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Weighted Composition and Volterra Operators on Banach Spaces of Analytic Functions: Compactness and Spectral Properties

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Preface

This thesis was carried out during the years 2014-2018 at the department of mathematics at Åbo Akademi University. I would like to thank the personnel at the department, and especially my supervisor professor Mikael Lindström for all the support I received during this process. I also want to express my gratitude to professor José Peláez and docent Jani Virtanen for their careful pre-examination of my thesis, and to professor José Bonet for agreeing to act as my opponent at the public defence of the thesis. Last but not least, I would like to thank my family and friends for everything during these years.

Åbo, October 2018

Ted Eklund

Svensk sammanfattning

Låt $\mathcal{H}(\mathbb{D})$ beteckna mängden av alla analytiska funktioner $f: \mathbb{D} \rightarrow \mathbb{C}$ definierade i den öppna enhetsdisken

$$\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$$

i det komplexa talplanet \mathbb{C} . Avsikten med föreliggande avhandling har varit att undersöka egenskaper hos vissa klassiska linjära operatorer $T: \mathcal{X} \rightarrow \mathcal{Y}$ som avbildar mellan olika Banachrum $\mathcal{X}, \mathcal{Y} \subset \mathcal{H}(\mathbb{D})$ av analytiska funktioner. Operatorerna som studerades var *viktade kompositionoperatorer* uC_φ samt *generaliserade Volterra operatorer* T_g^φ . Dessa definieras via givna funktioner $u, \varphi, g \in \mathcal{H}(\mathbb{D})$, $\varphi(\mathbb{D}) \subset \mathbb{D}$, enligt

$$uC_\varphi(f)(z) = u(z)f(\varphi(z)), \quad z \in \mathbb{D}, f \in \mathcal{H}(\mathbb{D}),$$

och

$$T_g^\varphi(f)(z) = \int_0^{\varphi(z)} f(\xi)g'(\xi)d\xi, \quad z \in \mathbb{D}, f \in \mathcal{H}(\mathbb{D}).$$

I avhandlingen undersöktes hur operatorteoritiska egenskaper, såsom begränsning, kompakthet, svag kompakthet, egenfunktioner och spektrum, för operatorerna uC_φ och T_g^φ beror av funktionsteoretiska egenskaper hos de induserande symbolerna u, φ och g , för olika val av funktionsrum $\mathcal{X}, \mathcal{Y} \subset \mathcal{H}(\mathbb{D})$.

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Chapter 1

Introduction

The collection of all analytic functions $f: \mathbb{D} \rightarrow \mathbb{C}$ on the *open unit disc*

$$\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$$

in the complex plane \mathbb{C} is denoted by $\mathcal{H}(\mathbb{D})$. Historically, one of the motivations for confining the study of complex valued functions to those that are defined on the unit disc is that, by the *Riemann mapping theorem*, any non-empty simply connected open subset \mathcal{O} of the complex plane such that $\mathcal{O} \neq \mathbb{C}$ can be mapped bijectively onto \mathbb{D} by means of some analytic function whose inverse is also analytic.

The purpose of this thesis has been to investigate properties of certain classical linear operators $T: \mathcal{X} \rightarrow \mathcal{Y}$ mapping between various Banach spaces $\mathcal{X}, \mathcal{Y} \subset \mathcal{H}(\mathbb{D})$ of analytic functions on the unit disc. Namely, we studied weighted composition operators uC_φ and generalized Volterra operators T_g^φ , focusing on operator theoretic properties such as boundedness, compactness, weak compactness, eigenfunctions and spectrum. All these notions, although certainly well-known to experts, will be defined and discussed in detail in chapters 2 and 3 below, in addition to other relevant concepts such as predual spaces and analytic selfmaps $\varphi: \mathbb{D} \rightarrow \mathbb{D}$ of the unit disc.

The text is organized as follows. In chapter 2 we recall some classical Banach spaces of analytic functions considered in this thesis, and also present a framework consisting of general axioms imposed on function spaces, as well as present some facts about predual spaces. Chapter 3 begins with an overview of basic operator theoretic properties, whereafter we study analytic selfmaps of the unit disc. These selfmaps are then used to define weighted composition operators, and the main results obtained in the papers constituting this thesis related to these operators are presented. After that we discuss the research conducted on the Königs eigenfunction, and finally our results obtained on the generalized Volterra operator are presented.

1.1 List of publications

The thesis is based on the following publications. The mathematical ideas presented in the papers I-IV were developed jointly with the co-authors, and the author of this thesis is responsible for a significant part of the research in each of these.

Paper I T. Eklund, M. Lindström and P. Mleczko, *Spectral properties of weighted composition operators on the Bloch and Dirichlet spaces*, *Studia Mathematica*, Volume 232, pp. 95-112, 2016.

Paper II T. Eklund, P. Galindo and M. Lindström, *Königs eigenfunction for composition operators on Bloch and H^∞ type spaces*, *Journal of Mathematical Analysis and Applications*, Volume 445, pp. 1300-1309, 2017.

Paper III T. Eklund, P. Galindo, M. Lindström and I. Nieminen, *Norm, essential norm and weak compactness of weighted composition operators between dual Banach spaces of analytic functions*, *Journal of Mathematical Analysis and Applications*, Volume 451, pp. 1-13, 2017.

Paper IV T. Eklund, M. Lindström, M.M. Pirasteh, A.H. Sanatpour and N. Wikman, *Generalized Volterra operators mapping between Banach spaces of analytic functions*, *Monatshefte für Mathematik*, pp. 1-19, 2018.
<https://doi.org/10.1007/s00605-018-1216-5>

Chapter 2

Banach spaces of analytic functions

2.1 Weights

Almost every classical Banach space of analytic functions on the unit disc is defined via some growth restriction on the functions contained in the space. For some function spaces, this growth is determined by a weight function. In the literature there are various definitions regarding which properties such a weight function should satisfy. In this thesis however, a *weight* is defined to be a continuous and strictly positive function $v: \mathbb{D} \rightarrow \mathbb{R}$ such that $\lim_{|z| \rightarrow 1} v(z) = 0$. In order to be able to prove more interesting results, one sometimes needs to equip the weights with additional regularity properties. The weight v is called *normal* if it is *radial*, in the sense that $v(z) = v(|z|)$ for every $z \in \mathbb{D}$, almost decreasing with respect to $|z|$ and satisfies the conditions

$$\inf_{n \in \mathbb{N}} \frac{v(1 - 2^{-n-1})}{v(1 - 2^{-n})} > 0 \quad \text{and} \quad \inf_{k \in \mathbb{N}} \limsup_{n \rightarrow \infty} \frac{v(1 - 2^{-n-k})}{v(1 - 2^{-n})} < 1. \quad (2.1.1)$$

Recall that a function $f: [a, b] \rightarrow \mathbb{R}$ is called *almost increasing* if there exists a constant $C > 0$ such that $f(x) \leq Cf(y)$ whenever $a \leq x < y \leq b$, and *almost decreasing* if the same holds when $a \leq y < x \leq b$. The concept of normality was introduced by Shields and Williams in the paper [46], where they stated that a radial weight v is normal if there exist numbers $\alpha > \beta > 0$ and $0 < r_0 < 1$ such that

$$\frac{v(r)}{(1-r)^\alpha} \nearrow \infty \quad \text{and} \quad \frac{v(r)}{(1-r)^\beta} \searrow 0 \quad \text{as } r_0 \leq r < 1 \text{ and } r \rightarrow 1^-.$$

Shields and Williams later on extended their definition of a normal weight in [47], where they defined a radial weight v to be normal if there are constants $\alpha > \beta > 0$ such that the function $r \mapsto \frac{v(r)}{(1-r)^\alpha}$ is almost increasing and the corresponding function $r \mapsto \frac{v(r)}{(1-r)^\beta}$ is almost decreasing on $[0, 1)$. In [18, Lemma 1], Domański and Lindström showed that this original definition of normal weights is equivalent to the one given in (2.1.1), under the assumption that the radial weight v is almost decreasing. The two conditions in (2.1.1) thus ensure that the weight v does not, in some sense, tend too slowly nor too rapidly to zero at the boundary of the unit disc. The *standard weights*

$v_\alpha(z) := (1 - |z|^2)^\alpha$ with $\alpha > 0$ are clearly normal, whereas for example the weights $v_{\exp,\alpha}(z) = \exp(-1/(1 - |z|^2)^\alpha)$ and $v_{\log,\alpha}(z) = (1 - \log(1 - |z|^2))^{-\alpha}$ for $\alpha > 0$ are not. The weight $v_{\exp,\alpha}$ fails to satisfy the first condition in (2.1.1) and $v_{\log,\alpha}$ does not fulfill the second requirement for normality, see [18, Section 3].

2.2 Classical function spaces

In this section we briefly discuss some basic properties of the classical Banach spaces of analytic functions on the unit disc considered in this thesis, and provide references for more detailed information on these spaces. To begin with, the *Hardy spaces* H^p for $1 \leq p < \infty$ consist of all functions $f \in \mathcal{H}(\mathbb{D})$ such that

$$\|f\|_{H^p} = \sup_{0 \leq r < 1} \left[\frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p d\theta \right]^{\frac{1}{p}} < \infty,$$

and the space H^∞ of *bounded analytic functions* on the unit disc is equipped with the supremum norm

$$\|f\|_\infty = \sup_{z \in \mathbb{D}} |f(z)|.$$

The Hardy spaces are decreasing in size as the exponent grows, meaning that $H^q \subset H^p$ whenever $1 \leq p \leq q \leq \infty$. If $f \in H^p$ for $1 \leq p < \infty$, then one has the pointwise estimate

$$|f(z)| \leq \frac{\|f\|_{H^p}}{(1 - |z|^2)^{\frac{1}{p}}}, \quad z \in \mathbb{D}. \quad (2.2.1)$$

Every function $f \in H^p$ has radial limit

$$f^*(\zeta) = \lim_{r \rightarrow 1^-} f(r\zeta)$$

at almost every boundary point $\zeta \in \partial\mathbb{D} = \{z \in \mathbb{C} : |z| = 1\}$, with respect to Lebesgue measure, and the original function can be recovered from its boundary values f^* by means of the *Poisson integral*

$$f(z) = \frac{1}{2\pi} \int_0^{2\pi} f^*(e^{i\theta}) P_z(\theta) d\theta, \quad z \in \mathbb{D},$$

where $P_z(\theta)$ is the *Poisson kernel*

$$P_z(\theta) = \frac{1 - |z|^2}{|1 - \bar{z}e^{i\theta}|^2}.$$

The books [19] by Duren, [26] by Garnett and [32] by Koosis serve as good references for the Hardy spaces. The *weighted Bergman spaces* A_α^p for $1 \leq p < \infty$ and $-1 < \alpha < \infty$ consist of all analytic functions in the Lebesgue space $L^p(\mathbb{D}, dA_\alpha)$, where

$$dA_\alpha(z) = (\alpha + 1)(1 - |z|^2)^\alpha dA(z)$$

and $dA(z) = \frac{1}{\pi} dx dy$ is the normalized area measure on \mathbb{D} . We also write A^p for the unweighted Bergman space A_0^p . The norm of $f \in A_\alpha^p$ is given by

$$\|f\|_{A_\alpha^p} = \left[(\alpha + 1) \int_{\mathbb{D}} |f(z)|^p (1 - |z|^2)^\alpha dA(z) \right]^{\frac{1}{p}},$$

and comparing the Hardy and Bergman norms one finds that $\|f\|_{A^p} \leq \|f\|_{H^p}$ for every $f \in H^p$ and $1 \leq p < \infty$, showing that $H^p \subset A^p$. As for the Hardy spaces, the weighted Bergman spaces are also decreasing in size as the power p grows, and if $f \in A_\alpha^p$ then a corresponding pointwise estimate holds:

$$|f(z)| \leq \frac{\|f\|_{A_\alpha^p}}{(1 - |z|^2)^{\frac{2+\alpha}{p}}}, \quad z \in \mathbb{D}. \quad (2.2.2)$$

For more information on Bergman spaces, see the books [20] by Duren and Schuster, [29] by Hedenmalm, Korenblum and Zhu, and [51] by Zhu. The *Dirichlet space* \mathcal{D} , treated for example in [25], consists of all functions $f \in \mathcal{H}(\mathbb{D})$ with derivative f' belonging to the Bergman space A^2 , and the norm is defined as

$$\|f\|_{\mathcal{D}} = \left[|f(0)|^2 + \int_{\mathbb{D}} |f'(z)|^2 dA(z) \right]^{\frac{1}{2}}.$$

The space \mathcal{D} is a dense subspace of the Hardy Hilbert space H^2 , but is not closed in H^2 . However, the Dirichlet space can be turned into a Hilbert space by defining the inner product as

$$\langle f, g \rangle_{\mathcal{D}} = f(0)\overline{g(0)} + \int_{\mathbb{D}} f'(z)\overline{g'(z)} dA(z), \quad f, g \in \mathcal{D}.$$

Among the most central spaces for this thesis are the *weighted Banach spaces of analytic functions* H_v^∞ and H_v^0 , defined by

$$H_v^\infty = \left\{ f \in \mathcal{H}(\mathbb{D}) : \|f\|_{H_v^\infty} := \sup_{z \in \mathbb{D}} v(z)|f(z)| < \infty \right\},$$

$$H_v^0 = \left\{ f \in H_v^\infty : \lim_{|z| \rightarrow 1^-} v(z)|f(z)| = 0 \right\},$$

where $v: \mathbb{D} \rightarrow \mathbb{R}$ is a given weight. These growth spaces were first studied by Rubel and Shields in 1970, see the paper [42]. When dealing with these spaces one sometimes needs to consider the corresponding *associated weights*

$$\tilde{v}(z) = \left(\sup \{ |f(z)| : f \in H_v^\infty \text{ and } \|f\|_{H_v^\infty} \leq 1 \} \right)^{-1},$$

which are in fact weights themselves. In the paper [8], Bierstedt, Bonet and Taskinen clarified the connection between the associated weights and the growth spaces. Among other things, they showed that if the weight v is replaced by its associated weight \tilde{v} , then the spaces H_v^∞ and $H_{\tilde{v}}^\infty$ are isometrically isomorphic, and so are the spaces H_v^0 and $H_{\tilde{v}}^0$. Moreover, Lusky [35] has shown that $H_v^\infty \approx \ell^\infty$ and $H_v^0 \approx c_0$ for a large class of

weights including the normal weights. By $\mathcal{X} \approx \mathcal{Y}$ we mean that the spaces \mathcal{X} and \mathcal{Y} are *isomorphic*, meaning that there exists a bijective linear map from \mathcal{X} onto \mathcal{Y} .

