu\in V_5$ that $(d\sigma)_p = d(g dx_j)_p = \{-g, g\}$ for all $p\in P_N$. Thus, $(d\sigma)_p$ does not depend on $p\in\hat{\partial} e$. For $e\in E_6$, fix $p_e\in\hat{\partial} e$. Since q is an oriented surface and E_6 only contains edges in the boundary of q, we obtain

$$\sum_{p\in\hat{\partial} e} q_p(d\sigma)_p = \sum_{p\in\hat{\partial} e} q_p(d\sigma)_{p_e} = \left(\sum_{p\in\hat{\partial} e} q_p\right)(d\sigma)_{p_e} = (d\sigma)_{p_e}, \quad \text{for} e\in E_6.$$

Therefore,

$$W_\gamma^6 = \rho\left(\sum_{e\in E_6} (d\sigma)_{p_e}\right).$$

If the event A_2 occurs, then $E_5 = E_6$ and $W_\gamma^5 = W_\gamma^6$. If A_2 does not occur, then $|W_\gamma^5 - W_\gamma^6| \leq 2$. Hence, by Lemma 6.7,

$$\mathbb{E}_{\beta,N}[|W_\gamma^5 - W_\gamma^6|] \leq 0 \cdot \mu_{\beta,N}(A_2) + 2\mu_{\beta,N}(A_2^c) = 2C_1 C_0^{(7)} \ell\lambda(\beta)^{14}. \tag{6.29}$$

The sixth expectation ($|\mathbb{E}_{\beta,N}[W_\gamma^6] - \theta^\ell|$)

The strategy will be to first compute the conditional expectation of W_γ^6 given $(\sigma_e)_{e\neq\pm\gamma_1}$ and then apply the law of total expectation. Given $(\sigma_e)_{e\notin\pm\gamma_1}$, denote the conditional probability by $\mu'_{\beta,N}$ and the conditional expectation by $\mathbb{E}'_{\beta,N}$.

We calculate the conditional expectation value of W_γ^6 given $(\sigma_e)_{e\neq\pm\gamma_1}$. By Lemma 6.9, we have that the spins $(\sigma_e)_{e\in\gamma_1}$ are independent. For $e \in \gamma_1 \setminus \gamma'$, define $\sigma^e := \sum_{\hat{e}\in p_e \smallsetminus \{e\}} \sigma_{\hat{e}}$ such that

$$(d\sigma)_p = \sigma^e + \sigma_e \quad \text{for each } p \in \hat{\partial} e. \tag{6.30}$$

Then,

$$\begin{aligned}
\mathbb{E}'_{\beta,N}[W_\gamma^6] &= \mathbb{E}'_{\beta,N}\left[\rho\left(\sum_{e\in E_6}(d\sigma)_{p_e}\right)\right] \\
&\overset{(6.28)}{=} \mathbb{E}'_{\beta,N}\left[\rho\left(\sum_{e\in\gamma_1\smallsetminus\gamma'}(d\sigma)_{p_e}\right)\right] \\
&= \mathbb{E}'_{\beta,N}\left[\prod_{e\in\gamma_1\smallsetminus\gamma'}\rho\left((d\sigma)_{p_e}\right)\right] \\
&\overset{\perp\!\!\!\perp}{=} \prod_{e\in\gamma_1\smallsetminus\gamma'} \mathbb{E}'_{\beta,N}\left[\rho((d\sigma)_{p_e})\right] \\
&\overset{(6.30)}{=} \prod_{e\in\gamma_1\smallsetminus\gamma'} \mathbb{E}'_{\beta,N}\left[\rho(\sigma^e + \sigma_e)\right] \\
&= \prod_{e\in\gamma_1\smallsetminus\gamma'}\left(\sum_{g\in G}\rho(\sigma^e+g)\frac{\phi_\beta(\sigma^e+g)^{12}}{\sum_{g\in G}\phi_\beta(\sigma^e+g)^{12}}\right) \\
&= \prod_{e\in\gamma_1\smallsetminus\gamma'}\left(\frac{\sum_{g\in G}\rho(g)\phi_\beta(g)^{12}}{\sum_{g\in G}\phi_\beta(g)^{12}}\right) \\
&= \left(\frac{\sum_{g\in G}\rho(g)\phi_\beta(g)^{12}}{\sum_{g\in G}\phi_\beta(g)^{12}}\right)^{|\gamma_1\smallsetminus\gamma'|}
\end{aligned}$$

$$= \theta(\beta)^{|\gamma_1 \setminus \gamma'|}.$$

By the law of total expectation, we have

$$\begin{aligned}\mathbb{E}_{\beta,N}[W_\gamma^6] &= \mathbb{E}_{\beta,N}[\mathbb{E}'_{\beta,N}[W_\gamma^6]] = \mathbb{E}_{\beta,N}[\theta(\beta)^{|\gamma_1 \setminus \gamma'|}] = \mathbb{E}_{\beta,N}[\theta(\beta)^{|\gamma_1|}\theta(\beta)^{-|\gamma'|}] \\ &= \theta(\beta)^{|\gamma_1|}\mathbb{E}_{\beta,N}[\theta(\beta)^{-|\gamma'|}]\end{aligned} \tag{6.31}$$

Note that since $\theta \leq 1$, it follows that

$$\theta(\beta)^{|\gamma_1|} \leq 1. \tag{6.32}$$

Since $|\gamma_1|$ is the number of non-corner edges in γ and ℓ_c is the number of corner edges in γ, we have

$$|\gamma_1| + \ell_c = \ell. \tag{6.33}$$

Let j be such that $0 \leq j \leq \ell$. Then,

$$\begin{aligned}|\mathbb{E}_{\beta,N}[W_\gamma^6] - \theta^\ell| &\overset{(6.31)}{=} |\theta^{|\gamma_1|}\mathbb{E}_{\beta,N}[\theta^{-|\gamma'|}] - \theta^\ell| \\ &\overset{(6.33)}{=} |\theta^{|\gamma_1|}\mathbb{E}_{\beta,N}[\theta^{-|\gamma'|}] - \theta^{|\gamma_1|} + \theta^{|\gamma_1|} - \theta^{|\gamma_1|+\ell_c}| \\ &= |\theta^{|\gamma_1|}(\mathbb{E}_{\beta,N}[\theta^{-|\gamma'|}] - 1) - \theta^{|\gamma_1|}(\theta^{\ell_c} - 1)| \\ &\leq |\theta^{|\gamma_1|}(\mathbb{E}_{\beta,N}[\theta^{-|\gamma'|}] - 1)| + |\theta^{|\gamma_1|}(\theta^{\ell_c} - 1)| \\ &\overset{(6.32)}{\leq} |\mathbb{E}_{\beta,N}[\theta^{-|\gamma'|}] - 1| + |\theta^{\ell_c} - 1| \\ &\overset{(6.12)}{\leq} |\mathbb{E}_{\beta,N}[\theta^{-|\gamma'|}] - 1| + 2C^*\ell_c\lambda(\beta)^{12}e^{C^*\ell\lambda(\beta)^{12}} \\ &= |\mathbb{E}_{\beta,N}[\theta^{-|\gamma'|}] - \mathbb{E}_{\beta,N}[\theta^{-|\gamma'|}\mathbb{I}_{\{|\gamma'|\leq j\}}] + \mathbb{E}_{\beta,N}[\theta^{-|\gamma'|}\mathbb{I}_{\{|\gamma'|\leq j\}}] - 1| \\ &\quad + 2C^*\ell_c\lambda(\beta)^{12}e^{C^*\ell\lambda(\beta)^{12}} \\ &\leq |\mathbb{E}_{\beta,N}[\theta^{-|\gamma'|}] - \mathbb{E}_{\beta,N}[\theta^{-|\gamma'|}\mathbb{I}_{\{|\gamma'|\leq j\}}]| + |\mathbb{E}_{\beta,N}[\theta^{-|\gamma'|}\mathbb{I}_{\{|\gamma'|\leq j\}}] - 1| \\ &\quad + 2C^*\ell_c\lambda(\beta)^{12}e^{C^*\ell\lambda(\beta)^{12}}\end{aligned} \tag{6.34}$$

We focus on the first term on the right-hand side above:

$$\begin{aligned}|\mathbb{E}_{\beta,N}[\theta^{-|\gamma'|}] - \mathbb{E}_{\beta,N}[\theta^{-|\gamma'|}\mathbb{I}_{\{|\gamma'|\leq j\}}]| &= \mathbb{E}_{\beta,N}[\theta^{-|\gamma'|}\mathbb{I}_{\{|\gamma'|>j\}}] \\ &\leq \theta^{-\ell}\mu_{\beta,N}(\{\sigma \in \Sigma_{E_N} : |\gamma'| > j\}) \\ &\leq \frac{\theta^{-\ell}\mathbb{E}_{\beta,N}[|\gamma'|]}{j} \\ &\overset{(6.23)}{\leq} \frac{\theta^{-\ell}C_0^{(6)}\ell\lambda(\beta)^{12}}{j} \\ &\overset{(6.11)}{\leq} \frac{2C_0^{(6)}\ell\lambda(\beta)^{12}e^{C^*\ell\lambda(\beta)^{12}}}{j}.\end{aligned} \tag{6.35}$$

Then we focus on the second term:

$$\begin{aligned} |\mathbb{E}_{\beta,N}[\theta^{-|\gamma'|}\mathbb{I}_{\{|\gamma'|\le j\}}]-1| &\le |\mathbb{E}_{\beta,N}[(\theta^{-|\gamma'|}-1)\mathbb{I}_{\{|\gamma'|\le j\}}]|+\mathbb{E}_{\beta,N}[\mathbb{I}_{\{|\gamma'|\le j\}}] \\ &\le (\theta^{-j}-1)+\frac{\mathbb{E}_{\beta,N}[|\gamma'|]}{j} \\ &\overset{(6.12),(6.23)}{\le} 2C^* j\lambda(\beta)^{12}e^{C^*\ell\lambda(\beta)^{12}}+\frac{C_0^{(6)}\ell\lambda(\beta)^{12}}{j}. \end{aligned} \tag{6.36}$$