The *Bloch-type spaces* \mathcal{B}_v^∞ and \mathcal{B}_v^0 are given by

$$\mathcal{B}_v^\infty = \left\{ f \in \mathcal{H}(\mathcal{D}) : \|f\|_{\mathcal{B}_v^\infty} := |f(0)| + \sup_{z \in \mathcal{D}} v(z)|f'(z)| < \infty \right\},$$

$$\mathcal{B}_v^0 = \left\{ f \in \mathcal{B}_v^\infty : \lim_{|z| \rightarrow 1^-} v(z)|f'(z)| = 0 \right\}.$$

The *classical Bloch space* $\mathcal{B}_{v_1}^\infty$ is denoted by \mathcal{B} and the *little Bloch space* $\mathcal{B}_{v_1}^0$ by \mathcal{B}_0 . If the weight v is normal then, by a result of Lusky [34] and using the weight $w(z) = (1-|z|)v(z)$, one can identify $H_v^\infty = \mathcal{B}_w^\infty$ and $H_v^0 = \mathcal{B}_w^0$. This result can be seen as a generalization of the well-known result by Hardy and Littlewood on the growth of a function and the growth of its derivative, stating that for any $f \in \mathcal{H}(\mathbb{D})$ and $\alpha > 0$ it holds that

$$f(z) = \mathcal{O}\left((1-|z|^2)^{-\alpha}\right) \text{ if and only if } f'(z) = \mathcal{O}\left((1-|z|^2)^{-(\alpha+1)}\right), \text{ as } |z| \rightarrow 1^-.$$

Finally, the *disc algebra* $A(\mathbb{D})$ is the space of functions analytic in the unit disc that extend continuously to the boundary $\partial\mathbb{D}$.

Common for all the spaces presented above is that their norm topologies are finer than the *compact open topology* $(\mathcal{H}(\mathbb{D}), co)$, which is defined to be the smallest topology on $\mathcal{H}(\mathbb{D})$ containing all sets of the form

$$\{f \in \mathcal{H}(\mathbb{D}) : f(K) \subset U\},$$

where $K \subset \mathbb{D}$ is compact and $U \subset \mathbb{C}$ is open. Convergence in the compact open topology is equivalent to uniform convergence on compact subset of the unit disc. Hence, if $\{f_n\}_{n=1}^\infty$ is a sequence in any of the Banach spaces \mathcal{X} above and $\|f_n\|_{\mathcal{X}} \rightarrow 0$ then $f_n \xrightarrow{co} 0$, in the sense that

$$\lim_{n \rightarrow \infty} \sup_{z \in K} |f_n(z)| = 0$$

for any compact subset $K \subset \mathbb{D}$. The compact open topology can be induced by a metric, or in other words, it is *metrizable*, see for example [45, Chapter 2, Exercise 7]. This means for example that it is enough to consider sequences instead of nets when studying concepts such as continuity of functions or compactness of sets in this topology. The spaces discussed above also have other properties in common, which is the topic of the next section.

2.3 General axioms

When studying operators mapping between Banach spaces of analytic functions one often only needs to use some of the properties that the spaces possess. Many of the spaces discussed in the previous section, as well as other types of function spaces, have certain general properties in common, and the calculations can therefore be performed in a general framework. To introduce the framework used in the papers constituting

this thesis, let $\mathcal{X} \subset \mathcal{H}(\mathbb{D})$ be a Banach space of analytic functions on the unit disc and let $\|\cdot\|_{\mathcal{X}}$ denote its norm. The *closed unit ball* of \mathcal{X} will be denoted by

$$B_{\mathcal{X}} = \{x \in \mathcal{X} : \|x\|_{\mathcal{X}} \leq 1\}.$$

For any $z \in \mathbb{D}$, the *point evaluation functional* $\delta_z: \mathcal{X} \rightarrow \mathbb{C}$ is defined as $\delta_z(f) = f(z)$ for $f \in \mathcal{X}$. The first assumption adopted is that \mathcal{X} contains the constant functions, and hence all δ_z are non-zero. We also considered different combinations of the following conditions on the space \mathcal{X} (see [14] or [22]). Below the notation $A \lesssim B$ is used to indicate that there is a positive constant c , not depending on properties of A or B , such that $A \leq cB$. We will also write $A \asymp B$ whenever both $A \lesssim B$ and $B \lesssim A$ hold.

- (I) The closed unit ball $B_{\mathcal{X}}$ of \mathcal{X} is compact with respect to the compact open topology.
- (II) The point evaluation functionals $\delta_z: \mathcal{X} \rightarrow \mathbb{C}$ satisfy $\lim_{|z| \rightarrow 1} \|\delta_z\|_{\mathcal{X} \rightarrow \mathbb{C}} = \infty$.
- (III) The linear operator $\mathcal{T}_r: \mathcal{X} \rightarrow \mathcal{X}$ mapping $f \mapsto f_r$, where $f_r(z) := f(rz)$, is compact for every $0 < r < 1$.
- (IV) The operators \mathcal{T}_r in (III) satisfy $\sup_{0 < r < 1} \|\mathcal{T}_r\|_{\mathcal{X} \rightarrow \mathcal{X}} < \infty$.
- (V) The pointwise multiplication operator $M_u: \mathcal{X} \rightarrow \mathcal{X}$ satisfies $\|M_u\|_{\mathcal{X} \rightarrow \mathcal{X}} \lesssim \|u\|_{\infty}$ for every $u \in H^{\infty}$.
- (VI) For every $f \in \mathcal{X}$ and $z \in \mathbb{D}$ it holds that $(1 - |z|^2)|f'(z)| \lesssim \|f\|_{\mathcal{X}} \|\delta_z\|_{\mathcal{X} \rightarrow \mathbb{C}}$.

Condition (I) holds for all the spaces mentioned in section 2.2, except for H_v^0 , \mathcal{B}_v^0 and $A(\mathbb{D})$. If some space \mathcal{X} satisfies condition (I), then the identity map

$$\text{id}: (\mathcal{X}, \|\cdot\|_{\mathcal{X}}) \rightarrow (\mathcal{X}, co)$$

is continuous and hence the norm topology of \mathcal{X} is finer than the compact open topology. Another consequence of this is that the point evaluation functionals are continuous with respect to the norm topology on \mathcal{X} , or in other words, each $\delta_z \in \mathcal{X}^*$. Moreover, since the functionals δ_z are *co*-continuous and $(B_{\mathcal{X}}, co)$ is compact, we have for every $z \in \mathbb{D}$ that the norm $\|\delta_z\|_{\mathcal{X} \rightarrow \mathbb{C}}$ is attained at some $f_z \in \mathcal{X}$ with $\|f_z\|_{\mathcal{X}} \leq 1$, that is, $\|\delta_z\|_{\mathcal{X} \rightarrow \mathbb{C}} = |f_z(z)|$. Condition (I) also ensures the existence of a predual space ${}^*\mathcal{X}$, to be explained in more detail in the next section. The Hardy and Bergman spaces H^p and A_{α}^p satisfy all conditions (I)-(VI) for $1 \leq p < \infty$ and $\alpha > -1$. For example, from (2.2.1) and (2.2.2) one obtains the functional norms $\|\delta_z\|_{H^p \rightarrow \mathbb{C}} = (1 - |z|^2)^{-\frac{1}{p}}$ and $\|\delta_z\|_{A_{\alpha}^p \rightarrow \mathbb{C}} = (1 - |z|^2)^{\frac{-2-\alpha}{p}}$, from which we see that (II) is satisfied. However, condition (II) is not satisfied in H^{∞} since $\|\delta_z\|_{H^{\infty} \rightarrow \mathbb{C}} = 1$. The growth spaces H_v^{∞} also satisfy all conditions (I)-(VI) if the weight v is normal and equivalent to its associated weight $\tilde{v}(z) = \|\delta_z\|_{H_v^{\infty} \rightarrow \mathbb{C}}^{-1}$. The classical Bloch space \mathcal{B} and the Dirichlet space \mathcal{D} are examples of spaces which fail condition (V). If some space \mathcal{X} does satisfy condition (V), then $H^{\infty} \subset \mathcal{X}$. For more detailed information on the conditions (I)-(VI), see for example [14, Section 2] or [22, Section 2].

2.4 The predual space ${}^*\mathcal{X}$

Every normed space \mathcal{X} can be thought of as a subset of its second dual space \mathcal{X}^{**} . This natural embedding is carried out by the *evaluation map* $Q(x) = \widehat{x}: \mathcal{X} \rightarrow \mathcal{X}^{**}$, where $\widehat{x}(\ell) = \ell(x)$ for $x \in \mathcal{X}$ and $\ell \in \mathcal{X}^*$. The evaluation map is always injective, since it is isometric, but not necessarily surjective. If Q does map \mathcal{X} onto \mathcal{X}^{**} , then the space \mathcal{X} is called *reflexive* and can thus be identified with its second dual. For example each finite-dimensional normed space is reflexive, and so is every Hilbert space because they are selfdual due to the Riesz representation theorem. However, usually the Banach spaces considered in functional analysis are not reflexive, in which case \mathcal{X}^* is not a predual of \mathcal{X} . To remedy this situation for Banach spaces $\mathcal{X} \subset \mathcal{H}(\mathbb{D})$, one can instead consider the subset of the dual space \mathcal{X}^* given by

$${}^*\mathcal{X} = \left\{ \ell \in \mathcal{X}^* : \ell|_{B_{\mathcal{X}}} \text{ is co-continuous} \right\}. \quad (2.4.1)$$

If \mathcal{X} satisfies condition (I) of the previous section then, by the *Dixmier-Ng theorem* [38], ${}^*\mathcal{X}$ becomes a Banach space endowed with the norm induced by the dual space \mathcal{X}^* , and the evaluation map $\Phi_{\mathcal{X}}: \mathcal{X} \rightarrow ({}^*\mathcal{X})^*$, defined as the restriction $\Phi_{\mathcal{X}}(f) = \widehat{f}|_{{}^*\mathcal{X}}$, turns out to be an onto isometric isomorphism. The space ${}^*\mathcal{X}$ can thus be thought of as a *predual* of \mathcal{X} . Moreover, it follows from the Hahn-Banach theorem that the linear span of the set of all point evaluation functionals is contained and norm dense in ${}^*\mathcal{X}$, that is

$${}^*\mathcal{X} = \overline{\text{span}}^{{}^*\mathcal{X}} \left\{ \delta_z : z \in \mathbb{D} \right\}.$$

See for example [10] for a proof of this fact. The existence of preduals is actually true in a more general setting than merely Banach spaces of analytic functions on the unit disc. Kung-Fu Ngs simplification of Jacques Dixmiers original theorem reads as follows:

Theorem 2.4.1. *Let $(\mathcal{X}, \|\cdot\|_{\mathcal{X}})$ be a normed space with closed unit ball $B_{\mathcal{X}}$. Suppose there exists a (Hausdorff) locally convex topology τ for \mathcal{X} such that $B_{\mathcal{X}}$ is τ -compact. Then \mathcal{X} itself is a Banach dual space, that is, there exists a Banach space V such that \mathcal{X} is isometrically isomorphic to the dual space V^* of V (in particular, \mathcal{X} is complete).*

Since $(\mathcal{H}(\mathbb{D}), co)$ is a *locally convex topology*, one is allowed to apply the Dixmier-Ng theorem to Banach spaces $\mathcal{X} \subset \mathcal{H}(\mathbb{D})$ satisfying condition (I). The predual space V is defined in the proof of the theorem above as

$$V = \left\{ \ell \in \mathcal{X}^* : \ell|_{B_{\mathcal{X}}} \text{ is } \tau\text{-continuous} \right\},$$

in resemblance with (2.4.1). We end this section with identifications of the preduals of the spaces H_v^∞ and \mathcal{B}_v^∞ , which are especially important in the context of this thesis.

Lemma 2.4.2. [9, Example 2.1] *The closed unit ball of H_v^0 (respectively \mathcal{B}_v^0) is co-dense in the closed unit ball of H_v^∞ (respectively \mathcal{B}_v^∞) for any radial weight v .*

Proof. The proof of the Bloch case is similar to the H_v^∞ case, so we only prove the latter. Choose $f \in B_{H_v^\infty}$ arbitrarily, let $\{r_n\}_{n=1}^\infty$ be some sequence in $(0, 1)$ such that $r_n \rightarrow 1$ as

$n \rightarrow \infty$ and define $f_n(z) := f(r_n z)$. Obviously $f_n \in H^\infty \subset H_v^0$ for every $n \in \mathbb{N}$. To show that the sequence $\{f_n\}_{n=1}^\infty$ is contained in the closed unit ball of H_v^0 , notice that for each $z \in \mathbb{D}$ there exists a number $\lambda_z \in \partial\mathbb{D}$ such that $|f(r_n z)| \leq |f(\lambda_z z)|$ for every $n \in \mathbb{N}$. This follows from the maximum modulus theorem. Hence, by the radiality of v , we have that

$$\|f_n\|_{H_v^\infty} = \sup_{z \in \mathbb{D}} v(z) |f(r_n z)| \leq \sup_{z \in \mathbb{D}} v(\lambda_z z) |f(\lambda_z z)| \leq \|f\|_{H_v^\infty} \leq 1,$$

showing that $\{f_n\}_{n=1}^\infty \subset B_{H_v^0}$. To finish the proof we need to show that $f_n \xrightarrow{co} f$, so let $K \subset \mathbb{D}$ be an arbitrary compact set, let $\varepsilon > 0$ and choose a radius $0 < r_K < 1$ such that $K \subset \overline{D(0, r_K)}$. Since f in particular is uniformly continuous on any compact subset of the unit disc, there exists some number $\delta_{\varepsilon, K} > 0$ such that $|f(z) - f(w)| < \varepsilon$ whenever $z, w \in \overline{D(0, r_K)}$ and $|z - w| < \delta_{\varepsilon, K}$. Now if $n_\varepsilon \in \mathbb{N}$ is such that $(1 - r_n)r_K < \delta_{\varepsilon, K}$ when $n > n_\varepsilon$, then for these integers n it also holds that

$$\sup_{z \in K} |f_n(z) - f(z)| \leq \sup_{|z| \leq r_K} |f(r_n z) - f(z)| < \varepsilon,$$

and the proof is complete. \square

Theorem 2.4.3. *If the weight v is normal, then the predual spaces ${}^*H_v^\infty$ and ${}^*\mathcal{B}_v^\infty$ are both isomorphic to the sequence space ℓ^1 .*

Proof. By Lemma 2.4.2 and [9, Theorem 1.1 (b)], the restriction map $\rho: {}^*H_v^\infty \rightarrow (H_v^0)^*$, mapping $\ell \mapsto \ell|_{H_v^0}$, is an isometric isomorphism. Since $H_v^0 \approx c_0$ for normal weights v , we conclude that ${}^*H_v^\infty \approx \ell^1$. The proof of ${}^*\mathcal{B}_v^\infty \approx \ell^1$ is similar, see for example the proof of [22, Theorem 5.1]. \square

Chapter 3

Operators on function spaces

This chapter contains the most central results on weighted composition operators, on the Königs eigenfunctions related to them, as well as on generalized Volterra operators obtained in the papers constituting this thesis. In order to discuss these topics, we first recall some basic operator theoretic properties such as compactness, weak compactness and spectrum, and also study analytic selfmaps of the unit disc which are used to define the mentioned operators.