We combine these two inequalities and choose $j=\sqrt{\ell}$:

$$\begin{aligned} &|\mathbb{E}_{\beta,N}[\theta^{-|\gamma'|}]-\mathbb{E}_{\beta,N}[\theta^{-|\gamma'|}\mathbb{I}_{\{|\gamma'|\le j\}}]|+|\mathbb{E}_{\beta,N}[\theta^{-|\gamma'|}\mathbb{I}_{\{|\gamma'|\le j\}}]-1| \\ &\overset{(6.35),(6.36)}{\le} \frac{2C_0^{(6)}\ell\lambda(\beta)^{12}e^{C^*\ell\lambda(\beta)^{12}}}{j}+2C^*j\lambda(\beta)^{12}e^{C^*\ell\lambda(\beta)^{12}}+\frac{C_0^{(6)}\ell\lambda(\beta)^{12}}{j} \\ &\overset{j=\sqrt{\ell}}{=} 2C_0^{(6)}\sqrt{\ell}\lambda(\beta)^{12}e^{C^*\ell\lambda(\beta)^{12}}+2C^*\sqrt{\ell}\lambda(\beta)^{12}e^{C^*\ell\lambda(\beta)^{12}}+C_0^{(6)}\sqrt{\ell}\lambda(\beta)^{12} \\ &= 2(C_0^{(6)}+C^*)\sqrt{\ell}\lambda(\beta)^{12}e^{C^*\ell\lambda(\beta)^{12}}+C_0^{(6)}\sqrt{\ell}\lambda(\beta)^{12}. \end{aligned} \tag{6.37}$$

We return to inequality (6.34) and obtain

$$\begin{aligned} |\mathbb{E}_{\beta,N}[W_\gamma^6]-\theta^\ell| &\overset{(6.37)}{\le} 2(C_0^{(6)}+C^*)\sqrt{\ell}\lambda(\beta)^{12}e^{C^*\ell\lambda(\beta)^{12}}+C_0^{(6)}\sqrt{\ell}\lambda(\beta)^{12} \\ &\quad +2\ell_c C^*\lambda(\beta)^{12}e^{c^*\ell\lambda(\beta)^{12}} \\ &= 2\left(\frac{C_0^{(6)}+C^*}{\sqrt{\ell}}+\frac{C^*\ell_c}{\ell}\right)\ell\lambda(\beta)^{12}e^{C^*\ell\lambda(\beta)^{12}}+C_0^{(6)}\sqrt{\ell}\lambda(\beta)^{12}. \end{aligned} \tag{6.38}$$

Combining the expectation values

We combine equations (6.24), (6.25), (6.26), (6.27), (6.29) and (6.38):

$$\begin{aligned} |\mathbb{E}_{\beta,N}[W_\gamma]-\theta^\ell| &\le |\mathbb{E}_{\beta,N}[W_\gamma^6]-\theta^\ell|+\mathbb{E}_{\beta,N}[|W_\gamma^6-W_\gamma^5|]+\mathbb{E}_{\beta,N}[|W_\gamma^5-W_\gamma^4|] \\ &\quad +\mathbb{E}_{\beta,N}[|W_\gamma^4-W_\gamma^3|]+\mathbb{E}_{\beta,N}[|W_\gamma^3-W_\gamma^0|]+\mathbb{E}_{\beta,N}[|W_\gamma^0-W_\gamma|] \\ &\le 2\left(\frac{C_0^{(6)}+C^*}{\sqrt{\ell}}+\frac{C^*\ell_c}{\ell}\right)\ell\lambda(\beta)^{12}e^{C^*\ell\lambda(\beta)^{12}}+C_0^{(6)}\sqrt{\ell}\lambda(\beta)^{12} \\ &\quad +2C_1C_0^{(7)}\ell\lambda(\beta)^{14}+2C_0^{(6)}\ell_c\lambda(\beta)^{12}+0+2C_1C_0^{(7)}\ell\lambda(\beta)^{14} \\ &\quad +2C_0^{(25)}\ell^4\lambda(\beta)^{50} \\ &= 2\left(\frac{C_0^{(6)}+C^*}{C^*\sqrt{\ell}}+\frac{\ell_c}{\ell}\right)C^*\ell\lambda(\beta)^{12}e^{C^*\ell\lambda(\beta)^{12}} \\ &\quad +\left(\frac{C_0^{(6)}}{2C^*\sqrt{\ell}}+\frac{2C_1C_0^{(7)}\lambda(\beta)^2}{C^*}+\frac{C_0^{(6)}\ell_c}{C^*\ell}\right)2C^*\ell\lambda(\beta)^{12} \\ &\quad +\frac{C_0^{(25)}\lambda(\beta)^2}{2^3(C^*)^4}(2C^*\ell\lambda(\beta)^{12})^4. \end{aligned}$$

Note that for $x > 0$ the two following inequalities hold:

$$x \leq e^x \text{ and } x^4 \leq 5e^x.$$

Hence,

$$C^* \ell \lambda(\beta)^{12} e^{C^* \ell \lambda(\beta)^{12}} \leq e^{2C^* \ell \lambda(\beta)^{12}},$$
$$2C^* \ell \lambda(\beta)^{12} \leq e^{2C^* \ell \lambda(\beta)^{12}}$$

and

$$(2C^* \ell \lambda(\beta)^{12})^4 \leq 5e^{2C^* \ell \lambda(\beta)^{12}}.$$

Furthermore, since $1 \leq \ell_c \leq \ell$, we have

$$\frac{1}{\sqrt{\ell}} \leq \sqrt{\frac{\ell_c}{\ell}} \text{ and } \frac{\ell_c}{\ell} \leq \sqrt{\frac{\ell_c}{\ell}}. \tag{6.39}$$

Therefore,

$$\begin{aligned}
&|\mathbb{E}_{\beta,N}[W_\gamma] - \theta^\ell| \\
&\leq \left(2\frac{C_0^{(6)} + C^*}{C^* \sqrt{\ell}} + \frac{2\ell_c}{\ell} + \frac{C_0^{(6)}}{2C^* \sqrt{\ell}} + \frac{2C_1 C_0^{(7)} \lambda(\beta)^2}{C^*} + \frac{C_0^{(6)} \ell_c}{C^* \ell} + \frac{5C_0^{(25)} \lambda(\beta)^2}{2^3 (C^*)^4} \right) \\
&\quad \cdot e^{2C^* \ell \lambda(\beta)^{12}} \\
&= \left(\frac{5C_0^{(6)} + 4C^*}{2C^* \sqrt{\ell}} + \frac{2\ell_c}{\ell} + \frac{2C_1 C_0^{(7)} \lambda(\beta)^2}{C^*} + \frac{C_0^{(6)} \ell_c}{C^* \ell} + \frac{5C_0^{(25)} \lambda(\beta)^2}{2^3 (C^*)^4} \right) \cdot e^{2C^* \ell \lambda(\beta)^{12}} \\
&\overset{(6.39)}{\leq} \left(\frac{5C_0^{(6)} + 4C^*}{2C^*} + 2 + \frac{2C_1 C_0^{(7)}}{C^*} + \frac{C_0^{(6)}}{C^*} + \frac{5C_0^{(25)}}{2^3 (C^*)^4} \right) \left(\sqrt{\frac{\ell_c}{\ell}} + \lambda(\beta)^2 \right) \\
&\quad \cdot e^{2C^* \ell \lambda(\beta)^{12}} \\
&= \left(\frac{7C_0^{(6)}}{2C^*} + 4 + \frac{2C_1 C_0^{(7)}}{C^*} + \frac{5C_0^{(25)}}{2^3 (C^*)^4} \right) \left(\sqrt{\frac{\ell_c}{\ell}} + \lambda(\beta)^2 \right) e^{2C^* \ell \lambda(\beta)^{12}} \\
&= (C_A - 1/2) \left(\sqrt{\frac{\ell_c}{\ell}} + \lambda(\beta)^2 \right) e^{2C^* \ell \lambda(\beta)^{12}},
\end{aligned} \tag{6.40}$$

where

$$C_A = \frac{7C_0^{(6)}}{2C^*} + \frac{2C_1 C_0^{(7)}}{C^*} + \frac{5C_0^{(25)}}{2^3 (C^*)^4} + \frac{9}{2}. \tag{6.41}$$

Finally, we can calculate the last inequality:

$$\begin{aligned}
|\mathbb{E}_{\beta,N}[W_\gamma] - e^{-\ell(1-\theta(\beta))}| &= |\mathbb{E}_{\beta,N}[W_\gamma] - \theta^\ell + \theta^\ell - e^{-\ell(1-\theta(\beta))}| \\
&\leq |\mathbb{E}_{\beta,N}[W_\gamma] - \theta^\ell| + |\theta^\ell - e^{-\ell(1-\theta(\beta))}|
\end{aligned}$$

$$\overset{(6.13),(6.40)}{\leq} (C_A - 1/2)\left(\sqrt{\frac{\ell_c}{\ell}} + \lambda(\beta)^2\right) e^{2C^*\ell\lambda(\beta)^{12}} + \frac{e^{2C^*\ell\lambda(\beta)^{12}}}{2\ell}$$

$$= \left(C_A - 1/2 + \frac{1}{2}\cdot\frac{1}{\ell\left(\sqrt{\frac{\ell_c}{\ell}} + \lambda(\beta)^2\right)}\right)\left(\sqrt{\frac{\ell_c}{\ell}} + \lambda(\beta)^2\right)$$

$$\cdot\, e^{2C^*\ell\lambda(\beta)^{12}}$$

$$\leq \left(C_A - 1/2 + \frac{1}{2}\right)\left(\sqrt{\frac{\ell_c}{\ell}} + \lambda(\beta)^2\right) e^{2C^*\ell\lambda(\beta)^{12}}$$

$$= C_A\left(\sqrt{\frac{\ell_c}{\ell}} + \lambda(\beta)^2\right) e^{2C^*\ell\lambda(\beta)^{12}}.$$

□

6.2 The case when $\ell\lambda(\beta)^{12}$ is large

We now consider the case when $\ell\lambda(\beta)^{12}$ is large. First, some notations and definitions of importance are stated.