3.1 General operator theoretic properties

Let \mathcal{X} and \mathcal{Y} be normed spaces and let $\mathcal{L}(\mathcal{X}, \mathcal{Y})$ denote the space of all continuous linear operators mapping from \mathcal{X} into \mathcal{Y} . We also denote $\mathcal{L}(\mathcal{X}) := \mathcal{L}(\mathcal{X}, \mathcal{X})$, with similar abbreviation for other collections of operators on the same space \mathcal{X} . It is a well-known fact that a linear operator $T: \mathcal{X} \rightarrow \mathcal{Y}$ is continuous if and only if it is *bounded*, in the sense that the image $T(B)$ of any bounded subset B of \mathcal{X} is a bounded subset of \mathcal{Y} , which in turn is precisely the same as stating that the *operator norm*

$$\|T\|_{\mathcal{X} \rightarrow \mathcal{Y}} = \sup \{ \|T(x)\|_{\mathcal{Y}} : \|x\|_{\mathcal{X}} \leq 1 \}$$

is finite. In fact, $\mathcal{L}(\mathcal{X}, \mathcal{Y})$ is a normed space equipped with this operator norm, and furthermore a Banach space if the target space \mathcal{Y} itself is a Banach space. In the same manner, an operator $T: \mathcal{X} \rightarrow \mathcal{Y}$ is said to be *bounded below* if there is some constant $c > 0$ such that $\|T(x)\|_{\mathcal{Y}} \geq c\|x\|_{\mathcal{X}}$ for every $x \in \mathcal{X}$. The *null space* of $T \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ is the closed subspace of \mathcal{X} given by

$$N(T) = \{x \in \mathcal{X} : T(x) = 0\}$$

and the *range* of T is denoted by

$$R(T) = \{T(x) : x \in \mathcal{X}\}.$$

If $\dim R(T)$ is finite then the operator $T: \mathcal{X} \rightarrow \mathcal{Y}$ is said to have *finite rank*, and the collection of all such operators is denoted by $\mathcal{F}(\mathcal{X}, \mathcal{Y})$.

When working with operators mapping between Banach spaces of analytic functions on the unit disc, one usually only needs to check that the operators are well-defined in order to ensure that they are bounded.

Theorem 3.1.1. *Let $\mathcal{X}, \mathcal{Y} \subset H(\mathbb{D})$ be Banach spaces with norm topologies finer than the compact open topology and assume that $T: \mathcal{H}(\mathbb{D}) \rightarrow \mathcal{H}(\mathbb{D})$ is a co-co continuous linear operator. Then $T \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ if and only if $T(\mathcal{X}) \subset \mathcal{Y}$. In other words, $T: \mathcal{X} \rightarrow \mathcal{Y}$ is bounded precisely when it is well-defined.*

Proof. Assume that $T: \mathcal{X} \rightarrow \mathcal{Y}$ is well-defined and let $\{f_n\}_{n=1}^\infty \subset \mathcal{X}$ be a sequence such that $f_n \rightarrow f$ in \mathcal{X} and $T(f_n) \rightarrow g$ in \mathcal{Y} . By the assumption on the norm topologies we then have that $f_n \xrightarrow{co} f$ and $T(f_n) \xrightarrow{co} g$. But since also $T(f_n) \xrightarrow{co} T(f)$, we must have that $T(f) = g$ and hence $T: \mathcal{X} \rightarrow \mathcal{Y}$ is bounded by the closed graph theorem. \square

An operator $T \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ is *compact* if the image $T(B)$ is a relatively compact subset of \mathcal{Y} , meaning that the closure $\overline{T(B)}$ is compact, whenever B is a bounded subset of \mathcal{X} . The set $\mathcal{K}(\mathcal{X}, \mathcal{Y})$ of compact operators mapping from \mathcal{X} into \mathcal{Y} is a closed subspace of $\mathcal{L}(\mathcal{X}, \mathcal{Y})$, and hence one can gather operators $T \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ into equivalence classes $T + \mathcal{K}(\mathcal{X}, \mathcal{Y})$ via the quotient space $\mathcal{L}(\mathcal{X}, \mathcal{Y})/\mathcal{K}(\mathcal{X}, \mathcal{Y})$, thus identifying operators differing only by a compact operator. The *essential norm* of a bounded linear operator $T: \mathcal{X} \rightarrow \mathcal{Y}$ is defined to be the quotient norm

$$\|T\|_{e, \mathcal{X} \rightarrow \mathcal{Y}} = \|T + \mathcal{K}(\mathcal{X}, \mathcal{Y})\|_{\mathcal{L}(\mathcal{X}, \mathcal{Y})/\mathcal{K}(\mathcal{X}, \mathcal{Y})} = \inf\{\|T - K\|_{\mathcal{X} \rightarrow \mathcal{Y}} : K \in \mathcal{K}(\mathcal{X}, \mathcal{Y})\}.$$

As the last expression suggests, the essential norm gives the distance from T to the compact operators. Notice that since $\mathcal{K}(\mathcal{X}, \mathcal{Y})$ is closed in $\mathcal{L}(\mathcal{X}, \mathcal{Y})$, an operator $T: \mathcal{X} \rightarrow \mathcal{Y}$ is compact if and only if $\|T\|_{e, \mathcal{X} \rightarrow \mathcal{Y}} = 0$. When the considered spaces \mathcal{X} and \mathcal{Y} are Banach spaces, one has the following equivalent characterizations of compact operators. Recall that a subset of a metric space is *totally bounded* if for every $\varepsilon > 0$ there exists a finite number of balls with radius equal to ε covering the set.

Theorem 3.1.2. *Let \mathcal{X} and \mathcal{Y} be Banach spaces and assume that $T \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$. Then the following conditions are equivalent:*

- (a) *The operator $T: \mathcal{X} \rightarrow \mathcal{Y}$ is compact.*
- (b) *The image $T(B)$ of any bounded subset B of \mathcal{X} is totally bounded in \mathcal{Y} .*
- (c) *Every bounded sequence $\{x_n\}_{n=1}^\infty$ in \mathcal{X} has a subsequence $\{x_{n_k}\}_{k=1}^\infty$ for which the image sequence $\{T(x_{n_k})\}_{k=1}^\infty$ converges in \mathcal{Y} .*

Every finite rank linear operator is obviously compact. A Banach space \mathcal{X} is said to have the *approximation property* if for every Banach space \mathcal{Y} the set of finite rank operators $\mathcal{F}(\mathcal{X}, \mathcal{Y})$ is dense in $\mathcal{K}(\mathcal{X}, \mathcal{Y})$. Alexander Grothendieck described these spaces in 1955 as the ones on which the identity operator can be uniformly approximated on compact subsets by finite rank operators:

Theorem 3.1.3. *A Banach space \mathcal{X} has the approximation property if and only if for every compact subset $K \subset \mathcal{X}$ and every $\varepsilon > 0$ there exists a finite rank operator $T_{K,\varepsilon} : \mathcal{X} \rightarrow \mathcal{X}$ such that*

$$\sup_{x \in K} \|T_{K,\varepsilon}(x) - x\|_{\mathcal{X}} < \varepsilon.$$

A stronger condition on the space \mathcal{X} is the λ -metric approximation property for a fixed constant $\lambda > 0$, where one requires the same thing as in Theorem 3.1.3, but also that $\|T_{K,\varepsilon}\|_{\mathcal{X} \rightarrow \mathcal{X}} \leq \lambda$. The sequence spaces c_0 and ℓ^p for $1 \leq p \leq \infty$ have the 1-metric approximation property, see [27, Section 18.5]. As a consequence, for example the growth space H_v^∞ has the λ -metric approximation property for some scalar $\lambda > 0$ if the weight v is normal, because then $H_v^\infty \approx \ell^\infty$. The reason why one cannot choose $\lambda = 1$ here is that the mentioned isomorphism is not necessarily isometric.

The *weak topology* of a normed space \mathcal{X} is the topology $\sigma(\mathcal{X}, \mathcal{X}^*)$ induced by the dual space \mathcal{X}^* as the topologizing family. It is thus the smallest topology on \mathcal{X} with respect to which every element in \mathcal{X}^* is continuous. A net $\{x_\alpha\}$ in \mathcal{X} is said to *converge weakly* to $x \in \mathcal{X}$, denoted by $x_\alpha \xrightarrow{w} x$, if $\lim_\alpha \ell(x_\alpha) = \ell(x)$ for every $\ell \in \mathcal{X}^*$. Using this weak topology, one defines an operator $T \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ to be *weakly compact* if the image $T(B)$ is a relatively weakly compact subset of \mathcal{Y} , in the sense that $\overline{T(B)}$ is weakly compact, whenever B is a bounded subset of \mathcal{X} . The collection of weakly compact operators $T : \mathcal{X} \rightarrow \mathcal{Y}$ is denoted by $\mathcal{W}(\mathcal{X}, \mathcal{Y})$, and since obviously all compact operators are weakly compact it holds that

$$\mathcal{K}(\mathcal{X}, \mathcal{Y}) \subset \mathcal{W}(\mathcal{X}, \mathcal{Y}) \subset \mathcal{L}(\mathcal{X}, \mathcal{Y}).$$

As in the compactness case, weak compactness of operators can also be characterized using sequences. The fact that one is allowed to consider sequences instead of nets is due to the *Eberlein-Šmulian theorem*.

Theorem 3.1.4. *Let \mathcal{X} and \mathcal{Y} be Banach spaces and assume that $T \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$. Then the following conditions are equivalent:*

- (a) *The operator $T : \mathcal{X} \rightarrow \mathcal{Y}$ is weakly compact.*
- (b) *Every bounded sequence $\{x_n\}_{n=1}^\infty$ in \mathcal{X} has a subsequence $\{x_{n_k}\}_{k=1}^\infty$ such that the image sequence $\{T(x_{n_k})\}_{k=1}^\infty$ converges weakly in \mathcal{Y} .*

When considering operators mapping between Banach spaces of analytic functions on the unit disc, one can under very general assumptions think of compactness and weak compactness of operators as properties that strengthen uniform convergence on compact subsets of the unit disc into norm respectively weak convergence in the target space.

Lemma 3.1.5. [15, Lemma 3.3] *Let $\mathcal{X} \subset \mathcal{H}(\mathbb{D})$ be a Banach space satisfying condition (I) and $\mathcal{Y} \subset \mathcal{H}(\mathbb{D})$ be a Banach space such that point evaluation functionals on \mathcal{Y} are bounded. Assume that $T : \mathcal{X} \rightarrow \mathcal{Y}$ is a co-co continuous linear operator. Then $T : \mathcal{X} \rightarrow \mathcal{Y}$ is compact (respectively weakly compact) if and only if $\{T(f_n)\}_{n=1}^\infty$ converges to zero in the norm (respectively in the weak topology) of \mathcal{Y} for each bounded sequence $\{f_n\}_{n=1}^\infty$ in \mathcal{X} such that $f_n \rightarrow 0$ uniformly on compact subsets of \mathbb{D} .*

The concepts of boundedness, compactness and weak compactness transfeere naturally to the situation of dual spaces. Namely, to any operator $T \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ there corresponds a *dual operator* $T^*: \mathcal{Y}^* \rightarrow \mathcal{X}^*$, defined for $\ell \in \mathcal{Y}^*$ and $x \in \mathcal{X}$ as

$$T^*(\ell)(x) = \ell(T(x)),$$

which preserves the mentioned properties. This is stated in Theorem 3.1.6 below, of which part (b) is due to Juliusz Schauder (1930) and part (c) is also known as the *Gantmacher theorem* after Vera Gantmacher (1940).

Theorem 3.1.6. *Let \mathcal{X} and \mathcal{Y} be Banach spaces and let $T: \mathcal{X} \rightarrow \mathcal{Y}$ be a linear operator. Then it holds that*

- (a) $T \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ if and only if $T^* \in \mathcal{L}(\mathcal{Y}^*, \mathcal{X}^*)$.
- (b) $T \in \mathcal{K}(\mathcal{X}, \mathcal{Y})$ if and only if $T^* \in \mathcal{K}(\mathcal{Y}^*, \mathcal{X}^*)$.
- (c) $T \in \mathcal{W}(\mathcal{X}, \mathcal{Y})$ if and only if $T^* \in \mathcal{W}(\mathcal{Y}^*, \mathcal{X}^*)$.

The *weak* topology* of the dual \mathcal{X}^* of a normed space \mathcal{X} is the topology $\sigma(\mathcal{X}^*, \mathcal{X})$ induced by the image $\widehat{\mathcal{X}} := Q(\mathcal{X}) = \{\widehat{x} : x \in \mathcal{X}\}$ of \mathcal{X} under the canonical embedding $Q: \mathcal{X} \rightarrow \mathcal{X}^{**}$ as the topologizing family. It is the smallest topology on \mathcal{X}^* with respect to which every element in $\widehat{\mathcal{X}}$ is continuous. A net $\{\ell_\alpha\} \subset \mathcal{X}^*$ converges *weak** to $\ell \in \mathcal{X}^*$ if it converges in the topology $\sigma(\mathcal{X}^*, \mathcal{X})$, meaning that $\ell_\alpha(x) \rightarrow \ell(x)$ for every $x \in \mathcal{X}$. The weak* topology $\sigma(\mathcal{X}^*, \mathcal{X})$ is always included in the weak topology $\sigma(\mathcal{X}^*, \mathcal{X}^{**})$ of the dual space \mathcal{X}^* , which in turn is included in the norm topology of \mathcal{X}^* . Banach spaces \mathcal{X} such that weak and weak* convergence of sequences in \mathcal{X}^* coincide are called *Grothendieck spaces*. For example ℓ^∞ has this property. One benefit of using topologies containing less open sets is that it enables more sets to become compact, as demonstrated by the *Banach-Alaoglu theorem*.

Theorem 3.1.7. *The closed unit ball $B_{\mathcal{X}^*}$ of the dual of a normed space \mathcal{X} is weak* compact.*

Using the weak* topology one can relate weak compactness of operators $T \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ to their dual operators in the following way:

Theorem 3.1.8. *Let \mathcal{X} and \mathcal{Y} be Banach spaces and assume that $T \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$. Then the following conditions are equivalent:*

- (a) *The operator $T: \mathcal{X} \rightarrow \mathcal{Y}$ is weakly compact.*
- (b) *The dual operator $T^*: \mathcal{Y}^* \rightarrow \mathcal{X}^*$ is $\sigma(\mathcal{Y}^*, \mathcal{Y}) - \sigma(\mathcal{X}^*, \mathcal{X}^{**})$ continuous.*
- (c) *The range of the second dual operator satisfies $T^{**}(\mathcal{X}^{**}) \subset \widehat{\mathcal{Y}}$.*

Again returning to the unit disc, and recalling the predual spaces ${}^*\mathcal{X}$ discussed in section 2.4, one can prove the existence of a *predual operator* $S = {}^*T$ under very general assumptions. The name comes from the fact that the operator $({}^*T)^*$ can be identified with T since $({}^*T)^* = \Phi_{\mathcal{Y}} \circ T \circ \Phi_{\mathcal{X}}^{-1}$, as seen below.