Let the set K be a finite and non-empty index set. The arguments of the maxima are the elements for which the value of the function is maximised and is denoted by arg max. Likewise, the arguments of the minima are the elements for which the value of the function is minimised and denoted by arg min. Given $g_k \in G$, define for each K, the set

$$G_0[(g_k)_{k\in K}] := \arg\max_{g\in G} \prod_{k\in K} \phi_\beta(g + g_k).$$

Next, an assumption is given and we prove that it is satisfied if β_0 is chosen large enough.

($\star$) For all $\beta \geq \beta_0$ and choices of $g_k \in G$ for $k \in K$ and any $g' \in G_0[(g_k)_{k\in K}]$, the following inequality holds:

$$\sum_{g\in G\smallsetminus G_0[(g_k)_{k\in K}]} \prod_{k\in K} \frac{\phi_\beta(g+g_k)^2}{\phi_\beta(g'+g_k)^2} \leq \frac{1-\cos(2\pi/n)}{8}.$$

Let $\beta \geq 0$ and $g \in G$. Since

$$\prod_{k\in K} \phi_\beta(g + g_k) = \prod_{k\in K} e^{\beta\Re(\rho(g+g_k))} = e^{\beta\sum_{k\in K}\Re(\rho(g+g_k))},$$

we obtain

$$G_0[(g_k)_{k\in K}] = \arg\max_{g\in G} e^{\beta \sum_{k\in K} \Re(\rho(g+g_k))} = \arg\max_{g\in G} \sum_{k\in K} \Re(\rho(g+g_k)).$$

We see that $G_0[(g_k)_{k\in K}]$ does not depend on β. Let $g' \in G_0[(g_k)_{k\in K}]$. Then g' is one of the elements in G for which $\sum_{k\in K} \Re(\rho(g+g_k))$ takes its maximum value. For $g \in G \setminus G_0[(g_k)_{k\in K}]$, we have

$$\sum_{k\in K} \Re(\rho(g'+g_k)) > \sum_{k\in K} \Re(\rho(g+g_k)).$$

Thus, by taking the limit when $\beta \to \infty$ we obtain

$$\lim_{\beta\to\infty} \prod_{k\in K} \frac{\phi_\beta(g+g_k)^2}{\phi_\beta(g'+g_k)^2} = \lim_{\beta\to\infty} \exp\left(2\beta \sum_{k\in K} (\Re\rho(g+g_k) - \Re\rho(g'+g_k))\right) = 0.$$

Thus, there exists a β_0 for which ($\star$) holds. This implies that there exists a β_0 for which ($\star$) holds for all K simultaneously, since β_0 can be chosen as the maximum of β_0 for all fixed K.

Proposition 6.11. *Consider lattice gauge theory with structure group $G = \mathbb{Z}_n$ and a one-dimensional faithful representation ρ of G. Let γ be a simple oriented loop in $\mathbb{Z}^4$, $\ell = |\gamma|$ the length of it and ℓ_c the number of corner edges in γ. Let N be large enough so that the edges of γ are internal edges of B_N. Let $\beta_0 > 0$ satisfy the assumption ($\star$) when applied with sets K with $|K| = 6$, and be such that $2\lambda(\beta_0)^{2|K|} \leq 1$. Then for all $\beta > \beta_0$, we have*

$$|\mathbb{E}_{\beta,N}[W_\gamma]| \leq e^{-C_*(\ell-\ell_c)\lambda(\beta)^{12}}, \quad \text{where } C_* := \frac{1-\cos(2\pi/n)}{4}. \tag{6.42}$$

To prove this proposition, three lemmas will be applied. We begin with stating the main lemma of this section, which will require two lemmas to be proved.

Lemma 6.12. *Let K be a finite and non-empty index set and assume that $\beta_0 = \beta_0(|K|) > 0$ satisfies ($\star$) applied with K and*

$$2\lambda(\beta_0)^{2|K|} \leq 1. \tag{6.43}$$

Then, for all choices of $g_k \in G$ *for* $k \in K$ *and in the setting of Lemma 6.11, we have*

$$\left| \frac{\sum_{g\in G} \rho(g) \prod_{k\in K} \phi_\beta(g+g_k)^2}{\sum_{g\in G} \prod_{k\in K} \phi_\beta(g+g_k)^2} \right| \leq 1 - C_* \lambda(\beta)^{2|K|}. \tag{6.44}$$

The left-hand side of inequality (6.44) is the conditional expected value of a single spin (see the beginning of the proof of this proposition). To lighten notations, we define

$$S_\beta((g_k)_{k\in K}) := \frac{\sum_{g\in G} \rho(g) \prod_{k\in K} \phi_\beta(g+g_k)^2}{\sum_{g\in G} \prod_{k\in K} \phi_\beta(g+g_k)^2} \tag{6.45}$$

and

$$\omega := 1 - S_\beta((g_k)_{k\in K}). \tag{6.46}$$

We state and prove two lemmas for the case when the set $G_0[(g_k)_{k\in K}]$ only consists of one element (We will later prove that $G_0[(g_k)_{k\in K}] = \{0\}$ in this case.). Thereafter, the proof of Lemma 6.12 is given.

Lemma 6.13. *Let* K *be a finite and non-empty index set and assume that* $\beta_0 = \beta_0(|K|) > 0$ *satisfies* $(\star)$ *applied with* K*. If* $G_0[(g_k)_{k\in K}] = \{0\}$*, then*

$$(i)\ |\omega|^2 \leq \Re\omega, \tag{6.47}$$

$$(ii)\ |S_\beta((g_k)_{k\in K})| \leq 1 - C_* \sum_{g\in G\setminus\{0\}} \frac{\prod_{k\in K} \phi_\beta(g+g_k)^2}{\prod_{k\in K} \phi_\beta(0+g_k)^2}. \tag{6.48}$$

Proof. (i) From equations (6.45) and (6.46) follows that

$$\begin{aligned} \omega &= 1 - \frac{\sum_{g\in G} \rho(g) \prod_{k\in K} \phi_\beta(g+g_k)^2}{\sum_{g\in G} \prod_{k\in K} \phi_\beta(g+g_k)^2} \\ &= \frac{\sum_{g\in G} (1-\rho(g)) \prod_{k\in K} \phi_\beta(g+g_k)^2}{\sum_{g\in G} \prod_{k\in K} \phi_\beta(g+g_k)^2} \\ &= \frac{\sum_{g\in G} (1-\rho(g)) \prod_{k\in K} \phi_\beta(g+g_k)^2}{\prod_{k\in K} \phi_\beta(0+g_k)^2} \cdot \frac{\prod_{k\in K} \phi_\beta(0+g_k)^2}{\sum_{g\in G} \prod_{k\in K} \phi_\beta(g+g_k)^2} \\ &= \left(\sum_{g\in G} (1-\rho(g)) \prod_{k\in K} \frac{\phi_\beta(g+g_k)^2}{\phi_\beta(0+g_k)^2} \right) \cdot \frac{\prod_{k\in K} \phi_\beta(0+g_k)^2}{\sum_{g\in G} \prod_{k\in K} \phi_\beta(g+g_k)^2}. \end{aligned} \tag{6.49}$$

Since $G_0[(g_k)_{k\in K}] = \{0\}$, by $(\star)$ we have

$$\sum_{g\in G\setminus\{0\}} \prod_{k\in K} \frac{\phi_\beta(g+g_k)^2}{\phi_\beta(0+g_k)^2} \leq \frac{1-\cos(2\pi/n)}{8}. \tag{6.50}$$

We focus on the second factor in (6.49):

$$
\begin{aligned}
1 &\geq \frac{\prod_{k\in K}\phi_\beta(0+g_k)^2}{\sum_{g\in G}\prod_{k\in K}\phi_\beta(g+g_k)^2} \\
&= 1 - \frac{\sum_{g\in G\setminus\{0\}}\prod_{k\in K}\phi_\beta(g+g_k)^2}{\sum_{g\in G}\prod_{k\in K}\phi_\beta(g+g_k)^2} \\
&> 1 - \frac{\sum_{g\in G\setminus\{0\}}\prod_{k\in K}\phi_\beta(g+g_k)^2}{\prod_{k\in K}\phi_\beta(0+g_k)^2} \\
&\overset{(6.50)}{\geq} 1 - \frac{1-\cos(2\pi/n)}{8} \\
&= \frac{7+\cos(2\pi/n)}{8} \\
&\geq \frac{6}{8} > \frac{1}{2}.
\end{aligned}
\tag{6.51}
$$

We calculate $\Re\omega$. Notice first that

$$
\min_{g\in G\setminus\{0\}}(1-\Re\rho(g)) = 1-\cos(2\pi/n) > 0. \tag{6.52}
$$

Thus,

$$
\begin{aligned}
\Re\omega &= \left(\sum_{g\in G}(1-\Re(\rho(g)))\frac{\prod_{k\in K}\phi_\beta(g+g_k)^2}{\prod_{k\in K}\phi_\beta(0+g_k)^2}\right)\cdot\frac{\prod_{k\in K}\phi_\beta(0+g_k)^2}{\sum_{g\in G}\prod_{k\in K}\phi_\beta(g+g_k)^2} \\
&= \left(\sum_{g\in G\setminus\{0\}}(1-\Re(\rho(g)))\frac{\prod_{k\in K}\phi_\beta(g+g_k)^2}{\prod_{k\in K}\phi_\beta(0+g_k)^2}\right)\cdot\frac{\prod_{k\in K}\phi_\beta(0+g_k)^2}{\sum_{g\in G}\prod_{k\in K}\phi_\beta(g+g_k)^2} \\
&\overset{(6.51),(6.52)}{>} (1-\cos(2\pi/n))\sum_{g\in G\setminus\{0\}}\frac{\prod_{k\in K}\phi_\beta(g+g_k)^2}{\prod_{k\in K}\phi_\beta(0+g_k)^2}\cdot\frac{1}{2} \\
&\overset{(6.50)}{\geq} \frac{1}{2}(1-\cos(2\pi/n))\frac{1-\cos(2\pi/n)}{8} = \frac{(1-\cos(2\pi/n))^2}{4^2}.
\end{aligned}
\tag{6.53}
$$

Since

$$
\max_{g\in G}|1-\rho(g)| \leq 1+1 = 2, \tag{6.54}
$$

we have

$$
\begin{aligned}
|\omega| &= \left| \left(\sum_{g \in G \setminus \{0\}} (1-\rho(g)) \frac{\prod_{k\in K} \phi_\beta(g+g_k)^2}{\prod_{k\in K} \phi_\beta(0+g_k)^2} \right) \cdot \frac{\prod_{k\in K} \phi_\beta(0+g_k)^2}{\sum_{g\in G} \prod_{k\in K} \phi_\beta(g+g_k)^2} \right| \\
&\leq \max_{g\in G} |1-\rho(g)| \left| \sum_{g \in G \setminus \{0\}} \prod_{k\in K} \frac{\phi_\beta(g+g_k)^2}{\phi_\beta(0+g_k)^2} \cdot \frac{\prod_{k\in K} \phi_\beta(0+g_k)^2}{\sum_{g\in G} \prod_{k\in K} \phi_\beta(g+g_k)^2} \right| \\
&\overset{(6.54)}{\leq} 2 \sum_{g \in G \setminus \{0\}} \prod_{k\in K} \frac{\phi_\beta(g+g_k)^2}{\phi_\beta(0+g_k)^2} \cdot \frac{\prod_{k\in K} \phi_\beta(0+g_k)^2}{\sum_{g\in G} \prod_{k\in K} \phi_\beta(g+g_k)^2} \\
&\overset{(6.51)}{\leq} 2 \sum_{g \in G \setminus \{0\}} \prod_{k\in K} \frac{\phi_\beta(g+g_k)^2}{\phi_\beta(0+g_k)^2} \cdot 1 \\
&\overset{(6.50)}{\leq} \frac{1-\cos(2\pi/n)}{4}.
\end{aligned}
$$

Therefore,

$$
|\omega|^2 \leq \left(\frac{1-\cos(2\pi/n)}{4} \right)^2 \overset{(6.53)}{\leq} \Re\omega.
$$

(ii) We calculate $|S_\beta((g_k)_{k\in K})| = |1-\omega|$:

$$
\begin{aligned}
|1-\omega| &= \sqrt{(1-\Re\omega)^2 + (\Im\omega)^2} = \sqrt{1-2\Re\omega+|\omega|^2} \\
&\leq \sqrt{1-2\Re\omega+|\omega|^2+\Re\omega(-|\omega|^2+\Re\omega)+\frac{|\omega|^4}{4}} \\
&= \sqrt{\left(1-\Re\omega+\frac{|\omega|^2}{2}\right)^2} = 1-\Re\omega+\frac{|\omega|^2}{2} \\
&\overset{(6.47)}{\leq} 1-\Re\omega+\frac{\Re\omega}{2} = 1-\frac{\Re\omega}{2} \\
&\overset{(6.53)}{\leq} 1-\frac{1}{2}\left(\frac{(1-\cos(2\pi/n))}{2} \sum_{g \in G \setminus \{0\}} \frac{\prod_{k\in K} \phi_\beta(g+g_k)^2}{\prod_{k\in K} \phi_\beta(0+g_k)^2} \right) \\
&= 1 - C_* \sum_{g \in G \setminus \{0\}} \frac{\prod_{k\in K} \phi_\beta(g+g_k)^2}{\prod_{k\in K} \phi_\beta(0+g_k)^2}.
\end{aligned}
$$

□

Before the following lemma, the notation $\angle(z_1, z_2)$ is introduced. Let $z_1, z_2 \in \mathbb{C} \setminus \{0\}$, then $\angle(z_1, z_2)$ is the absolute value of the smallest angle between these numbers.

Lemma 6.14. *Let K be a finite and non-empty index set and assume that $\beta_0 = \beta_0(|K|) > 0$ satisfies $(\star)$ applied with K. For each $k \in K$, let $g_k \in G$. If*

$G_0[(g_k)_{k\in K}] = \{0\}$, *then*

$$\max_{g\in G\setminus\{0\}} \prod_{k\in K} \frac{\phi_\beta(g+g_k)}{\phi_\beta(0+g_k)} \geq \max_{g\in G\setminus\{0\}} \prod_{k\in K} \frac{\phi_\beta(g+0)}{\phi_\beta(0+0)}. \tag{6.55}$$

Proof. Since

$$\prod_{k\in K} \phi_\beta(g+g_k) = \exp(\beta\Re(\rho(g)\sum_{k\in K}\rho(g_k))),$$

inequality (6.55) can be written as

$$\max_{g\in G\setminus\{0\}} \frac{\exp\left(\beta\Re(\rho(g)\sum_{k\in K}\rho(g_k))\right)}{\exp\left(\beta\Re(\rho(0)\sum_{k\in K}\rho(g_k))\right)} \geq \max_{g\in G\setminus\{0\}} \frac{\exp\left(\beta\Re(\rho(g)\sum_{k\in K}\rho(0))\right)}{\exp\left(\beta\Re(\rho(0)\sum_{k\in K}\rho(0))\right)},$$

which is equivalent to

$$\begin{aligned} &\max_{g\in G\setminus\{0\}} \exp\left(\beta\left(\Re(\rho(g)\sum_{k\in K}\rho(g_k)) - \Re(\rho(0)\sum_{k\in K}\rho(g_k))\right)\right) \\ &\geq \max_{g\in G\setminus\{0\}} \exp\left(\beta\left(\Re(\rho(g)\sum_{k\in K}\rho(0)) - \Re(\rho(0)\sum_{k\in K}\rho(0))\right)\right) \end{aligned}$$

and

$$\begin{aligned} &\max_{g\in G\setminus\{0\}} \left(\Re(\rho(g)\sum_{k\in K}\rho(g_k)) - \Re(\rho(0)\sum_{k\in K}\rho(g_k))\right) \\ &\geq \max_{g\in G\setminus\{0\}} \left(\Re(\rho(g)\sum_{k\in K}\rho(0)) - \Re(\rho(0)\sum_{k\in K}\rho(0))\right). \end{aligned} \tag{6.56}$$

Since $\rho(g)$ and $\sum_{k\in K}\rho(-g_k)$ are non-zero complex numbers, we have

$$\begin{aligned} \Re\left(\rho(g)\sum_{k\in K}\rho(g_k)\right) &= \Re\left(\sum_{k\in K}\overline{\rho(-g_k)}\rho(g)\right) \\ &= \left|\sum_{k\in K}\rho(g_k)\right||\rho(g)|\cos\left(\angle\left(\sum_{k\in K}\rho(-g_k),\rho(g)\right)\right) \\ &= \left|\sum_{k\in K}\rho(g_k)\right|\cos\left(\angle\left(\sum_{k\in K}\rho(-g_k),\rho(g)\right)\right). \end{aligned} \tag{6.57}$$

Fix $\hat{g}\in G\setminus\{0\}$. Then $\hat{g}\notin G_0[(g_k)_{k\in K}]$ and

$$\prod_{k\in K}\phi_\beta(\hat{g}+g_k) < \prod_{k\in K}\phi_\beta(0+g_k),$$

which is equivalent to

$$\Re\left(\rho(g)\sum_{k\in K}\rho(g_k)\right) < \Re\left(\rho(0)\sum_{k\in K}\rho(g_k)\right) = \Re\left(1\cdot\sum_{k\in K}\rho(g_k)\right). \tag{6.58}$$

From equation (6.57) follows that inequality (6.58) is equivalent to

$$\begin{aligned}&\left|\sum_{k\in K}\rho(g_k)\right|\cos\left(\angle\left(\sum_{k\in K}\rho(-g_k),\rho(g)\right)\right)\\ &<\left|\sum_{k\in K}\rho(g_k)\right|\cos\left(\angle\left(\sum_{k\in K}\rho(-g_k),1\right)\right).\end{aligned}$$

Thus,

$$\cos\left(\angle\left(\sum_{k\in K}\rho(-g_k),\rho(g)\right)\right) - \cos\left(\angle\left(\sum_{k\in K}\rho(-g_k),1\right)\right) < 0. \tag{6.59}$$

We write $G_0[(g_k)_{k\in K}]$ with arg max and arg min. Note that $\cos(x)$ is decreasing in $[0,\pi]$ and therefore takes it largest value when the angle is as small as possible. Therefore,

$$\begin{aligned}\{0\} &= G_0[(g_k)_{k\in K}]\\ &= \arg\max_{g\in G}\exp\left(\beta\Re\left(\rho(g)\sum_{k\in K}(g+g_k)\right)\right)\\ &\overset{(6.57)}{=} \arg\max_{g\in G}\exp\left(\beta\left|\sum_{k\in K}\rho(g_k)\right|\cos\left(\angle\left(\sum_{k\in K}\rho(-g_k),\rho(g)\right)\right)\right)\\ &= \arg\max_{g\in G}\cos\left(\angle\left(\sum_{k\in K}\rho(-g_k),\rho(g)\right)\right)\\ &= \arg\min_{g\in G}\angle\left(\sum_{k\in K}\rho(-g_k),\rho(g)\right).\end{aligned}$$

Thus, $\rho(0)$ is the point in $\rho(G)$ that is closest to $\sum_{k\in K}\rho(-g_k)$. Similarly, we have for $\hat{g}$ that

$$\hat{g} \in \arg\max_{g\in G\setminus G_0[(g_k)_{k\in K}]}\prod_{k\in K}\phi_\beta(g+g_k) = \arg\min_{g\in G\setminus\{0\}}\angle\left(\sum_{k\in K}\rho(-g_k),\rho(g)\right).$$

Hence, $\rho(\hat{g})$ is the point second closest to $\sum_{k\in K}\rho(-g_k)$. These results for 0 and $\hat{g}$ are illustrated in Figure 6.1. Note that there are two possibilities for where $\sum_{k\in K}\rho(-g_k)$ is.

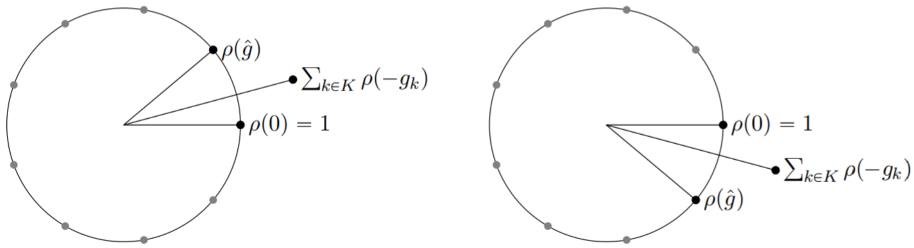

Figure 6.1: The points in $\rho(G)$ and the two possibilities for $\sum_{k\in K}\rho(-g_k)$. The figure is from [1].