Lemma 3.1.9. *Let $\mathcal{X}, \mathcal{Y} \subset H(\mathbb{D})$ be Banach spaces satisfying condition (I). If the operator $T: \mathcal{X} \rightarrow \mathcal{Y}$ is bounded and the restriction $T|_{B_{\mathcal{X}}}$ is co-co continuous, then the operator*

$$\Phi_{\mathcal{Y}} \circ T \circ \Phi_{\mathcal{X}}^{-1}: (*\mathcal{X})^* \rightarrow (*\mathcal{Y})^*$$

is $\sigma((\mathcal{X})^*, *\mathcal{X}) - \sigma((*\mathcal{Y})^*, *\mathcal{Y})$ continuous, and consequently there exists a bounded operator $S: *\mathcal{Y} \rightarrow *\mathcal{X}$ such that $T = \Phi_{\mathcal{Y}}^{-1} \circ S^* \circ \Phi_{\mathcal{X}}$.*

Proof. Choose an arbitrary net $\{\ell_{\gamma} = \Phi_{\mathcal{X}}(f_{\gamma})\} \subset (*\mathcal{X})^*$ such that $\ell_{\gamma} \xrightarrow{w^*} 0$, where each $f_{\gamma} \in \mathcal{X}$, and let $u \in *\mathcal{Y}$. We have that $u \circ T \in *\mathcal{X}$, since if $\{h_n\}_{n=1}^{\infty} \subset B_{\mathcal{X}}$ is such that $h_n \xrightarrow{co} 0$ then $T(h_n) \xrightarrow{co} 0$ by assumption, and hence

$$\lim_{n \rightarrow \infty} u(T(h_n)) = \|T\|_{\mathcal{X} \rightarrow \mathcal{Y}} \lim_{n \rightarrow \infty} u \left(\frac{T(h_n)}{\|T\|_{\mathcal{X} \rightarrow \mathcal{Y}}} \right) = 0$$

because $u|_{B_{\mathcal{Y}}}$ is co-continuous and

$$\left\| \frac{T(h_n)}{\|T\|_{\mathcal{X} \rightarrow \mathcal{Y}}} \right\|_{\mathcal{Y}} \leq \|h_n\|_{\mathcal{X}} \leq 1.$$

Now from

$$\begin{aligned} ((\Phi_{\mathcal{Y}} \circ T \circ \Phi_{\mathcal{X}}^{-1})(\ell_{\gamma}))(u) &= ((\Phi_{\mathcal{Y}} \circ T \circ \Phi_{\mathcal{X}}^{-1})(\Phi_{\mathcal{X}}(f_{\gamma})))(u) \\ &= (\Phi_{\mathcal{Y}} \circ T(f_{\gamma}))(u) = u(T(f_{\gamma})) \\ &= (\Phi_{\mathcal{X}}(f_{\gamma}))(u \circ T) = \ell_{\gamma}(u \circ T) \end{aligned}$$

it follows that

$$(\Phi_{\mathcal{Y}} \circ T \circ \Phi_{\mathcal{X}}^{-1})(\ell_{\gamma}) \xrightarrow{w^*} 0,$$

and hence $\Phi_{\mathcal{Y}} \circ T \circ \Phi_{\mathcal{X}}^{-1}: (*\mathcal{X})^* \rightarrow (*\mathcal{Y})^*$ is w^* - w^* continuous. The last statement of the lemma follows from [36, Theorem 3.1.11]. \square

Now we turn to discuss some basic spectral theory of linear operators. A bounded linear operator $T: \mathcal{X} \rightarrow \mathcal{Y}$ between normed spaces \mathcal{X} and \mathcal{Y} is said to be *invertible* in $\mathcal{L}(\mathcal{X}, \mathcal{Y})$ if there exists an operator $T^{-1} \in \mathcal{L}(\mathcal{Y}, \mathcal{X})$ such that $T \circ T^{-1} = I_{\mathcal{Y}}$ and $T^{-1} \circ T = I_{\mathcal{X}}$, where $I_{\mathcal{X}}$ denotes the identity operator on the space \mathcal{X} . If such an operator T^{-1} exists then it has to be unique and is called the *inverse operator* of T . Using the open mapping theorem, and assuming that \mathcal{X} and \mathcal{Y} are Banach spaces, one can show that $T \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ is invertible if and only if it is both injective and surjective, or in other words, bijective. It is a well-known fact that operators mapping between finite dimensional vector spaces \mathcal{V} and \mathcal{W} are surjective precisely when they are injective. Such operators can also be viewed as matrices, and if $T \in \mathcal{L}(\mathcal{V})$ then one can study its *eigenvalues*, that is, the numbers $\lambda \in \mathbb{C}$ such that $T(v) = \lambda v$ for some non-zero vector $v \in \mathcal{V}$. The concept of eigenvalues is generalized to the situation of arbitrary Banach spaces \mathcal{X} by defining the

spectrum of an operator $T \in \mathcal{L}(\mathcal{X})$ to be the non-empty, compact subset of the complex plane given by

$$\sigma_{\mathcal{X}}(T) = \left\{ \lambda \in \mathbb{C} : T - \lambda I_{\mathcal{X}} \text{ is not invertible in } \mathcal{L}(\mathcal{X}) \right\}.$$

The corresponding *spectral radius*

$$r_{\mathcal{X}}(T) = \max \{ |\lambda| : \lambda \in \sigma_{\mathcal{X}}(T) \}$$

is the radius of the smallest disc centered at the origin that contains the spectrum, and it can be computed using the *Gelfand formula*

$$r_{\mathcal{X}}(T) = \lim_{n \rightarrow \infty} \|T^n\|_{\mathcal{X} \rightarrow \mathcal{X}}^{\frac{1}{n}} = \inf_{n \in \mathbb{N}} \|T^n\|_{\mathcal{X} \rightarrow \mathcal{X}}^{\frac{1}{n}},$$

from which one sees that $r_{\mathcal{X}}(T) \leq \|T\|_{\mathcal{X} \rightarrow \mathcal{X}}$. The complement $\rho_{\mathcal{X}}(T) = \mathbb{C} \setminus \sigma_{\mathcal{X}}(T)$ of the spectrum is called the *resolvent set* and on this one defines the *resolvent function* $R_{\mathcal{X}}: \rho_{\mathcal{X}}(T) \times \mathcal{L}(\mathcal{X}) \rightarrow \mathcal{L}(\mathcal{X})$, mapping $(\lambda, T) \mapsto (T - \lambda I_{\mathcal{X}})^{-1}$, which can be expanded into a *Neumann series*

$$R_{\mathcal{X}}(\lambda, T) = (T - \lambda I_{\mathcal{X}})^{-1} = - \sum_{n=0}^{\infty} \lambda^{-(n+1)} T^n$$

converging in operator norm for $|\lambda| > r_{\mathcal{X}}(T)$. The resolvent function plays a central role in functional calculus when defining arbitrary functions $f(T)$ of operators $T \in \mathcal{L}(\mathcal{X})$. Namely, let $\mathcal{F}_{\mathcal{X}}(T)$ be the collection of all complex-valued functions that are analytic in some open set containing the spectrum $\sigma_{\mathcal{X}}(T)$. Then one defines the image of T under $f \in \mathcal{F}_{\mathcal{X}}(T)$ via the Cauchy-type integral

$$f(T) = -\frac{1}{2\pi i} \int_{\Gamma} f(\lambda) R_{\mathcal{X}}(\lambda, T) d\lambda,$$

where the integration is performed along some Jordan curve Γ surrounding $\sigma_{\mathcal{X}}(T)$ that is contained in the domain of analyticity of f . The development of this theory results for example in the *spectral mapping theorem*.

Theorem 3.1.10. *Let \mathcal{X} be a Banach space and assume that $T \in \mathcal{L}(\mathcal{X})$. Then for every $f \in \mathcal{F}_{\mathcal{X}}(T)$ it holds that*

$$\sigma_{\mathcal{X}}(f(T)) = f(\sigma_{\mathcal{X}}(T)) = \left\{ f(\lambda) : \lambda \in \sigma_{\mathcal{X}}(T) \right\}.$$

As an application of the spectral mapping theorem, we have the following result that might be useful if one for example is about to prove that some spectrum takes the form of an annulus.

Corollary 3.1.11. *Let \mathcal{X} be a Banach space and $T \in \mathcal{L}(\mathcal{X})$. If $\mu \in \mathbb{C}$ is a number such that $|\mu| = r_{\mathcal{X}}(T)$ and $r_{\mathcal{X}}(T + \mu I_{\mathcal{X}}) \geq 2r_{\mathcal{X}}(T)$, then $\mu \in \sigma_{\mathcal{X}}(T)$.*

Proof. By the spectral mapping theorem applied to the function $f(z) = z + \mu$ it holds that

$$\sigma_{\mathcal{X}}(T + \mu I_{\mathcal{X}}) = \{\lambda + \mu : \lambda \in \sigma_{\mathcal{X}}(T)\},$$

and hence one can choose $\lambda_0 \in \sigma_{\mathcal{X}}(T)$ such that $|\lambda_0 + \mu| = r_{\mathcal{X}}(T + \mu I_{\mathcal{X}}) = 2r_{\mathcal{X}}(T)$, where the last equality follows from the assumptions on the spectral radii. Using the triangle inequality one gets that $|\lambda_0| = r_{\mathcal{X}}(T)$, and since also $|\mu| = r_{\mathcal{X}}(T)$ and $|\lambda_0 + \mu| = 2r_{\mathcal{X}}(T)$ we must have that $\mu = -(-\lambda_0) = \lambda_0 \in \sigma_{\mathcal{X}}(T)$. \square

When studying the spectrum $\sigma_{\mathcal{X}}(T)$ it is sometimes useful to divide it into separate parts, depending on how the operator $T - \lambda I_{\mathcal{X}}$ fails to be invertible. The *point spectrum*

$$\sigma_{p,\mathcal{X}}(T) = \{\lambda \in \sigma_{\mathcal{X}}(T) : T - \lambda I_{\mathcal{X}} \text{ is not injective}\}$$

consists of the eigenvalues of T , and for the values of $\lambda \in \sigma_{\mathcal{X}}(T)$ for which the operator $T - \lambda I_{\mathcal{X}}$ is injective but not surjective one distinguishes between the cases when it has dense range or not, to obtain the *continuous spectrum*

$$\sigma_{c,\mathcal{X}}(T) = \{\lambda \in \sigma_{\mathcal{X}}(T) \setminus \sigma_{p,\mathcal{X}}(T) : \overline{(T - \lambda I_{\mathcal{X}})(\mathcal{X})} = \mathcal{X}\}$$

and the *residual spectrum*

$$\sigma_{r,\mathcal{X}}(T) = \{\lambda \in \sigma_{\mathcal{X}}(T) \setminus \sigma_{p,\mathcal{X}}(T) : \overline{(T - \lambda I_{\mathcal{X}})(\mathcal{X})} \neq \mathcal{X}\}.$$

By definition the sets $\sigma_{p,\mathcal{X}}(T)$, $\sigma_{c,\mathcal{X}}(T)$ and $\sigma_{r,\mathcal{X}}(T)$ are pairwise disjoint and

$$\sigma_{\mathcal{X}}(T) = \sigma_{p,\mathcal{X}}(T) \cup \sigma_{c,\mathcal{X}}(T) \cup \sigma_{r,\mathcal{X}}(T).$$

Another way to look at the spectrum is to write it as the not necessarily disjoint union

$$\sigma_{\mathcal{X}}(T) = \sigma_{a,\mathcal{X}}(T) \cup \sigma_{r,\mathcal{X}}(T),$$

where the *approximate point spectrum*

$$\sigma_{a,\mathcal{X}}(T) = \{\lambda \in \mathbb{C} : T - \lambda I_{\mathcal{X}} \text{ is not bounded below}\}$$

consists of all numbers $\lambda \in \mathbb{C}$ for which there exist some sequence $\{x_n\}_{n=1}^{\infty} \subset \mathcal{X}$ of unit vectors such that $T(x_n) - \lambda x_n \rightarrow 0$ in the norm of \mathcal{X} as $n \rightarrow \infty$. For this reason they are also known as *approximate eigenvalues*, and we obviously have that $\sigma_{p,\mathcal{X}}(T) \subset \sigma_{a,\mathcal{X}}(T)$, and also $\sigma_{c,\mathcal{X}}(T) \subset \sigma_{a,\mathcal{X}}(T)$. The approximate point spectrum is interesting partly because it is a closed set containing the boundary of the spectrum, that is $\partial\sigma_{\mathcal{X}}(T) \subset \sigma_{a,\mathcal{X}}(T)$.

As an important example of computing the spectrum, we give the spectrum for compact operators. As mentioned before, compact operators can be thought of as generalizations of finite rank operators, which is also illustrated through the simplicity of their spectrum.

Theorem 3.1.12. *Let \mathcal{X} be an infinite dimensional Banach space and assume that the operator $T \in \mathcal{L}(\mathcal{X})$ is compact. Then the spectrum of T is a countable set given by $\sigma_{\mathcal{X}}(T) = \{0\} \cup \sigma_{p,\mathcal{X}}(T)$. If the spectrum is countably infinite, say $\sigma_{\mathcal{X}}(T) = \{\lambda_n : n \in \mathbb{N}\}$, then $\lambda_n \rightarrow 0$ as $n \rightarrow \infty$.*

Now let \mathcal{X} be a Banach space and consider the quotient space $\mathcal{C}(\mathcal{X}) := \mathcal{L}(\mathcal{X})/\mathcal{K}(\mathcal{X})$, also known as the *Calkin algebra*, and the corresponding quotient map $\pi_{\mathcal{X}}: \mathcal{L}(\mathcal{X}) \rightarrow \mathcal{C}(\mathcal{X})$, mapping $T \mapsto T + \mathcal{K}(\mathcal{X})$. The algebraic operations in $\mathcal{C}(\mathcal{X})$ are defined for $S, T \in \mathcal{L}(\mathcal{X})$ and $\lambda \in \mathbb{C}$ as follows:

$$\pi_{\mathcal{X}}(S) + \pi_{\mathcal{X}}(T) = \pi_{\mathcal{X}}(S + T), \quad \lambda \cdot \pi_{\mathcal{X}}(T) = \pi_{\mathcal{X}}(\lambda T) \quad \text{and} \quad \pi_{\mathcal{X}}(S)\pi_{\mathcal{X}}(T) = \pi_{\mathcal{X}}(S \circ T),$$

and the element $\pi_{\mathcal{X}}(I_{\mathcal{X}})$ serves as the unit of the Calkin algebra. An operator $T \in \mathcal{L}(\mathcal{X})$ is said to be *essentially invertible* if $\pi_{\mathcal{X}}(T)$ is invertible in $\mathcal{C}(\mathcal{X})$, or equivalently if there exists an operator $S \in \mathcal{L}(\mathcal{X})$ such that both $S \circ T - I_{\mathcal{X}} \in \mathcal{K}(\mathcal{X})$ and $T \circ S - I_{\mathcal{X}} \in \mathcal{K}(\mathcal{X})$. It can be proven that if $T \in \mathcal{L}(\mathcal{X})$ is essentially invertible, then the dimension of the null space $N(T)$ is finite and there exist a closed subspace \mathcal{V} and a finite dimensional subspace \mathcal{W} of \mathcal{X} such that

$$N(T) \oplus \mathcal{V} = \mathcal{X} = R(T) \oplus \mathcal{W}.$$

The restriction $T|_{\mathcal{V}}: \mathcal{V} \rightarrow R(T)$ is bijective, and the operator T is hence in some sense close to being an isomorphism. The essentially invertible operators actually coincide with the *Fredholm operators*, which are defined through the requirements that the *nullity* $n(T) := \dim N(T)$ and *defect* $d(T) := \dim \mathcal{X}/R(T)$ are both finite.