We see that the angle $\angle(1,\rho(\hat{g}))$ can be written as a sum of two angles:

$$\angle\left(\sum_{k\in K}\rho(-g_k),1\right)+\angle\left(\sum_{k\in K}\rho(-g_k),\rho(\hat{g})\right)=\angle(1,\rho(\hat{g})).$$

Since $\cos(x)$ is decreasing on $[0,\pi]$, we have

$$\cos\left(\angle\left(\sum_{k\in K}\rho(-g_k),\rho(\hat{g})\right)\right)\geq\cos(\angle(1,\rho(\hat{g})).$$

Furthermore,

$$1=\cos(\angle(1,1))\geq\cos\left(\angle\left(\sum_{k\in K}\rho(-g_k),1\right)\right).$$

Therefore, we have

$$\begin{aligned}&\cos(\angle(1,1))+\cos\left(\angle\left(\sum_{k\in K}\rho(-g_k),\rho(\hat{g})\right)\right)\\&\geq\cos\left(\angle\left(\sum_{k\in K}\rho(-g_k),1\right)\right)+\cos(\angle(1,\rho(\hat{g}))\end{aligned}$$

and

$$\begin{aligned}&\cos\left(\angle\left(\sum_{k\in K}\rho(-g_k),\rho(\hat{g})\right)\right)-\cos\left(\angle\left(\sum_{k\in K}\rho(-g_k),1\right)\right)\\&\geq\cos\left(\angle(1,\rho(\hat{g}))\right)-\cos\left(\angle(1,1)\right).\end{aligned}\tag{6.60}$$

Since

$$\left|\sum_{k\in K}\rho(g_k)\right|\leq\sum_{k\in K}|\rho(g_k)|\leq\sum_{k\in K}1=|K|,\tag{6.61}$$

we have

$$\begin{aligned}
&\Re(\rho(\hat{g})\sum_{k\in K}\rho(g_k)) - \Re(\rho(0))\sum_{k\in K}\rho(g_k))\\
&\overset{(6.57)}{=}\left|\sum_{k\in K}\rho(g_k)\right|\cos\left(\angle(\sum_{k\in K}\rho(-g_k),\rho(\hat{g}))\right)\\
&\quad-\left|\sum_{k\in K}\rho(g_k)\right|\cos\left(\angle(\sum_{k\in K}\rho(-g_k),\rho(1))\right)\\
&\overset{(6.59),(6.61)}{\geq}|K|\left(\cos\left(\angle(\sum_{k\in K}\rho(-g_k),\rho(\hat{g}))\right)-\cos\left(\angle(\sum_{k\in K}\rho(-g_k),\rho(1))\right)\right)\\
&\overset{(6.60)}{\geq}|K|\cos\left(\angle(1,\rho(\hat{g}))-\cos(\angle(1,1))\right)\\
&=\left|\sum_{k\in K}\rho(0)\right|\left(\cos\left(\angle(\sum_{k\in K}\rho(0),\rho(\hat{g}))\right)-\cos\left(\angle(\sum_{k\in K}\rho(0),\rho(0))\right)\right)\\
&\overset{(6.57)}{=}\Re(\rho(\hat{g})\sum_{k\in K}\rho(0))-\Re(\rho(0))\sum_{k\in K}\rho(0)).
\end{aligned}$$

Since $\hat{g} \in G \setminus \{0\}$, it follows that

$$\begin{aligned}
&\max_{g\in G\setminus\{0\}}\left(\Re(\rho(g)\sum_{k\in K}\rho(g_k))-\Re(\rho(0))\sum_{k\in K}\rho(g_k)\right)\\
&\geq\max_{g\in G\setminus\{0\}}\left(\Re(\rho(g)\sum_{k\in K}\rho(0))-\Re(\rho(0))\sum_{k\in K}\rho(0)\right).
\end{aligned}$$

This is the same as inequality (6.56), which was shown to be equivalent to inequality (6.55). □

We continue to the proof of Lemma 6.12.

Proof of Lemma 6.12. We investigate how many elements $G_0[(g_k)_{k\in K}]$ can consist of:

$$\begin{aligned}
\prod_{k\in K}\phi_\beta(g+g_k)^2 &= \exp\left(\beta\sum_{k\in K}\Re(\rho(g+g_k))\right)=\exp\left(\beta\sum_{k\in K}\Re(\rho(g)\rho(g_k))\right)\\
&=\exp\left(\beta\Re(\rho(g)\sum_{k\in K}\rho(g_k))\right).
\end{aligned}$$

Recall that $G_0[(g_k)_{k\in K}]$ is the set of the elements $g \in G$ for which the above function is maximised.

First, if $\sum_{k\in K}\rho(g_k)=0$, then $\prod_{k\in K}\phi_\beta(g+g_k)^2=1$ for all $g\in G$. Thus, $G_0[(g_k)_{k\in K}]$ consists of G elements.

Second, we consider the case when $\sum_{k\in K}\rho(g_k)\in\mathbb{R}\setminus\{0\}$. If $\sum_{k\in K}\rho(g_k)>0$, then the function is maximised when $\Re\rho(g)$ is maximised, i.e. $g=0$. Otherwise, it is maximised when $\Re\rho(g)$ is minimised. This happens when $\rho(g)=e^{g\cdot 2\pi im/n}$ is as close to $e^{\pi i}$ as possible. Thus, there is 1 or 2 elements in $G_0[(g_k)_{k\in K}]$.

Third, if $\sum_{k\in K}\rho(g_k)\notin\mathbb{R}$, then $\sum_{k\in K}\rho(g_k)=re^{i\theta}$. Hence,

$$\Re(\rho(g)\sum_{k\in K}\rho(g_k))=r\Re(\rho(g)e^{i\theta}).$$

This is maximised when $\rho(g)=e^{g\cdot 2\pi im/n}$ is as close as possible to $e^{-i\theta}$. Thus, we have that there are 1 or 2 elements in $G_0[(g_k)_{k\in K}]$.

To conclude, there are three possibilities for the numbers of elements in $G_0[(g_k)_{k\in K}]$: 1, 2 or G elements. We continue by proving that inequality (6.44) holds in all three cases.

First case: G elements

If $|G_0[(g_k)_{k\in K}]|=G$, then $\prod_{k\in K}\phi_\beta(g+g_k)^2=1$ for all $g\in G$. Hence

$$\begin{aligned}\left|\frac{\sum_{g\in G}\rho(g)\prod_{k\in K}\phi_\beta(g+g_k)^2}{\sum_{g\in G}\prod_{k\in K}\phi_\beta(g+g_k)^2}\right| &= \frac{|\sum_{g\in G}\rho(g)|}{|G|}=\frac{|\sum_{k=0}^{n-1}e^{k\cdot 2\pi im/n}|}{|G|}\\ &= \frac{1}{|G|}\frac{e^{2\pi im}-1}{e^{2\pi im/n}-1}=0\le 1-C_*\lambda(\beta)^{2|K|}.\end{aligned}$$

Second case: two elements

If $|G_0[(g_k)_{k\in K}]|=2$, then

$$\begin{aligned}|S_\beta((g_k)_{k\in K})| &= \left|\frac{\sum_{g\in G}\rho(g)\prod_{k\in K}\phi_\beta(g+g_k)^2}{\sum_{g\in G}\prod_{k\in K}\phi_\beta(g+g_k)^2}\right|\\ &\le \left|\frac{\sum_{g\in G}\rho(g)\prod_{k\in K}\phi_\beta(g+g_k)^2}{\sum_{g\in G_0[(g_k)_{k\in K}]}\prod_{k\in K}\phi_\beta(g+g_k)^2}\right|\\ &\le \left|\frac{\sum_{g\in G_0[(g_k)_{k\in K}]}\rho(g)\prod_{k\in K}\phi_\beta(g+g_k)^2}{\sum_{g\in G_0[(g_k)_{k\in K}]}\prod_{k\in K}\phi_\beta(g+g_k)^2}\right|\\ &\quad+\left|\frac{\sum_{g\in G\setminus G_0[(g_k)_{k\in K}]}\rho(g)\prod_{k\in K}\phi_\beta(g+g_k)^2}{\sum_{g\in G_0[(g_k)_{k\in K}]}\prod_{k\in K}\phi_\beta(g+g_k)^2}\right|.\end{aligned}$$

Notice that for every $g\in G_0[(g_k)_{k\in K}]$, the function $\phi_\beta(g+g_k)$ takes the same value. Let $g' \in G_0[(g_k)_{k\in K}]$. Then,

$$
\begin{aligned}
|S_\beta((g_k)_{k\in K})| &\le \left|\frac{\sum_{g\in G_0[(g_k)_{k\in K}]}\rho(g)}{G_0[(g_k)_{k\in K}]}\right| + \left|\frac{\sum_{g\in G\setminus G_0[(g_k)_{k\in K}]}\prod_{k\in K}\phi_\beta(g+g_k)^2}{\sum_{g\in G_0[(g_k)_{k\in K}]}\prod_{k\in K}\phi_\beta(g+g_k)^2}\right| \\
&\quad\cdot|\rho(g)| \\
&\overset{(2.3)}{=} \frac{|\sum_{g\in G_0[(g_k)_{k\in K}]}\rho(g)|}{2} + \left|\frac{\sum_{g\in G\setminus G_0[(g_k)_{k\in K}]}\prod_{k\in K}\phi_\beta(g+g_k)^2}{\sum_{g\in G_0[(g_k)_{k\in K}]}\prod_{k\in K}\phi_\beta(g+g_k)^2}\right| \\
&\le \frac{|\sum_{g\in G_0[(g_k)_{k\in K}]}\rho(g)|}{2} + \left|\sum_{g\in G\setminus G_0[(g_k)_{k\in K}]}\prod_{k\in K}\frac{\phi_\beta(g+g_k)^2}{\phi_\beta(g'+g_k)^2}\right| \\
&\overset{(\star)}{\le} \frac{|\sum_{g\in G_0[(g_k)_{k\in K}]}\rho(g)|}{2} + \frac{1-\cos(2\pi/n)}{8}.
\end{aligned}
\tag{6.62}
$$

Since $G_0[(g_k)_{k\in K}] = \{j, j+1\}$ for a $j\in\mathbb{Z}_n$, we have

$$
\begin{aligned}
\frac{|\sum_{g\in G_0[(g_k)_{k\in K}]}\rho(g)|}{2} &= \frac{|\rho(j)+\rho(j+1)|}{2} \\
&\le \frac{\sqrt{(1+\cos(2\pi/n))^2+\sin^2(2\pi/n)}}{2} \\
&= \frac{\sqrt{1+2\cos(2\pi/n)+\cos^2(2\pi/n)+\sin^2(2\pi/n)}}{2} \\
&= \sqrt{\frac{1+\cos(2\pi/n)}{2}} \\
&= \sqrt{1-\frac{1-\cos(2\pi/n)}{2}} \\
&\le \sqrt{1-\frac{1-\cos(2\pi/n)}{4}} \\
&\le 1-\frac{1-\cos(2\pi/n)}{4}.
\end{aligned}
$$

Combining this with inequality (6.62), we have

$$
\begin{aligned}
|S_\beta((g_k)_{k\in K})| &\le 1-\frac{1-\cos(2\pi/n)}{4} + \frac{1-\cos(2\pi/n)}{8} \\
&= 1-\frac{1-\cos(2\pi/n)}{8} = 1-\frac{1}{2}C_* \\
&\overset{(6.43)}{\le} 1-C_*\lambda(\beta_0)^{2|K|}.
\end{aligned}
$$

Third case: one element

Last, we focus on the case when $|G_0[(g_k)_{k\in K}]| = 1$. First, we investigate which element is in $G_0[(g_k)_{k\in K}]$. For any $g' \in G$, we have

$$\begin{aligned}
|S_\beta((g'+g)_{k\in K})| &= \left|\frac{\sum_{g\in G}\rho(g)\prod_{k\in K}\phi_\beta(g+(g'+g_k))^2}{\sum_{g\in G}\prod_{k\in K}\phi_\beta(g+(g'+g_k))^2}\right| \\
&= |\rho(g')|\left|\frac{\sum_{g\in G}\rho(g)\prod_{k\in K}\phi_\beta(g+(g'+g_k))^2}{\sum_{g\in G}\prod_{k\in K}\phi_\beta(g+(g'+g_k))^2}\right| \\
&= \left|\frac{\sum_{g\in G}\rho(g)\rho(g')\prod_{k\in K}\phi_\beta(g+(g'+g_k))^2}{\sum_{g\in G}\prod_{k\in K}\phi_\beta(g+(g'+g_k))^2}\right| \\
&= \left|\frac{\sum_{g\in G}\rho(g+g')\prod_{k\in K}\phi_\beta((g+g')+g_k)^2}{\sum_{g\in G}\prod_{k\in K}\phi_\beta((g+g')+g_k)^2}\right| \\
&= \left|\frac{\sum_{g\in G}\rho(g)\prod_{k\in K}\phi_\beta(g+g_k)^2}{\sum_{g\in G}\prod_{k\in K}\phi_\beta(g+g_k)^2}\right| \\
&= |S_\beta((g_k)_{k\in K})|.
\end{aligned}$$

Thus, it implies that $0 \in G_0[(g_k)_{k\in K}]$, which only consists of one element. Therefore, $G_0[(g_k)_{k\in K}] = \{0\}$. Then, from Lemma 6.13 follows that

$$|S_\beta((g_k)_{k\in K})| \le 1 - C_* \sum_{g\in G\setminus\{0\}} \frac{\prod_{k\in K}\phi_\beta(g+g_k)^2}{\prod_{k\in K}\phi_\beta(0+g_k)^2}.$$

Therefore, inequality (6.44) holds if

$$\begin{aligned}
\sum_{g\in G\setminus\{0\}} \frac{\prod_{k\in K}\phi_\beta(g+g_k)^2}{\prod_{k\in K}\phi_\beta(0+g_k)^2} \ge \lambda(\beta)^{2|K|} &= \left(\max_{g\in G\setminus\{0\}} \frac{\phi_\beta(g)}{\phi_\beta(0)}\right)^{2|K|} \\
&= \max_{g\in G\setminus\{0\}} \prod_{k\in K} \frac{\phi_\beta(g+0)^2}{\phi_\beta(0+0)^2},
\end{aligned}$$

which holds if

$$\max_{g\in G\setminus\{0\}} \prod_{k\in K} \frac{\phi_\beta(g+g_k)}{\phi_\beta(0+g_k)} \ge \max_{g\in G\setminus\{0\}} \prod_{k\in K} \frac{\phi_\beta(g+0)}{\phi_\beta(0+0)}. \tag{6.63}$$

Inequality (6.63) was proved to be true in Lemma 6.14. Thus,

$$\begin{aligned}
|S_\beta((g_k)_{k\in K})| &\le 1 - C_* \sum_{g\in G\setminus\{0\}} \frac{\prod_{k\in K}\phi_\beta(g+g_k)^2}{\prod_{k\in K}\phi_\beta(0+g_k)^2} \\
&\le 1 - C_* \sum_{g\in G\setminus\{0\}} \frac{\prod_{k\in K}\phi_\beta(g+0)^2}{\prod_{k\in K}\phi_\beta(0+0)^2} \\
&= 1 - C_*\lambda(\beta)^{2|K|}.
\end{aligned}$$

□

We continue to the proof of Proposition 6.11.

Proof of Proposition 6.11. Recall, by Lemma 6.9 follows that the spins $(\sigma_e)_{e\in\gamma_1}$ are independent if the spins $(\sigma_e)_{e\notin\pm\gamma_1}$ are given. Take an edge $e \in \gamma_1$. For $p \in \hat{\partial} e$, let $\sigma_p^e := \sum_{e'\in\partial p\setminus\{e\}} \sigma_{e'}$. For $g \in G$, the conditional probability when $\sigma_e = g$ is

$$\mu'_{\beta,N}(\sigma_e = g) = \frac{\prod_{p\in\hat{\partial}e} \phi_\beta(\sigma_p^e + g)^2}{\sum_{g'\in G}\prod_{p\in\hat{\partial}e} \phi_\beta(\sigma_p^e + g')^2}$$

and the conditional expectation of $\rho(\sigma_e)$ is

$$\mathbb{E}'_{\beta,N}[\rho(\sigma_e)] = \frac{\sum_{g\in G}\rho(g)\prod_{p\in\hat{\partial}e} \phi_\beta(\sigma_p^e + g)^2}{\sum_{g\in G}\prod_{p\in\hat{\partial}e} \phi_\beta(\sigma_p^e + g)^2}.$$

Since the spins $(\sigma_e)_{e\in\gamma_1}$ are independent, we obtain

$$\mathbb{E}'_{\beta,N}\left[\prod_{e\in\gamma_1}\rho(\sigma_e)\right] = \prod_{e\in\gamma_1}\mathbb{E}'_{\beta,N}[\rho(\sigma_e)]. \tag{6.64}$$

Before calculating the expectation of W_γ, we apply Lemma 6.12 for $K = \hat{\partial}e$ and $g_k = \sigma_p^e$. Since $|K| = 6$, we obtain

$$\begin{aligned} |\mathbb{E}'_{\beta,N}[\rho(\sigma_e)]| &= \left|\frac{\sum_{g\in G}\rho(g)\prod_{p\in\hat{\partial}e} \phi_\beta(\sigma_p^e + g)^2}{\sum_{g\in G}\prod_{p\in\hat{\partial}e} \phi_\beta(\sigma_p^e + g)^2}\right| \\ &\overset{(6.44}{\leq} 1 - C_*\lambda(\beta)^{2|K|} \\ &= 1 - C_*\lambda(\beta)^{12} \\ &\leq e^{-C_*\lambda(\beta)^{12}}. \end{aligned} \tag{6.65}$$

Thus, the upper bound for the expectation is

$$\begin{aligned} |\mathbb{E}_{\beta,N}[W_\gamma]| &= \left|\mathbb{E}_{\beta,N}\left[\rho\left(\sum_{e\in\gamma}\sigma_e\right)\right]\right| = \left|\mathbb{E}_{\beta,N}\left[\rho\left(\sum_{e\in\gamma\setminus\gamma_1}\sigma_e\right)\rho\left(\sum_{e\in\gamma_1}\sigma_e\right)\right]\right| \\ &= \left|\mathbb{E}_{\beta,N}\left[\prod_{e\in\gamma\setminus\gamma_1}\rho(\sigma_e)\prod_{e\in\gamma_1}\rho(\sigma_e)\right]\right| \\ &= \left|\mathbb{E}_{\beta,N}\left[\mathbb{E}'_{\beta,N}\left[\prod_{e\in\gamma\setminus\gamma_1}\rho(\sigma_e)\prod_{e\in\gamma_1}\rho(\sigma_e)\right]\right]\right| \\ &= \left|\mathbb{E}_{\beta,N}\left[\prod_{e\in\gamma\setminus\gamma_1}\rho(\sigma_e)\,\mathbb{E}'_{\beta,N}\left[\prod_{e\in\gamma_1}\rho(\sigma_e)\right]\right]\right| \\ &\leq \mathbb{E}_{\beta,N}\left[\left|\prod_{e\in\gamma\setminus\gamma_1}\rho(\sigma_e)\,\mathbb{E}'_{\beta,N}\left[\prod_{e\in\gamma_1}\rho(\sigma_e)\right]\right|\right] \end{aligned}$$

$$
\begin{aligned}
&= \mathbb{E}_{\beta,N}\left[\left|\prod_{e\in\gamma\smallsetminus\gamma_1}\rho(\sigma_e)\right|\cdot\left|\mathbb{E}'_{\beta,N}\left[\prod_{e\in\gamma_1}\rho(\sigma_e)\right]\right|\right]\\
&= \mathbb{E}_{\beta,N}\left[1\cdot\left|\mathbb{E}'_{\beta,N}\left[\prod_{e\in\gamma_1}\rho(\sigma_e)\right]\right|\right]\\
&\overset{(6.64)}{=} \mathbb{E}_{\beta,N}\left[\prod_{e\in\gamma_1}\left|\mathbb{E}'_{\beta,N}\left[\rho(\sigma_e)\right]\right|\right]\\
&\overset{(6.65)}{\leq} \mathbb{E}_{\beta,N}\left[\prod_{e\in\gamma_1}e^{-C_*\lambda(\beta)^{12}}\right]\\
&= \mathbb{E}_{\beta,N}\left[e^{-C_*\lambda(\beta)^{12}|\gamma_1|}\right] = e^{-C_*\lambda(\beta)^{12}|\gamma_1|}\\
&= e^{-C_*(\ell-\ell_c)\lambda(\beta)^{12}}.
\end{aligned}
$$