Using the Calkin algebra, one defines the *essential spectrum* of an operator $T \in \mathcal{L}(\mathcal{X})$ to be the spectrum of $\pi_{\mathcal{X}}(T)$ in $\mathcal{C}(\mathcal{X})$. In other words, it is the non-empty, compact subset of $\sigma_{\mathcal{X}}(T)$ given by

$$\sigma_{e,\mathcal{X}}(T) = \left\{ \lambda \in \mathbb{C} : \pi_{\mathcal{X}}(T - \lambda I_{\mathcal{X}}) \text{ is not invertible in } \mathcal{C}(\mathcal{X}) \right\},$$

thus consisting of the numbers $\lambda \in \mathbb{C}$ such that $T - \lambda I_{\mathcal{X}}$ is not a Fredholm operator. The *essential spectral radius*

$$r_{e,\mathcal{X}}(T) = \max \left\{ |\lambda| : \lambda \in \sigma_{e,\mathcal{X}}(T) \right\}$$

can be computed in a similar fashion as the spectral radius by iterating the operator and taking limits

$$r_{e,\mathcal{X}}(T) = \lim_{n \rightarrow \infty} \|T^n\|_{e,\mathcal{X} \rightarrow \mathcal{X}}^{\frac{1}{n}} = \inf_{n \in \mathbb{N}} \|T^n\|_{e,\mathcal{X} \rightarrow \mathcal{X}}^{\frac{1}{n}},$$

and hence it holds that $r_{e,\mathcal{X}}(T) \leq \|T\|_{e,\mathcal{X} \rightarrow \mathcal{X}}$, and obviously also $r_{e,\mathcal{X}}(T) \leq r_{\mathcal{X}}(T)$. Points $\lambda \in \sigma_{\mathcal{X}}(T)$ such that $|\lambda| > r_{e,\mathcal{X}}(T)$ are necessarily eigenvalues of the operator T as the following result shows.

Theorem 3.1.13. *Let $T: \mathcal{X} \rightarrow \mathcal{X}$ be a bounded linear operator on a complex Banach space \mathcal{X} and assume that $\lambda \in \sigma_{\mathcal{X}}(T) \setminus \sigma_{e,\mathcal{X}}(T)$. If there is a path lying outside of $\sigma_{e,\mathcal{X}}(T)$ joining λ with a point in the resolvent set $\rho_{\mathcal{X}}(T)$, then λ is an isolated point of the spectrum and hence belongs to $\sigma_{p,\mathcal{X}}(T)$.*

For more thorough treatments of linear operators mapping between Banach spaces, see for example the books by Abramovich and Aliprantis [1], Megginson [36] and Müller [37]. A good reference for operator theory in function spaces is the monograph [51] by Zhu.

3.2 Analytic selfmaps of the unit disc

One of the main objects of study in this thesis are composition operators, which are defined via *analytic selfmaps φ of the unit disc*. As the name itself tells, we are requiring that such a function satisfies $\varphi \in \mathcal{H}(\mathbb{D})$ and $\varphi(\mathbb{D}) \subset \mathbb{D}$. In this subsection we discuss some basic properties of these selfmaps, mainly focusing on their iterative behaviour. To begin with, recall that the bijective analytic selfmaps $\varphi: \mathbb{D} \rightarrow \mathbb{D}$, also known as *automorphisms of the unit disc*, are given precisely by the linear fractional transformations of the form

$$z \mapsto \lambda \psi_p(z),$$

where $p \in \mathbb{D}$ and $|\lambda| = 1$ are given constants and

$$\psi_p(z) = \frac{p - z}{1 - \bar{p}z}. \quad (3.2.1)$$

The automorphism ψ_p is particularly useful because it is selfinverse, that is $\psi_p^{-1} = \psi_p$, and interchanges the points 0 and p in the sense that $\psi_p(0) = p$ and $\psi_p(p) = 0$. The collection of disc automorphisms will be denoted by $\text{Aut}(\mathbb{D})$. Every non-constant linear fractional transformation, of the form

$$z \mapsto \frac{az + b}{cz + d}$$

where $a, b, c, d \in \mathbb{C}$ are constants such that $ad - bc \neq 0$, has at least one and at most two fixed points in the extended complex plane $\mathbb{C} \cup \{\infty\}$, except for the identity map $\text{id}(z) = z$ which of course fixes every point. The non-trivial elements of $\text{Aut}(\mathbb{D})$ are classified based on their fixed point configuration as follows: $\varphi \in \text{Aut}(\mathbb{D}) \setminus \{\text{id}\}$ is called *elliptic* if it has a fixed point in \mathbb{D} and another fixed point in the extended complex plane outside of the closed unit disc $\overline{\mathbb{D}}$, *parabolic* if it has only one fixed point that lies on the boundary $\partial\mathbb{D}$, and *hyperbolic* if it has two fixed points on the unit circle $\partial\mathbb{D}$. These are the only possibilities for automorphic selfmaps. The mentioned fixed points will be discussed in more detail in connection with the *Denjoy-Wolff theorem*, which is presented at the end of this section, see Theorem 3.2.3.

The situation becomes more complicated when the analytic selfmap $\varphi: \mathbb{D} \rightarrow \mathbb{D}$ is not assumed to be bijective. However, there are still much that can be said in this general case. For example, the analytic selfmaps are always contractive in the *pseudo-hyperbolic metric*

$$\rho(z, w) = |\psi_w(z)| = \left| \frac{w - z}{1 - \bar{w}z} \right|,$$

as shown by the *Schwarz-Pick inequality*.

Theorem 3.2.1. *If $\varphi: \mathbb{D} \rightarrow \mathbb{D}$ is an analytic selfmap of the unit disc, then for every $z, w \in \mathbb{D}$ we have that*

$$\rho(\varphi(z), \varphi(w)) \leq \rho(z, w),$$

where equality holds for some pair $z \neq w$ if and only if φ is an automorphism.

The pseudohyperbolic metric has proven to be the appropriate distance measure for studying selfmaps of the unit disc. Among other things, it is a central tool in the machinery used to prove the *Julia-Carathéodory theorem*, given in Theorem 3.2.2 below. In order to state this result, we need some additional terminology. For a given boundary point $\zeta \in \partial\mathbb{D}$ and angle $0 < \alpha < \pi$, we define the *non-tangential approach region* $\Omega(\zeta, \alpha)$ to be the convex hull

$$\Omega(\zeta, \alpha) = \text{conv}\left(D\left(0, \sin\left(\frac{\alpha}{2}\right)\right) \cup \{\zeta\}\right),$$

which is a subset of the unit disc that forms a sector with angle α at the vertex ζ . A function $f: \mathbb{D} \rightarrow \mathbb{C}$ is said to have *non-tangential limit* L at $\zeta \in \partial\mathbb{D}$, denoted by $\angle \lim_{z \rightarrow \zeta} f(z) = L$, if the limit

$$\lim_{z \rightarrow \zeta, z \in \Omega(\zeta, \alpha)} f(z)$$

exists and equals L for every angle $0 < \alpha < \pi$. An analytic selfmap φ of the unit disc is defined to have *angular derivative* $\varphi'(\zeta)$ at a boundary point $\zeta \in \partial\mathbb{D}$ if the non-tangential limit

$$\varphi'(\zeta) = \angle \lim_{z \rightarrow \zeta} \frac{\varphi(z) - \eta}{z - \zeta} \quad (3.2.2)$$

exists for some $\eta \in \partial\mathbb{D}$. An obvious necessary condition for the limit (3.2.2) to exist is that $\angle \lim_{z \rightarrow \zeta} \varphi(z) = \eta$, and hence there is no ambiguity in the definition of the derivative $\varphi'(\zeta)$. The existence of angular derivatives is clarified by the Julia-Carathéodory theorem.

Theorem 3.2.2. *Let $\varphi: \mathbb{D} \rightarrow \mathbb{D}$ be an analytic selfmap of the unit disc and let $\zeta \in \partial\mathbb{D}$. Then the following statements are equivalent:*

- (1) *The limit infimum $d(\zeta) := \liminf_{z \rightarrow \zeta} \frac{1 - |\varphi(z)|}{1 - |z|}$, where z approaches ζ unrestrictedly in the unit disc, is finite.*
- (2) *The angular derivative $\varphi'(\zeta) = \angle \lim_{z \rightarrow \zeta} \frac{\varphi(z) - \eta}{z - \zeta}$ of φ exists at ζ for some $\eta \in \partial\mathbb{D}$.*
- (3) *Both φ and φ' have non-tangential limits at ζ , and $\angle \lim_{z \rightarrow \zeta} \varphi(z) \in \partial\mathbb{D}$.*

When these conditions hold, then $\angle \lim_{z \rightarrow \zeta} \varphi(z) = \eta$ and

$$\angle \lim_{z \rightarrow \zeta} \varphi'(z) = \varphi'(\zeta) = \angle \lim_{z \rightarrow \zeta} \frac{\varphi(z) - \eta}{z - \zeta} = d(\zeta) \bar{\zeta} \eta,$$

where $d(\zeta)$ also can be obtained as the non-tangential limit

$$d(\zeta) = \angle \lim_{z \rightarrow \zeta} \frac{1 - |\varphi(z)|}{1 - |z|}.$$

When dealing with selfmaps $\varphi: \mathbb{D} \rightarrow \mathbb{D}$, it is customary to denote the n -fold composition of φ with itself by φ_n , that is

$$\varphi_n := \varphi^{\circ n} = \underbrace{\varphi \circ \varphi \circ \cdots \circ \varphi}_{n \text{ times}}, \quad n \in \mathbb{N},$$

with φ_0 representing the identity map. This convention will only be adopted for the symbol φ , otherwise subscripts are used for standard enumeration of sequences. The characterization of angular derivatives above has played a key role in the study of the asymptotic behaviour of the sequence $\{\varphi_n\}_{n=0}^{\infty}$. It turns out that the iterations of almost any analytic selfmap of the unit disc, except for the elliptic automorphisms discussed earlier, converge to some fixed point in $\overline{\mathbb{D}}$, even uniformly on compact subsets of the unit disc provided that φ is not the identity map, as the Denjoy-Wolff theorem below summarizes. For a proof, see the book by Shapiro [45, Chapter 5]. With a *boundary fixed point* of φ we mean a point $\zeta \in \partial\mathbb{D}$ such that $\angle \lim_{z \rightarrow \zeta} \varphi(z) = \zeta$.

Theorem 3.2.3. *Suppose φ is an analytic selfmap of \mathbb{D} that is not the identity map nor an elliptic automorphism.*

- (1) *If φ has a fixed point $p \in \mathbb{D}$, then it is unique, $|\varphi'(p)| < 1$ and $\varphi_n \xrightarrow{co} p$.*
- (2) *If φ has no fixed point in \mathbb{D} , then there is a unique point $\omega \in \partial\mathbb{D}$, called the Denjoy-Wolff point of φ , such that $\varphi_n \xrightarrow{co} \omega$. Furthermore, ω is a boundary fixed point of φ and the angular derivative exists at ω with $0 < \varphi'(\omega) \leq 1$.*
- (3) *If $\omega \in \partial\mathbb{D}$ is the Denjoy-Wolff point of φ and $\varphi'(\omega) < 1$, then the sequence $\{\varphi_n(z)\}_{n=0}^{\infty}$ of iterates converges non-tangentially to ω for each $z \in \mathbb{D}$.*

By means of the Denjoy-Wolff theorem, one can actually say more about the fixed points of parabolic and hyperbolic automorphisms discussed earlier. Namely, if $\varphi \in \text{Aut}(\mathbb{D})$ is parabolic, then the unique fixed point ω on the unit circle $\partial\mathbb{D}$ is actually the Denjoy-Wolff point of φ , and for this it holds that $\varphi'(\omega) = 1$. The iterations $\{\varphi_n(z)\}_{n=0}^{\infty}$ approach ω along an *oricycle*, that is, along the boundary circle of a *horodisc*

$$H(\omega, \lambda) = D\left(\frac{\omega}{1+\lambda}, \frac{\lambda}{1+\lambda}\right),$$

which is contained in \mathbb{D} and tangent to the unit circle at ω . Here $\lambda > 0$ is some constant determined by the initial point of the iteration. If φ is a hyperbolic automorphism, with Denjoy-Wolff point $\omega \in \partial\mathbb{D}$ and the other fixed point $\gamma \in \partial\mathbb{D}$, then $0 < \varphi'(\omega) < 1$ and $\varphi'(\gamma) = 1/\varphi'(\omega)$. In this case, ω is also called the *attractive fixed point* and γ the *repulsive fixed point* of φ , and the convergence of iterations to ω is described in Theorem 3.2.3 (3). When it comes to elliptic automorphisms φ , the iteration sequences $\{\varphi_n(z)\}_{n=0}^{\infty}$ are never convergent, except when starting at the fixed point $p \in \mathbb{D}$. Instead, the iterations wander around this fixed point on the boundary of a *pseudohyperbolic disc*

$$\Delta(p, r) = \{z \in \mathbb{D} : \rho(z, p) < r\},$$

with orbit either being a finite or dense subset of this boundary. The radius r of the pseudohyperbolic disc above is again determined by the starting point of the iteration.

As an application of the Julia-Carathéodory and Denjoy-Wolff theorems, we end the discussion on selfmaps of the unit disc with a useful limit formula, which for example is valid for parabolic and hyperbolic automorphisms. The proof is borrowed from [16].

Lemma 3.2.4. *If φ is an analytic selfmap of \mathbb{D} with Denjoy-Wolff point $\omega \in \partial\mathbb{D}$ then*

$$\lim_{n \rightarrow \infty} \left(1 - |\varphi_n(a)|\right)^{\frac{1}{n}} = \varphi'(\omega),$$

for any given starting point $a \in \mathbb{D}$.

Proof. Since the Denjoy-Wolff point ω in particular is a boundary fixed point of φ , we have by the Julia-Carathéodory theorem applied with $\zeta = \eta = \omega \in \partial\mathbb{D}$ that

$$\varphi'(\omega) = \liminf_{z \rightarrow \omega} \frac{1 - |\varphi(z)|}{1 - |z|} = \angle \lim_{z \rightarrow \omega} \frac{1 - |\varphi(z)|}{1 - |z|}. \quad (3.2.3)$$

Now if $0 < \varphi'(\omega) < 1$, then the iterates $\{\varphi_k(a)\}_{k=0}^{\infty}$ converge non-tangentially to ω by Theorem 3.2.3 (3). Hence, using the limit in (3.2.3) we see that the sequence $\{z_k\}_{k=1}^{\infty}$, defined as

$$z_k = \log \left(\frac{1 - |\varphi_k(a)|}{1 - |\varphi_{k-1}(a)|} \right) = \log \left(\frac{1 - |\varphi(\varphi_{k-1}(a))|}{1 - |\varphi_{k-1}(a)|} \right), \quad (3.2.4)$$

converges to $\log \varphi'(\omega)$, and so does the corresponding sequence of arithmetic means. Using this one obtains the desired limit

$$\lim_{n \rightarrow \infty} \left(1 - |\varphi_n(a)|\right)^{\frac{1}{n}} = \lim_{n \rightarrow \infty} \left(\prod_{k=1}^n \frac{1 - |\varphi_k(a)|}{1 - |\varphi_{k-1}(a)|} \right)^{\frac{1}{n}} = \lim_{n \rightarrow \infty} e^{\frac{1}{n} \sum_{k=1}^n z_k} = e^{\log \varphi'(\omega)} = \varphi'(\omega).$$

If on the other hand $\varphi'(\omega) = 1$, then by (3.2.3) it holds that

$$1 = \varphi'(\omega) = \liminf_{z \rightarrow \omega} \frac{1 - |\varphi(z)|}{1 - |z|} \leq \liminf_{k \rightarrow \infty} \frac{1 - |\varphi_k(a)|}{1 - |\varphi_{k-1}(a)|}.$$

Let $\{z_k\}_{k=1}^{\infty}$ again denote the sequence in (3.2.4). From the above estimate, and the obvious fact that $1 - |\varphi_n(a)| < 1$ for every $n \in \mathbb{N}$, it follows that

$$\begin{aligned} 1 &\geq \limsup_{n \rightarrow \infty} \left(1 - |\varphi_n(a)|\right)^{\frac{1}{n}} \geq \liminf_{n \rightarrow \infty} \left(1 - |\varphi_n(a)|\right)^{\frac{1}{n}} \\ &= e^{\liminf_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n z_k} \geq e^{\liminf_{k \rightarrow \infty} z_k} = \liminf_{k \rightarrow \infty} \frac{1 - |\varphi_k(a)|}{1 - |\varphi_{k-1}(a)|} \geq 1, \end{aligned}$$

showing that

$$\lim_{n \rightarrow \infty} \left(1 - |\varphi_n(a)|\right)^{\frac{1}{n}} = 1 = \varphi'(\omega),$$

and the proof is complete. \square

3.3 Weighted composition operators

The first main object of study in this thesis is the *weighted composition operator* uC_φ , which is a linear operator on $\mathcal{H}(\mathbb{D})$ defined by a given function $u \in \mathcal{H}(\mathbb{D})$ and an analytic selfmap $\varphi: \mathbb{D} \rightarrow \mathbb{D}$ of the unit disc as

$$uC_\varphi(f) = u \cdot (f \circ \varphi).$$

The choice $\varphi(z) = z$ leads to the *multiplication operator* M_u , whereas letting $u \equiv 1$ one obtains the *composition operator*

$$C_\varphi f = f \circ \varphi.$$

Among the earliest research related to unweighted composition operators C_φ are the papers written by Hardy, Littlewood and Riesz in the beginning of the 20th century. As an example, one can use the *Littlewood subordination theorem* in [33] to study boundedness of composition operators on the Hardy spaces. In fact, the operator $C_\varphi: H^p \rightarrow H^p$ is always bounded for any analytic selfmap φ of the unit disc, as seen below.

Theorem 3.3.1. *Let $\varphi: \mathbb{D} \rightarrow \mathbb{D}$ be an analytic selfmap of the unit disc and let $0 < p < \infty$. Then for every $f \in H^p$ it holds that*

$$\int_0^{2\pi} |f(\varphi(e^{i\theta}))|^p d\theta \leq \frac{1 + |\varphi(0)|}{1 - |\varphi(0)|} \int_0^{2\pi} |f(e^{i\theta})|^p d\theta.$$

From the theorem above one gets the following estimate of the operator norm

$$\|C_\varphi\|_{H^p \rightarrow H^p} \leq \left(\frac{1 + |\varphi(0)|}{1 - |\varphi(0)|} \right)^{\frac{1}{p}}. \quad (3.3.1)$$

However, the precise computation of the norm $\|C_\varphi\|_{H^p \rightarrow H^p}$ is still today an open problem, which reflects the degree of difficulty that questions related to composition operators might have. Of course, there are good estimates of this operator norm, and in special cases even exact formulas. For example, Eric Nordgren proved that one has equality in (3.3.1) if φ is an inner function, see [39]. This paper, entitled *Composition operators* and written by Nordgren in 1968, might be seen as the starting point of the systematic study of composition operators. To mention another milestone in the history of composition operators, Joel Shapiro gave a complete characterization of the compact composition operators on the Hardy space H^2 in 1987 by proving the essential norm formula

$$\|C_\varphi\|_{e, H^2 \rightarrow H^2} = \limsup_{|z| \rightarrow 1} \sqrt{\frac{N_\varphi(z)}{\log \frac{1}{|z|}}},$$

where

$$N_\varphi(z) = \begin{cases} \sum_{w \in \varphi^{-1}(z)} \log \frac{1}{|w|}, & \text{if } z \in \varphi(\mathbb{D}) \\ 0, & \text{if } z \in \mathbb{D} \setminus \varphi(\mathbb{D}) \end{cases}$$

is the *Nevanlinna counting function*, see [44]. For general information of composition operators on classical spaces of analytic functions the reader is referred to the monographs by Cowen and MacCluer [17] and Shapiro [45]. Among other things, it is easy to see after iterating the weighted composition operator $n \in \mathbb{N}$ times that

$$(uC_\varphi)^n f(z) = u(z)u(\varphi(z)) \cdots u(\varphi_{n-1}(z))f(\varphi_n(z)), \quad f \in \mathcal{H}(\mathbb{D}), z \in \mathbb{D},$$

which also can be stated as

$$(uC_\varphi)^n = u_{(n)}C_{\varphi_n},$$

where $u_{(0)} \equiv 1$ and

$$u_{(n)}(z) := \prod_{j=0}^{n-1} u \circ \varphi_j(z), \quad n \in \mathbb{N}, z \in \mathbb{D}.$$

These formulas are important for example when investigating spectral properties of uC_φ , such as computing spectral and essential spectral radii which requires iteration of the operator. When it comes to compactness characterizations, one can use Lemma 3.1.5 due to the following result.

Lemma 3.3.2. *Let $u \in \mathcal{H}(\mathbb{D})$ and φ be an analytic selfmap of \mathbb{D} . If $\{f_n\}_{n=1}^\infty \subset \mathcal{H}(\mathbb{D})$ and $f_n \xrightarrow{co} 0$, then $uC_\varphi(f_n) \xrightarrow{co} 0$. In other words, $uC_\varphi: \mathcal{H}(\mathbb{D}) \rightarrow \mathcal{H}(\mathbb{D})$ is co-co continuous.*

Proof. This follows immediately from the estimate

$$\sup_{z \in K} |uC_\varphi(f_n)(z)| = \sup_{z \in K} |u(z)f_n(\varphi(z))| \leq \max_{z \in K} |u(z)| \cdot \sup_{z \in \varphi(K)} |f_n(z)|,$$

which is valid for any compact subset $K \subset \mathbb{D}$. □

When studying weighted composition operators $uC_\varphi: \mathcal{X} \rightarrow \mathcal{Y}$ mapping between Banach spaces $\mathcal{X}, \mathcal{Y} \subset \mathcal{H}(\mathbb{D})$, one usually tries to characterize operator theoretic properties, such as boundedness, compactness and spectrum, in terms of function theoretic properties of the inducing symbols u and φ . In the first paper [23] constituting this thesis, we studied the spectrum of *invertible* weighted composition operators uC_φ acting on the classical Bloch space \mathcal{B} and the Dirichlet space \mathcal{D} . Previously, Hyvärinen, Lindström, Nieminen and Saukko had carried out a similar study on a very general class of Banach spaces of analytic functions, see [31] and in particular section 2.2 therein containing their axioms. However, the Bloch and Dirichlet spaces are not contained in this class since the bounded analytic functions H^∞ are not contained in the multiplier space

$$\mathcal{M}(\mathcal{X}) := \left\{ u \in \mathcal{H}(\mathbb{D}) : M_u : \mathcal{X} \rightarrow \mathcal{X} \text{ is bounded} \right\}$$

when \mathcal{X} is \mathcal{B} or \mathcal{D} , or in other words, these spaces do not satisfy condition (V) as noted previously in section 2.3. In the Bloch case for example, one has that $u \in \mathcal{M}(\mathcal{B})$ if and only if $u \in H^\infty$ and

$$\sup_{z \in \mathbb{D}} (1 - |z|^2) |u'(z)| \log \frac{e}{1 - |z|^2} < \infty,$$

see [40]. As a tool when working with invertible operators uC_φ on the Bloch and Dirichlet spaces, one has the following description by Bourdon from [12, Corollary 2.3].

Theorem 3.3.3. *Assume that \mathcal{X} is either the Bloch space \mathcal{B} or the Dirichlet space \mathcal{D} , and let $uC_\varphi: \mathcal{X} \rightarrow \mathcal{X}$ be a bounded weighted composition operator on \mathcal{X} . Then uC_φ is invertible on \mathcal{X} if and only if $u \in \mathcal{M}(\mathcal{X})$, u is bounded away from zero on \mathbb{D} and φ is an automorphism of \mathbb{D} . In such a case the inverse operator of $uC_\varphi: \mathcal{X} \rightarrow \mathcal{X}$ is also a weighted composition operator, given by*

$$(uC_\varphi)^{-1} = \frac{1}{u \circ \varphi^{-1}} C_{\varphi^{-1}}.$$

The investigation of the spectrum of invertible weighted composition operators uC_φ has to be divided into three cases, depending on the type of the automorphic symbol φ , that is, one has to treat the parabolic, hyperbolic and elliptic automorphisms separately. In [13], Chalendar, Gallardo-Gutiérrez and Partington gave complete characterizations of the spectrum $\sigma_{\mathcal{D}}(uC_\varphi)$ for invertible operators $uC_\varphi: \mathcal{D} \rightarrow \mathcal{D}$ on the Dirichlet space when the automorphism φ is parabolic or elliptic, and also provided the following inclusion in the hyperbolic case:

$$\sigma_{\mathcal{D}}(uC_\varphi) \subset \left\{ \lambda \in \mathbb{C} : \min\{|u(\omega)|, |u(\gamma)|\} \mu \leq |\lambda| \leq \max\{|u(\omega)|, |u(\gamma)|\} \frac{1}{\mu} \right\},$$

where $\omega, \gamma \in \partial\mathbb{D}$ are the fixed points of φ , and $0 < \mu < 1$ is such that φ is conjugate to the automorphism

$$\psi(z) = \frac{(1 + \mu)z + (1 - \mu)}{(1 - \mu)z + (1 + \mu)}.$$

In our paper [23], we improved this inclusion and computed the spectral radius:

Theorem 3.3.4. [23, Theorem 5.2] *Suppose that the operator $uC_\varphi: \mathcal{D} \rightarrow \mathcal{D}$ is invertible on the Dirichlet space and assume that the automorphism φ is hyperbolic, with attractive fixed point $\omega \in \partial\mathbb{D}$ and repulsive fixed point $\gamma \in \partial\mathbb{D}$. If $u \in A(\mathbb{D})$, then*

$$r_{\mathcal{D}}(uC_\varphi) = \max\{|u(\omega)|, |u(\gamma)|\}$$

and

$$\sigma_{\mathcal{D}}(uC_\varphi) \subset \left\{ \lambda \in \mathbb{C} : \min\{|u(\omega)|, |u(\gamma)|\} \leq |\lambda| \leq \max\{|u(\omega)|, |u(\gamma)|\} \right\}.$$

For the Bloch space \mathcal{B} , our results are summarized below:

Theorem 3.3.5. [23, Theorems 4.3, 4.5 and 4.8] *Suppose that the weighted composition operator $uC_\varphi: \mathcal{B} \rightarrow \mathcal{B}$ is invertible on the Bloch space and assume that $u \in A(\mathbb{D})$.*

(1) *If the automorphism φ is parabolic, with the unique fixed point $\omega \in \partial\mathbb{D}$, then*

$$\sigma_{\mathcal{B}}(uC_\varphi) = \left\{ \lambda \in \mathbb{C} : |\lambda| = |u(\omega)| \right\}.$$

(2) *If the automorphism φ is hyperbolic, with the attractive fixed point $\omega \in \partial\mathbb{D}$ and the repulsive fixed point $\gamma \in \partial\mathbb{D}$, then $r_{\mathcal{B}}(uC_\varphi) = \max\{|u(\omega)|, |u(\gamma)|\}$ and*

$$\sigma_{\mathcal{B}}(uC_\varphi) \subset \left\{ \lambda \in \mathbb{C} : \min\{|u(\omega)|, |u(\gamma)|\} \leq |\lambda| \leq \max\{|u(\omega)|, |u(\gamma)|\} \right\}.$$

(3) If the automorphism is elliptic, with the unique fixed point $p \in \mathbb{D}$, and if $\varphi_n \neq \text{id}$ for every $n \in \mathbb{N}$, then

$$\sigma_{\mathcal{B}}(uC_{\varphi}) = \{\lambda \in \mathbb{C} : |\lambda| = |u(p)|\}.$$

If in the elliptic case (3) there does exist a positive integer n such that $\varphi_n = \text{id}$, then letting m be the smallest such integer and assuming only boundedness of $uC_{\varphi}: \mathcal{B} \rightarrow \mathcal{B}$, we have

$$\sigma_{\mathcal{B}}(uC_{\varphi}) = \overline{\{\lambda \in \mathbb{C} : \lambda^m = u_{(m)}(z) \text{ for some } z \in \mathbb{D}\}}.$$

Among the central ingredients in the proofs of Theorems 3.3.4 and 3.3.5 are Corollary 3.1.11, Lemma 3.2.4 and the fact that iteration sequences $\{\varphi_n(a)\}_{n=0}^{\infty}$ for any given starting point $a \in \mathbb{D}$ are interpolating for H^{∞} if the automorphism φ is parabolic or hyperbolic, see [16, Proposition 4.9] and [17, Theorem 2.65]. The complete characterization of the spectrum of invertible operators uC_{φ} in the hyperbolic case is still an open problem both on the Bloch and Dirichlet space, but we believe the obtained inclusions actually are equalities. One fact that points in that direction is [23, Theorem 4.6], in which the mentioned equality is established for the Bloch space under the assumption that $|u(\omega)| = |u(\gamma)|$.