□

6.3 The last part of the proof

Recall that the inequality we want to prove is inequality (2.12):

$$|\langle W_\gamma\rangle_\beta - e^{-\ell(1-\theta(\beta))}| \leq C'\left[\frac{\sqrt{\ell_c}}{\ell}+\lambda(\beta)^2\right]^{C''}.$$

We begin with defining the constants C' and C''. Let

$$C' := \sqrt{2}(C_A 2^{4C^*/C_*})^{1/(1+4C^*/C_*)} \text{ and } C'' := 1/(1+4C^*/C_*). \tag{6.66}$$

The proof is divided in two cases, one when $\ell_c \geq \frac{\ell}{2}$ and one when $\ell_c < \frac{\ell}{2}$. The inequality

$$|\mathbb{E}_{\beta,N}[W_\gamma] - e^{-\ell(1-\theta(\beta))}| \leq C'\left[\frac{\sqrt{\ell_c}}{\ell}+\lambda(\beta)^2\right]^{C''} \tag{6.67}$$

will be proved, which then implies inequality (2.12). Since we earlier proved that the limit $\langle W_\gamma\rangle$ exists and both the right-hand side of the inequality and $e^{-\ell(1-\theta(\beta))}$ are independent of N, taking the limits of inequality (6.67) when N goes to infinity proves inequality (2.12).

6.3.1 First case: $\ell_c \geq \frac{\ell}{2}$

Assume that $\ell_c \geq \frac{\ell}{2}$. First, a lower bound for the right-hand side of (6.67) is calculated. From the definitions of C^* and C_* (equations (6.42) and (6.1)) follows that they are positive. Therefore, $4C^*/C_* > 0$ and $C'' \in]0,1[$. From the definition of C_A (equation (6.41)) follows that $C_A \geq \frac{9}{2}$. Thus,

$$\begin{aligned} C' &= \sqrt{2}(C_A 2^{4C^*/C_*})^{1/(1+4C^*/C_*)} \\ &= \sqrt{2}(C_A)^{1/(1+4C^*/C_*)} \cdot (2^{4C^*/C_*})^{1/(1+4C^*/C_*)} \\ &\geq \sqrt{2} \cdot 1 \cdot 2 = 2\sqrt{2}. \end{aligned}$$

From the assumption that $\ell_c \geq \frac{\ell}{2}$ follows that

$$\sqrt{\frac{l_c}{l}} \geq \sqrt{\frac{1}{2}}.$$

Therefore, the lower bound of the right-hand side of inequality (6.67) is

$$C' \left[\sqrt{\frac{l_c}{l}} + \lambda(\beta)^2 \right]^{C''} \geq 2\sqrt{2} \left(\sqrt{\frac{1}{2}} + 0 \right)^{C''} = 2^{\frac{3}{2} - \frac{C''}{2}} \geq 2.$$

For the left-hand side of inequality (6.67), we obtain

$$\begin{aligned} |\mathbb{E}_{\beta,N}[W_\gamma] - e^{-\ell(1-\theta(\beta))}| &\leq |\mathbb{E}_{\beta,N}[W_\gamma]| + |e^{-\ell(1-\theta(\beta))}| \\ &= \left| \mathbb{E}_{\beta,N} \left[\rho \left(\sum_{e \in \gamma} \sigma_e \right) \right] \right| + |e^{-\ell(1-\theta(\beta))}| \\ &\leq \mathbb{E}_{\beta,N} \left[\left| \rho \left(\sum_{e \in \gamma} \sigma_e \right) \right| \right] + 1 \\ &\leq \mathbb{E}_{\beta,N} [1] + 1 = 1 + 1 = 2. \end{aligned}$$

Hence, we obtain

$$|\mathbb{E}_{\beta,N}[W_\gamma] - e^{-\ell(1-\theta(\beta))}| \leq 2 \leq C' \left(\frac{\sqrt{\ell_c}}{\ell} + \lambda(\beta)^2 \right)^{C''}.$$

6.3.2 Second case: $\ell_c < \frac{\ell}{2}$

Assume that $\ell_c < \frac{\ell}{2}$. Then $\ell - \ell_c \geq \frac{1}{2}\ell$. From this assumption and Proposition 6.11 follows that

$$|\mathbb{E}_{\beta,N}[W_\gamma]| \leq e^{-C_*(\ell-\ell_c)\lambda(\beta)^{12}} \leq e^{-\frac{1}{2}C_*\ell\lambda(\beta)^{12}}.$$

Therefore,

$$\begin{aligned} |\mathbb{E}_{\beta,N}[W_\gamma] - e^{-\ell(1-\theta(\beta))}| &\leq |\mathbb{E}_{\beta,N}[W_\gamma]| + |e^{-\ell(1-\theta(\beta))}| \\ &\leq e^{-\frac{1}{2}C_*\ell\lambda(\beta)^{12}} + e^{-\ell(1-\theta(\beta))}. \end{aligned} \tag{6.68}$$

Since $5(|G|-1)\lambda(\beta)^2 < 1$ and $0 < \lambda(\beta) < 1$, we have

$$\begin{aligned}
& 1 - 5(|G|-1)\lambda(\beta)^2 > 0 \\
\Leftrightarrow\quad & \frac{\lambda(\beta)^{10}}{5} - (|G|-1)\lambda(\beta)^{12} \geq 0 \\
\Rightarrow\quad & \frac{1}{5} - (|G|-1)\lambda(\beta)^{12} \geq 0 \\
\Leftrightarrow\quad & 1 - (|G|-1)\lambda(\beta)^{12} \geq \frac{4}{5} \geq \frac{1}{2}.
\end{aligned} \tag{6.69}$$

The lower bound for $1-\theta(\beta)$ is then

$$\begin{aligned}
1-\theta(\beta) &\overset{(6.2)}{=} 1 - \frac{\sum_{g\in G} \Re(\rho(g)) e^{12\beta\Re\rho(g)}}{\sum_{g\in G} e^{12\beta\Re\rho(g)}} \\
&= \frac{\sum_{g\in G}(1-\Re(\rho(g)))e^{12\beta\Re\rho(g)}}{\sum_{g\in G} e^{12\beta\Re\rho(g)}} \\
&= \frac{\sum_{g\in G}(1-\Re(\rho(g)))e^{12\beta(\Re\rho(g)-1)}}{\sum_{g\in G} e^{12\beta(\Re\rho(g)-1)}} \\
&= \frac{\sum_{g\in G\setminus\{0\}}(1-\Re(\rho(g)))e^{12\beta(\Re\rho(g)-1)}}{1+\sum_{g\in G\setminus\{0\}} e^{12\beta(\Re\rho(g)-1)}} \\
&\overset{(6.52)}{\geq} \frac{\sum_{g\in G\setminus\{0\}}(1-\cos(2\pi/n))e^{12\beta(\Re\rho(g)-1)}}{1+\sum_{g\in G\setminus\{0\}} \max_{g\in G\setminus\{0\}} e^{12\beta(\Re\rho(g)-1)}} \\
&= \frac{(1-\cos(2\pi/n))\sum_{g\in G\setminus\{0\}} e^{12\beta(\Re\rho(g)-1)}}{1+(|G|-1)\max_{g\in G\setminus\{0\}} e^{12\beta(\Re\rho(g)-1)}} \\
&\geq \frac{(1-\cos(2\pi/n))\max_{g\in G\setminus\{0\}} e^{12\beta(\Re\rho(g)-1)}}{1+(|G|-1)\max_{g\in G\setminus\{0\}} e^{12\beta(\Re\rho(g)-1)}} \\
&\overset{(2.8)}{=} \frac{(1-\cos(2\pi/n))\lambda(\beta)^{12}}{1+(|G|-1)\lambda(\beta)^{12}} \\
&= \frac{(1-\cos(2\pi/n))\lambda(\beta)^{12}(1-(|G|-1)\lambda(\beta)^{12})}{(1+(|G|-1)\lambda(\beta)^{12})(1-(|G|-1)\lambda(\beta)^{12})} \\
&= \frac{(1-\cos(2\pi/n))\lambda(\beta)^{12}(1-(|G|-1)\lambda(\beta)^{12})}{1-((|G|-1)\lambda(\beta)^{12})^2)} \\
&\geq (1-\cos(2\pi/n))\lambda(\beta)^{12}(1-(|G|-1)\lambda(\beta)^{12}) \\
&= 4C_*\lambda(\beta)^{12}(1-(|G|-1)\lambda(\beta)^{12}) \\
&\overset{(6.69)}{\geq} 2C_*\lambda(\beta)^{12}.
\end{aligned}$$

Thus,

$$e^{-\ell(1-\theta)} \leq e^{-2C_*\ell\lambda(\beta)^{12}} \tag{6.70}$$

and

$$
\begin{aligned}
|\mathbb{E}_{\beta,N}[W_\gamma] - e^{-\ell(1-\theta(\beta))}| &\overset{(6.68),(6.70)}{\leq} e^{-\frac{1}{2}C_*\ell\lambda(\beta)^{12}} + e^{-2\ell C_*\lambda(\beta)^{12}} \\
&\leq e^{-\frac{1}{2}\ell C_*\lambda(\beta)^{12}} + e^{-\frac{1}{2}\ell C_*\lambda(\beta)^{12}} \\
&= 2e^{-\frac{1}{2}\ell C_*\lambda(\beta)^{12}}.
\end{aligned} \tag{6.71}
$$