Turning to another subject, in our paper [22] we also studied norms and essential norms of weighted composition operators mapping into the spaces H_v^{∞} and \mathcal{B}_v^{∞} for very general domain spaces \mathcal{X} , using the axioms discussed in section 2.3. This work was inspired by the paper [14], where Colonna and Tjani carried out a similar investigation, but assuming among other things reflexivity of the space \mathcal{X} , meaning that our framework is much broader. Also, we found a mistake in the proof of Theorem 3.4 in [14], which forced us to choose another approach and led to the application of predual operators. Namely, by a slight modification of the proof of [6, Theorem 3] one can show that the essential norm of any bounded operator can be used to estimate the corresponding essential norm of its dual operator.

Lemma 3.3.6. *Let \mathcal{X} and \mathcal{Y} be Banach spaces and assume that \mathcal{X}^* has the λ -metric approximation property for some $\lambda > 0$. If $T: \mathcal{X} \rightarrow \mathcal{Y}$ is any bounded operator, then*

$$\frac{1}{1+\lambda} \|T\|_{e, \mathcal{X} \rightarrow \mathcal{Y}} \leq \|T^*\|_{e, \mathcal{Y}^* \rightarrow \mathcal{X}^*} \leq \|T\|_{e, \mathcal{X} \rightarrow \mathcal{Y}}.$$

Applying this estimate on the predual operator ${}^*T: {}^*\mathcal{Y} \rightarrow {}^*\mathcal{X}$ guaranteed by Lemma 3.1.9 one arrives at the following useful result, see the proof of [22, Theorem 4.2].

Lemma 3.3.7. *Assume that $\mathcal{X}, \mathcal{Y} \subset \mathcal{H}(\mathbb{D})$ are Banach spaces satisfying condition (I), and suppose that \mathcal{Y} has the λ -metric approximation property for some $\lambda > 0$. If the operator $T: \mathcal{X} \rightarrow \mathcal{Y}$ is bounded and the restriction $T|_{\mathcal{B}_{\mathcal{X}}}$ is co-co continuous, then*

$$\|T\|_{e, \mathcal{X} \rightarrow \mathcal{Y}} \asymp \inf \left\{ \|T - \Phi_{\mathcal{Y}}^{-1} \circ K^* \circ \Phi_{\mathcal{X}}\|_{\mathcal{X} \rightarrow \mathcal{Y}} : K: {}^*\mathcal{Y} \rightarrow {}^*\mathcal{X} \text{ is compact} \right\}.$$

Using Lemma 3.3.7 and Carleson's interpolation theorem, we obtained this corrected and generalized version of the above mentioned Theorem 3.4 in [14].

Theorem 3.3.8. [22, Corollary 3.2 and Theorem 4.3] Assume that $\mathcal{X} \subset \mathcal{H}(\mathbb{D})$ is a Banach space satisfying condition (I). Then the operator norm of $uC_\varphi: \mathcal{X} \rightarrow H_v^\infty$ is given by

$$\|uC_\varphi\|_{\mathcal{X} \rightarrow H_v^\infty} = \sup_{z \in \mathbb{D}} v(z)|u(z)|\|\delta_{\varphi(z)}\|_{\mathcal{X} \rightarrow \mathbb{C}}.$$

If the space \mathcal{X} in addition satisfies conditions (II)-(V) and the operator $uC_\varphi: \mathcal{X} \rightarrow H_v^\infty$ is bounded, then

$$\|uC_\varphi\|_{e, \mathcal{X} \rightarrow H_v^\infty} \asymp \limsup_{|\varphi(z)| \rightarrow 1} v(z)|u(z)|\|\delta_{\varphi(z)}\|_{\mathcal{X} \rightarrow \mathbb{C}}.$$

For the corresponding results on norms and essential norms of the weighted composition operator uC_φ mapping from a general space $\mathcal{X} \subset \mathcal{H}(\mathbb{D})$ into the Bloch-type space \mathcal{B}_v^∞ , the reader is referred to the paper [22] and Corollary 3.3 and Theorem 4.4 therein.

3.4 Königs eigenfunction

Computing the point spectrum $\sigma_{p, \mathcal{X}}(C_\varphi)$ of a composition operator $C_\varphi: \mathcal{X} \rightarrow \mathcal{X}$ on some Banach space $\mathcal{X} \subset \mathcal{H}(\mathbb{D})$ constitutes of finding scalars $\lambda \in \mathbb{C}$ with corresponding non-zero functions $f \in \mathcal{X}$ such that

$$f \circ \varphi = \lambda f. \quad (3.4.1)$$

This eigenvalue equation is known as *Schröder's functional equation* after Ernest Schröder who studied it in the early 1870's, see [43]. In 1884 Gabriel Königs solved equation (3.4.1) under the assumptions that the analytic selfmap φ is non-automorphic and has a fixed point $p \in \mathbb{D}$, allowing any solutions $f \in \mathcal{H}(\mathbb{D})$. If $\varphi'(p) = 0$ then the only solution is $\lambda = 1$, with f equal to some constant function. In the interesting case when the derivative $\varphi'(p)$ is non-zero, Königs showed that the eigenvalues of $C_\varphi: \mathcal{H}(\mathbb{D}) \rightarrow \mathcal{H}(\mathbb{D})$ are precisely the numbers $\{\varphi'(p)^n\}_{n=0}^\infty$, where each eigenvalue has multiplicity one. Moreover, if $\sigma \in \mathcal{H}(\mathbb{D})$ is the unique eigenfunction with $\sigma'(p) = 1$ for the eigenvalue $\lambda = \varphi'(p)$, then σ^n spans the eigenspace for the eigenvalue $\lambda = \varphi'(p)^n$ for every positive integer n . The function σ is called the *Königs function* for φ or the *principal eigenfunction* of C_φ . When proving the existence of the Königs function it is actually enough to assume that the fixed point of φ is $p = 0$, because otherwise one can use the involutive automorphism ψ_p in (3.2.1) and consider the new selfmap $\phi = \psi_p \circ \varphi \circ \psi_p$, which has zero as its fixed point. The main step in the proof is then to show that the sequence of normalized iterates

$$\sigma_n = \frac{\varphi_n}{\varphi'(0)^n} \quad (3.4.2)$$

converges uniformly on compact subsets of the unit disc, since the analytic limit σ will then satisfy equation (3.4.1) for $\lambda = \varphi'(0)$ due to the relation $\sigma_n \circ \varphi = \varphi'(0)\sigma_{n+1}$, which is valid for every $n \in \mathbb{N}$. For a proof of the convergence, see [45, pp. 91-92]. The Königs function σ also plays a central role in connection with the point spectrum of weighted composition operators, as seen below.

Theorem 3.4.1. *Let $\varphi: \mathbb{D} \rightarrow \mathbb{D}$ be an analytic selfmap of the unit disc such that $\varphi(0) = 0$ and $0 < |\varphi'(0)| < 1$. Let $u \in \mathcal{H}(\mathbb{D})$ and assume that $u(0) \neq 0$. Then there exists a unique function $v \in \mathcal{H}(\mathbb{D})$ such that $v(0) = 1$ and for every $z \in \mathbb{D}$*

$$u(z)v(\varphi(z)) = u(0)v(z).$$

Moreover, the eigenvalues of the weighted composition operator $uC_\varphi: \mathcal{H}(\mathbb{D}) \rightarrow \mathcal{H}(\mathbb{D})$ are given by $\{u(0)\varphi'(0)^n\}_{n=0}^\infty$, where each eigenspace is one-dimensional. The eigenspace for the eigenvalue $\lambda = u(0)\varphi'(0)^n$ is spanned by the function $v\sigma^n$, where σ is the Königs function for φ .

This generalization of Königs's theorem is proved for example in [30], where the function v is obtained as the *co*-limit of the sequence $\{v_n\}_{n=1}^\infty$ of functions

$$v_n(z) = \frac{u_{(n)}(z)}{u(0)^n} = \frac{u(z)u(\varphi(z)) \cdots u(\varphi_{n-1}(z))}{u(0)^n}.$$

Under the given assumptions on the inducing symbols u and φ , Theorem 3.4.1 above completely characterizes the eigenvalues of weighted composition operators considered on $\mathcal{H}(\mathbb{D})$. However, if the operator $uC_\varphi: \mathcal{X} \rightarrow \mathcal{X}$ is considered on some Banach space $\mathcal{X} \subset \mathcal{H}(\mathbb{D})$, then the point spectrum $\sigma_{p,\mathcal{X}}(uC_\varphi)$ might be smaller than the whole sequence $\{u(0)\varphi'(0)^n\}_{n=0}^\infty$, since some eigenfunction $v\sigma^n$ might not belong to the space \mathcal{X} . Therefore it is of interest to determine when for example the Königs function σ belongs to different function spaces, in terms of function theoretic properties of the inducing symbol φ . To address this problem, notice that if $|\varphi'(0)| > r_{e,\mathcal{X}}(C_\varphi)$ then $\varphi'(0)$ is an eigenvalue of $C_\varphi: \mathcal{X} \rightarrow \mathcal{X}$ by Theorem 3.1.13, and hence $\sigma \in \mathcal{X}$. A natural follow-up question is whether the opposite implication holds, that is, whether it is true that

$$\sigma \in \mathcal{X} \iff |\varphi'(0)| > r_{e,\mathcal{X}}(C_\varphi). \quad (3.4.3)$$

A more well-posed question would be for which spaces \mathcal{X} the above equivalence holds. In the paper [11], Bourdon showed that condition (3.4.3) holds for the space $H_{v_\alpha}^0$ with $\alpha > 0$, and since $\mathcal{B}_{v_{\alpha+1}}^0 = H_{v_\alpha}^0$ for these values of α , we also have that

$$\sigma \in \mathcal{B}_{v_\alpha}^0 \iff |\varphi'(0)| > r_{e,\mathcal{B}_{v_\alpha}^0}(C_\varphi)$$

whenever $\alpha > 1$ and $C_\varphi: \mathcal{B}_{v_\alpha}^0 \rightarrow \mathcal{B}_{v_\alpha}^0$ is bounded. The starting point of our paper [21] was to investigate if (3.4.3) also holds for the classical Bloch space \mathcal{B} . However, by studying the *lens map*

$$\varphi(z) = \frac{\phi(z)^t - 1}{\phi(z)^t + 1},$$

where $\phi(z) = \frac{1+z}{1-z}$ and $0 < t < 1$, we found that the Königs function $\sigma(z) = \log \frac{1+z}{1-z}$ belongs to \mathcal{B} even though $r_{e,\mathcal{B}}(C_\varphi) = |\varphi'(0)|$, contradicting (3.4.3), see [21, Example 2.9]. In the mentioned paper [21], we continued studying the relationship of the Königs function to other Bloch- and H^∞ -type spaces, obtaining the following characterization.

Theorem 3.4.2. [21, Theorem 2.7] *Assume that the weight v is radial and non-increasing with respect to $|z|$. Then the Königs function σ belongs to H_v^∞ if and only if the sequence $\{\sigma_n\}_{n=1}^\infty$ of normalized iterates in (3.4.2) is bounded in H_v^∞ . The same statement holds for the spaces $\mathcal{B}_{v_\alpha}^\infty$ when $\alpha > 1$, and H^∞ .*

However, it is still an open problem to characterize when the Königs function belongs to the Bloch-type spaces $\mathcal{B}_{v_\alpha}^\infty$ for $0 < \alpha \leq 1$, among these the classical Bloch space \mathcal{B} . In the case of bounded analytic functions, one can actually say more. Namely, Zheng has proven that the essential spectral radius $r_{e, H^\infty}(C_\varphi)$ of $C_\varphi: H^\infty \rightarrow H^\infty$ is either 0 or 1, see [50]. In the former case, the Königs function belongs to H^∞ , whereas in the latter case it cannot be bounded on the unit disc. We end this section with the complete description.

Theorem 3.4.3. [21, Theorem 3.1] *The following statements are equivalent:*

- (1) $\sigma \in H^\infty$.
- (2) $\sigma^n \in H^\infty$ for all $n \in \mathbb{N}$.
- (3) $\sigma^n \in H^\infty$ for some $n \in \mathbb{N}$.
- (4) There is a positive integer n such that $\|\varphi_n\|_\infty < 1$.
- (5) The essential spectral radius $r_{e, H^\infty}(C_\varphi) = 0$.
- (6) The sequence $\{\sigma_n\}_{n=1}^\infty$ is bounded in H^∞ .

3.5 Generalized Volterra operators

The second main object studied in this thesis, in addition to weighted composition operators, is the *generalized Volterra operator* T_g^φ defined for a fixed function $g \in \mathcal{H}(\mathbb{D})$ and analytic selfmap $\varphi: \mathbb{D} \rightarrow \mathbb{D}$ as

$$T_g^\varphi(f)(z) = \int_0^{\varphi(z)} f(\xi)g'(\xi)d\xi, \quad z \in \mathbb{D}, f \in \mathcal{H}(\mathbb{D}). \quad (3.5.1)$$

The *classical Volterra operator*

$$T_g(f)(z) = \int_0^z f(\xi)g'(\xi)d\xi, \quad z \in \mathbb{D}, f \in \mathcal{H}(\mathbb{D}),$$

obtained by choosing $\varphi(z) = z$ in (3.5.1), has been extensively studied on various spaces of analytic functions during the past decades, starting from the paper [41] by Pommerenke. Aleman, Cima and Siskakis continued this investigation in the papers [3], [4] and [2], completing the characterization of boundedness and compactness of the Volterra operator on the Hardy spaces. They proved that $T_g: H^p \rightarrow H^p$ is bounded for $0 < p < \infty$ if and only if the inducing symbol g belongs to BMOA, and compact precisely when $g \in \text{VMOA}$. Among the open problems related to the classical Volterra operator is

the description of all functions $g \in \mathcal{H}(\mathbb{D})$ that induce bounded operators $T_g: H^\infty \rightarrow H^\infty$. The study of this problem was begun in the paper [5], where Anderson, Jovovic and Smith conjectured that the set

$$T[H^\infty] = \left\{ g \in \mathcal{H}(\mathbb{D}) : T_g: H^\infty \rightarrow H^\infty \text{ is bounded} \right\}$$

would coincide with the space of functions analytic in \mathbb{D} with bounded radial variation

$$\text{BRV} = \left\{ f \in \mathcal{H}(\mathbb{D}) : \sup_{0 \leq \theta < 2\pi} \int_0^1 |f'(re^{i\theta})| dr < \infty \right\}.$$

It is easily seen that $\text{BRV} \subset T[H^\infty]$, but the reverse inclusion was proven to be false due to a counterexample by Smith, Stolyarov and Volberg in [48]. In the same paper however, the authors showed that the conjecture is true if the inducing symbol g is univalent, that is

$$T[H^\infty] \cap \left\{ g \in \mathcal{H}(\mathbb{D}) : g \text{ is univalent} \right\} = \text{BRV}. \quad (3.5.2)$$

The proof of (3.5.2) leans heavily on the following approximation result. Given positive constants β and r , let $\mathcal{B}(\Omega_\beta^r)$ denote the class of all functions F , analytic in the open sector

$$\Omega_\beta^r := \left\{ z \in \mathbb{C} : 0 < |z| < r \text{ and } -\frac{\beta}{2} < \arg(z) < \frac{\beta}{2} \right\},$$

such that

$$C_F := \sup_{z \in \Omega_\beta^r} |zF'(z)| < \infty.$$

In the approximation theorem below \tilde{u} denotes the harmonic conjugate of u with $\tilde{u}\left(\frac{1}{2}\right) = 0$.