From Proposition 6.1 follows that

$$
|\mathbb{E}_{\beta,N}[W_\gamma] - e^{-\ell(1-\theta(\beta))}| \leq C_A e^{2C^*\ell\lambda(\beta)^{12}} \left(\sqrt{\frac{\ell_c}{\ell}} + \lambda(\beta)^2\right). \tag{6.72}
$$

By combining this with inequality (6.71), we obtain

$$
\begin{aligned}
&|\mathbb{E}_{\beta,N}[W_\gamma] - e^{-\ell(1-\theta(\beta))}|^{1+4C^*/C_*} \\
&\overset{(6.72),(6.71)}{\leq} \left(C_A e^{2C^*\ell\lambda(\beta)^{12}} \left(\sqrt{\frac{\ell_c}{\ell}} + \lambda(\beta)^2\right)\right)^1 \cdot \left(2e^{-\frac{1}{2}\ell C_*\lambda(\beta)^{12}}\right)^{4C^*/C_*} \\
&\leq C_A 2^{4C^*/C_*} \left(\sqrt{\frac{\ell_c}{\ell}} + \lambda(\beta)^2\right) \left(e^{2C^*\ell\lambda(\beta)^{12}} \cdot e^{-2C^*\ell\lambda(\beta)^{12}}\right) \\
&= C_A 2^{4C^*/C_*} \left(\sqrt{\frac{\ell_c}{\ell}} + \lambda(\beta)^2\right).
\end{aligned}
$$

Thus,

$$
\begin{aligned}
|\mathbb{E}_{\beta,N}[W_\gamma] - e^{-\ell(1-\theta(\beta))}| &\leq \left(C_A 2^{4C^*/C_*} \left(\sqrt{\frac{\ell_c}{\ell}} + \lambda(\beta)^2\right)\right)^{1/(1+4C^*/C_*)} \\
&= \left(C_A 2^{4C^*/C_*}\right)^{1/(1+4C^*/C*)} \left(\sqrt{\frac{\ell_c}{\ell}} + \lambda(\beta)^2\right)^{1/(1+4C^*/C_*)} \\
&\leq \sqrt{2}\left(C_A 2^{4C^*/C_*}\right)^{1/(1+4C^*/C*)} \\
&\quad \cdot \left(\sqrt{\frac{\ell_c}{\ell}} + \lambda(\beta)^2\right)^{1/(1+4C^*/C_*)} \\
&= C' \left(\sqrt{\frac{\ell_c}{\ell}} + \lambda(\beta)^2\right)^{C''}.
\end{aligned}
$$

We have proved the main theorem.

Chapter 7

Svensk sammanfattning

Denna avhandling är baserad på artikeln "Wilson loops in finite Abelian lattice gauge theories" av M. Forsström, J. Lenells och F. Viklund [1], där väntevärdet för en Wilsonloopobservabel uppskattas. Avhandlingens mål är att förklara artikeln så att matematikstuderande på magisternivå kan förstå den. I denna avhandling behandlas därför den matematiska teori som behövs för att förstå artikeln. Därtill ges bevisen i artikeln i en mera detaljerad och förklarande form.

Det första kapitlet i avhandlingen är en introduktion som ger en bakgrund till problemet som behandlas i denna avhandling. Matematiska modeller inom gittergaugeteorier har länge studerats och målet med dem är att försöka förklara kvantfältteorier inom standardmodellen. Även om dessa modeller inte ännu kan användas inom kvantfysiken är de värdefulla också som matematiska modeller.

Det andra kapitlet introducerar artikelns huvudsats. Den teori som behövs för att satsen ska kunna förstås diskuteras och satsen ges. Grundläggande teori om grupper och representationer behandlas och både gruppen och representationen som används i artikeln definieras. Gruppen som används är $G = (\mathbb{Z}_n, +)$ som är en ändlig abelsk grupp. Representationen som används är endimensionell, unitär och injektiv. Det visas att om en representation uppfyller dessa krav för den valda gruppen så ges den av $\rho(g) = e^{g \cdot 2\pi i m/n}$, där g tillhör gruppen $\mathbb{Z}_n$ och $m \in \{0, 1, \ldots, n-1\}$ är relativt primt till n. Därtill definieras det fyrdimensionella gittret $\mathbb{Z}^4$ vars noder är i varje heltalspunkt. Delmängder av detta gitter och begrepp som hör till det ges, bland annat plaketter[1] som bildas av två riktade bågar. Därefter tas begrepp för loopar och bågar upp innan ett mått

[1] En svensk term är inte etablerad (eng. plaquette).

med Wilsonverkan ges och Wilsonloopobservabeln W_γ definieras. Gränsvärdet för denna observabels väntevärde betecknas med $\langle W_\gamma \rangle_\beta$. Kapitlet avslutas med att satsen ges och kommenteras. Som följande ges satsen.

Sats. *Låt n vara ett heltal med $n \geq 2$. Låt gittergaugeteorin ha strukturgruppen $G = \mathbb{Z}_n$. Låt representationen ρ vara en endimensionell och injektiv representation av G. Låt $\gamma \in \mathbb{Z}^4$ vara en riktad cykel, ℓ dess längd och ℓ_c antalet hörnbågar i γ. Då existerar gränsvärdet av Wilsonloopobservabelns väntevärde och β_0 kan väljas så att det existerar konstanter $C'(\beta_0)$ och $C''(\beta_0)$ för vilka följande olikhet gäller för alla $\beta \geq \beta_0$:*

$$|\langle W_\gamma \rangle_\beta - e^{-\ell(1-\theta(\beta))}| \leq C' \left(\frac{\sqrt{\ell_c}}{\ell} + \lambda(\beta)^2 \right)^{C''}.$$

Funktionen $\theta(\beta)$ är definierad i ekvation (2.7) och $\lambda(\beta)$ i ekvation (2.8).

Från denna sats kan dras som slutsats att väntevärdet på Wilsonloopobservabeln tar ett värde väldigt nära 0 om $\ell(1 - \theta(\beta))$ är stort och ett värde väldigt nära 1 om $\ell(1 - \theta(\beta))$ är litet.

Det tredje kapitlet behandlar teori om diskret yttre algebra då $\mathbb{Z}^r$ har valts som gitter. Denna teori diskuteras alltså mera generellt än vad som skulle behövas för resten av avhandlingen. Riktade differentialceller och -former både definieras och ges exempel på (se Figur 3.1 för exempel på differentialceller). Dessa kallas också för k-celler och k-former. En riktad 1-cell är detsamma som en båge i gittret $\mathbb{Z}^r$ och en riktad 2-cell detsamma som en plakett. Därefter ges två operatorer för k-former. Dessa är den yttre derivatan, som avbildar en k-form på en $k+1$-form, och koderivatan som avbildar en k-form på en $k-1$-form. Exempel illustrerar hur dessa används. Ränder och koränder för k-former och speciellt en plakett behandlas. Dessutom ges två olika versioner av Poincarés lemma som både appliceras på en viss mängd av 2-former och används för att visa att måttet med Wilsonverkan kan skrivas om så att det blir ett mått för plakettkonfigurationer. Till sist diskuteras den diskreta Hodge dual-operatorn och bijektionen mellan det ursprungliga gittret och det som skapas av Hodge dual-operatorn. Lemman med Hodge dual-operatorn som är nödvändiga för fjärde kapitlets bevis ges.

Det fjärde kapitlet behandlar teori för virvlar[2] och riktade ytor. Dessa begrepp definieras och illustreras, observera att dessa definitioner kan variera mellan

[2]En svensk term är inte etablerad (eng. vortex).

olika källor. Först behandlas uppdelningar av virvlar, sedan definieras minimala virvlar och lemman med dessa som behövs till huvudbeviset ges och bevisas. Dessutom ges en proposition för en viss fördelning för virvlar som har en stor roll i huvudbeviset. Till sist ges några lemman för riktade ytor och en riktad ytas koppling till loopar förklaras. Dessutom definieras inre bågar och inre plaketter och ett lemma för riktade ytor ges.

I det femte kapitlet behandlas gränsvärdet för väntevärdet av en Wilson-loopobservabel. Det viktigaste resultatet i detta kapitel är en sats som säger att måtten $\mathbb{E}_{\beta,N}$ konvergerar svagt i topologin av lokal konvergens när N närmar sig oändligheten och det begränsade måttet är translationsinvariant i $\mathbb{Z}^r$. Denna sats bevisas med hjälp av Ginibres olikhet. Till sist tillämpas satsen på Wilson-loopobservabeln för att visa att dess väntevärdes gränsvärde både existerar och är invariant under translation.

I det sjätte kapitlet bevisas slutligen huvudsatsen med hjälp av den teori och de resultat som getts i de tidigare kapitlen. Beviset delas upp i två fall, först i Proposition 6.1 där $\ell\lambda(\beta)^{12}$ är stort och därefter i Proposition 6.11 där $\ell\lambda(\beta)^{12}$ är litet. Beviset för Proposition 6.1 är uppdelat i flera delar där olika väntevärden beräknas innan de kombineras för att bevisa den önskade olikheten. Till beviset behövs flera lemman som ges innan. Dessa lemman är främst olikheter för funktionen θ samt sannolikheter för olika händelser, vilka beräknas med hjälp av propositionen från det fjärde kapitlet. Därefter bevisas Proposition 6.11. Till detta bevis behövs några lemman som först ges och bevisas och även vissa slutsatser från beviset för Proposition 6.1 används. Därefter kombineras dessa två fall för att bevisa olikheten

$$|\mathbb{E}_{\beta,N}[W_\gamma] - e^{-\ell(1-\theta(\beta))}| \leq C' \left(\frac{\sqrt{\ell_c}}{\ell} + \lambda(\beta)^2 \right)^{C''} .$$

Med hjälp av resultatet för gränsvärdets existens för Wilsonloopobservabeln kan satsen därefter bevisas genom att ta gränsvärdet då N närmar sig oändligheten.

För de som är intresserade av liknande problem finns bland annat artiklarna "Wilson loops in Ising lattice gauge theory" av S. Chatterjee [2] och "Wilson loop expectations in lattice gauge theories with finite gauge groups" av S. Cao [3] som båda publicerades år 2020.

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