Theorem 3.5.1. [48, Theorem 1.2] *Let $0 < \gamma < \beta < \pi$ and $\varepsilon > 0$. Then there is a number $\delta(\varepsilon) > 0$ such that for each $F \in \mathcal{B}(\Omega_\gamma^{1/2})$ there exists a harmonic function $u: \Omega_\beta^1 \rightarrow \mathbb{R}$ with the properties*

- (1) $|\text{Re}(F(x)) - u(x)| \leq \varepsilon$, for $x \in (0, \delta(\varepsilon))$,
- (2) $|\tilde{u}(z)| \leq C(\varepsilon, \gamma, \beta, C_F) < \infty$, for $z \in \Omega_\beta^1$.

In our paper [24], we generalized the boundedness result (3.5.2) to Volterra operators $T_g: H_{v_\alpha}^\infty \rightarrow H^\infty$ for $0 \leq \alpha < 1$, giving the following description in the univalent case:

Theorem 3.5.2. [24, Theorem 2.2] *If $g \in \mathcal{H}(\mathbb{D})$ is univalent and $0 \leq \alpha < 1$, then $T_g: H_{v_\alpha}^\infty \rightarrow H^\infty$ is bounded if and only if*

$$\sup_{0 \leq \theta < 2\pi} \int_0^1 \frac{|g'(re^{i\theta})|}{(1-r^2)^\alpha} dr < \infty.$$

The reason why the remaining parameter values $\alpha \geq 1$ were not considered is that Contreras, Peláez, Pommerenke and Rättyä had already shown earlier in [15, Section 2.2] that the only operator $T_g: H_{v_\alpha}^\infty \rightarrow H^\infty$ that can be bounded for those values of α is the zero operator, regardless of whether g is univalent or not.

Naturally, since the boundedness of the operator $T_g: H^\infty \rightarrow H^\infty$ still remains as an unsolved problem in the general case for non-univalent symbols g , also the compactness is an open problem. In the mentioned paper [5], the authors suggested that the space

$$\text{BRV}_0 = \left\{ f \in \mathcal{H}(\mathbb{D}) : \lim_{t \rightarrow 1^-} \sup_{0 \leq \theta < 2\pi} \int_t^1 |f'(re^{i\theta})| dr = 0 \right\}$$

of functions analytic in the unit disc with derivative uniformly integrable on radii would be the right candidate for the set of functions $g \in \mathcal{H}(\mathbb{D})$ inducing compact operators $T_g: H^\infty \rightarrow H^\infty$. In our paper [24], we gave the following partial answer, confirming the conjecture in the univalent case.

Theorem 3.5.3. [24, Theorem 2.3] *If $g \in \mathcal{H}(\mathbb{D})$ is univalent and $0 \leq \alpha < 1$, then $T_g: H_{v_\alpha}^\infty \rightarrow H^\infty$ is compact if and only if*

$$\lim_{t \rightarrow 1^-} \sup_{0 \leq \theta < 2\pi} \int_t^1 \frac{|g'(re^{i\theta})|}{(1-r^2)^\alpha} dr = 0.$$

The proof is based on the approximation Theorem 3.5.1, and uses the compactness characterization in Lemma 3.1.5, which can be applied to the Volterra operator due to the following result.

Lemma 3.5.4. *Let $g \in \mathcal{H}(\mathbb{D})$ and φ be an analytic selfmap of \mathbb{D} . If $\{f_n\}_{n=1}^\infty \subset \mathcal{H}(\mathbb{D})$ and $f_n \xrightarrow{co} 0$, then $T_g^\varphi(f_n) \xrightarrow{co} 0$. In other words, $T_g^\varphi: \mathcal{H}(\mathbb{D}) \rightarrow \mathcal{H}(\mathbb{D})$ is co-co continuous.*

Proof. Assume that $f_n \xrightarrow{co} 0$ and choose an arbitrary compact subset $K \subset \mathbb{D}$. Then the image $\varphi(K)$ is contained in a closed disc $\overline{D(0, r_K)}$ centered at the origin with some radius $0 < r_K < 1$. Since the mapping $z \mapsto \int_0^z |g'(\xi)| d|\xi|$ is continuous on \mathbb{D} and

$$\begin{aligned} \sup_{z \in K} |T_g^\varphi(f_n)(z)| &= \sup_{z \in K} \left| \int_0^{\varphi(z)} f_n(\xi) g'(\xi) d\xi \right| \leq \sup_{z \in \varphi(K)} \int_0^z |f_n(\xi)| |g'(\xi)| d|\xi| \\ &\leq \sup_{\omega \in \overline{D(0, r_K)}} |f_n(\omega)| \sup_{z \in \varphi(K)} \int_0^z |g'(\xi)| d|\xi|, \end{aligned}$$

it follows that $T_g^\varphi(f_n) \xrightarrow{co} 0$. □

In order to see how the situation changes when the target space of the operator T_g changes from the space of bounded analytic functions H^∞ , as in Theorems 3.5.2 and 3.5.3 above, to a weighted Banach space $H_{v_\alpha}^\infty$, we also studied the generalized Volterra operator $T_g^\varphi: \mathcal{X} \rightarrow H_{v_\alpha}^\infty$ using the axioms discussed in section 2.3 on the space \mathcal{X} . Since the differentiated Volterra operator $D \circ T_g^\varphi = (g \circ \varphi)' C_\varphi$ is a weighted composition operator, one can apply our results from Theorem 3.3.8 to get the following estimates of the norm and essential norm.

Theorem 3.5.5. [24, Theorems 3.1 and 3.5] Let \mathcal{X} be a Banach space of analytic functions on \mathbb{D} satisfying condition (I) and assume that the weight v is normal and that $\varphi(0) = 0$. Then

$$\|T_g^\varphi\|_{\mathcal{X} \rightarrow H_v^\infty} \asymp \sup_{z \in \mathbb{D}} (1 - |z|)v(z)|(g \circ \varphi)'(z)| \|\delta_{\varphi(z)}\|_{\mathcal{X} \rightarrow \mathbb{C}}.$$

If the space \mathcal{X} in addition satisfies conditions (II)-(V) and the operator $T_g^\varphi: \mathcal{X} \rightarrow H_v^\infty$ is bounded, then

$$\|T_g^\varphi\|_{e, \mathcal{X} \rightarrow H_v^\infty} \asymp \limsup_{|\varphi(z)| \rightarrow 1} (1 - |z|)v(z)|(g \circ \varphi)'(z)| \|\delta_{\varphi(z)}\|_{\mathcal{X} \rightarrow \mathbb{C}}.$$

For the rest of this section we explained in more detail how to obtain the equivalences in [24, Corollary 3.12], which generalizes [7, Theorem 2] by Basallote et.al. and concludes this thesis. The first result stated below can be proven using the essential norm estimate of Theorem 3.5.5.

Theorem 3.5.6. [24, Theorem 3.10] Assume that $\mathcal{X} \subset \mathcal{H}(\mathbb{D})$ is a Banach space satisfying conditions (I)-(V), let v be a normal weight and assume that $\varphi(0) = 0$. Then the operator $T_g^\varphi: \mathcal{X} \rightarrow H_v^\infty$ is compact if and only if $T_g^\varphi: \mathcal{X} \rightarrow H_v^0$ is compact.

Using the predual operator S from Lemma 3.1.9 one can show that compactness and weak compactness coincide for a large class of operators mapping from H_v^∞ into a Banach space $\mathcal{Y} \subset \mathcal{H}(\mathbb{D})$. The proof relies on the fact that the predual space ${}^*H_v^\infty$ is isomorphic to the Schur space ℓ^1 by Theorem 2.4.3.

Theorem 3.5.7. [24, Theorem 3.3] Let v be a normal weight and assume that the Banach space $\mathcal{Y} \subset \mathcal{H}(\mathbb{D})$ satisfies condition (I). If the restriction $T|_{B_{H_v^\infty}}$ is co-co continuous then $T: H_v^\infty \rightarrow \mathcal{Y}$ is compact if and only if it is weakly compact.

The next lemma, where \mathcal{P} denotes the set of all complex polynomials on \mathbb{D} , is a slight generalization of [15, Proposition 2.1].

Lemma 3.5.8. Let $\mathcal{X} \subset \mathcal{H}(\mathbb{D})$ be a Banach space containing the disc algebra and satisfying conditions (I) and (IV). Let $\mathcal{Y} \subset \mathcal{H}(\mathbb{D})$ be a Banach space satisfying condition (I). If the operator $T: \mathcal{H}(\mathbb{D}) \rightarrow \mathcal{H}(\mathbb{D})$ is co-co continuous, then the following statements are equivalent:

- (1) $T: \mathcal{X} \rightarrow \mathcal{Y}$ is bounded,
- (2) $T: \overline{\mathcal{P}}^{\mathcal{X}} \rightarrow \mathcal{Y}$ is bounded,

and so are the operator norms: $\|T\|_{\mathcal{X} \rightarrow \mathcal{Y}} \asymp \|T\|_{\overline{\mathcal{P}}^{\mathcal{X}} \rightarrow \mathcal{Y}}$.

Proof. Clearly (1) implies (2). Conversely, assume that the operator $T: \overline{\mathcal{P}}^{\mathcal{X}} \rightarrow \mathcal{Y}$ is bounded, choose $f \in \mathcal{X}$ and let $\{r_n\}_{n=1}^\infty \subset (0, 1)$ be a sequence such that $r_n \rightarrow 1$ as $n \rightarrow \infty$. Then $f_{r_n} \xrightarrow{co} f$, and hence $T(f_{r_n}) \xrightarrow{co} T(f)$. Also, since \mathcal{X} contains the disc algebra, we have that

$$f_{r_n} \in A(\mathbb{D}) = \overline{\mathcal{P}}^{\|\cdot\|_{A(\mathbb{D})}} \subset \overline{\mathcal{P}}^{\mathcal{X}},$$

showing that the sequence $\{T(f_{r_n})\}_{n=1}^\infty$ is contained in \mathcal{Y} . In fact, it is even bounded in \mathcal{Y} , since by condition (IV) for \mathcal{X} it holds that

$$\begin{aligned} \|T(f_{r_n})\|_{\mathcal{Y}} &= \|T(\mathcal{T}_{r_n}(f))\|_{\mathcal{Y}} \leq \|T\|_{\overline{\mathcal{P}}^{\mathcal{X}} \rightarrow \mathcal{Y}} \|\mathcal{T}_{r_n}\|_{\mathcal{X} \rightarrow \mathcal{X}} \|f\|_{\mathcal{X}} \\ &\leq \|T\|_{\overline{\mathcal{P}}^{\mathcal{X}} \rightarrow \mathcal{Y}} \sup_{0 < r < 1} \|\mathcal{T}_r\|_{\mathcal{X} \rightarrow \mathcal{X}} \|f\|_{\mathcal{X}}. \end{aligned}$$

This implies that the sequence $\{T(f_{r_n})\}_{n=1}^\infty$ belongs to the closed ball $B_{\mathcal{Y}}(0, R)$ with radius $R = \|T\|_{\overline{\mathcal{P}}^{\mathcal{X}} \rightarrow \mathcal{Y}} \sup_{0 < r < 1} \|\mathcal{T}_r\|_{\mathcal{X} \rightarrow \mathcal{X}} \|f\|_{\mathcal{X}}$. Thus, since \mathcal{Y} satisfies condition (I), there exist $g \in B_{\mathcal{Y}}(0, R)$ and a subsequence $\{T(f_{r_{n_k}})\}_{k=1}^\infty$ such that $T(f_{r_{n_k}}) \xrightarrow{co} g$. Therefore $T(f) = g \in \mathcal{Y}$, meaning that $T: \mathcal{X} \rightarrow \mathcal{Y}$ is well-defined and hence bounded by the closed graph theorem. Moreover, since $g \in B_{\mathcal{Y}}(0, R)$, we have

$$\|T(f)\|_{\mathcal{Y}} = \|g\|_{\mathcal{Y}} \leq R = \|T\|_{\overline{\mathcal{P}}^{\mathcal{X}} \rightarrow \mathcal{Y}} \sup_{0 < r < 1} \|\mathcal{T}_r\|_{\mathcal{X} \rightarrow \mathcal{X}} \|f\|_{\mathcal{X}},$$

which implies that $\|T\|_{\mathcal{X} \rightarrow \mathcal{Y}} \lesssim \|T\|_{\overline{\mathcal{P}}^{\mathcal{X}} \rightarrow \mathcal{Y}}$ and completes the proof. \square

Remark 3.5.9. Lemma 3.5.8 above can for example be applied to the generalized Volterra operator $T_g^\varphi: \mathcal{X} \rightarrow \mathcal{Y}$ when $\mathcal{X} = H_v^\infty$, in which case $\overline{\mathcal{P}}^{\mathcal{X}} = H_v^0$.

Recall from Lemma 2.4.2 that the closed unit ball of H_v^0 is *co*-dense in the closed unit ball of H_v^∞ if the weight v is radial, and hence the restriction map $\rho: {}^*H_v^\infty \rightarrow (H_v^0)^*$, mapping $\ell \mapsto \ell|_{H_v^0}$, is an isometric isomorphism by Theorem 2.4.3. Using the evaluation map $\Phi_{H_v^\infty}: H_v^\infty \rightarrow ({}^*H_v^\infty)^*$ one then obtains an isometric isomorphism $\Lambda: H_v^\infty \rightarrow (H_v^0)^{**}$ by defining

$$\Lambda(f) := (\widehat{f}^{H_v^\infty} | {}^*H_v^\infty) \circ \rho^{-1}. \quad (3.5.3)$$

Here we use the notation $\widehat{f}^{\mathcal{X}}$ to emphasize which space \mathcal{X} the evaluation map is acting on, that is, $\widehat{f}^{\mathcal{X}} \in \mathcal{X}^{**}$ whenever $f \in \mathcal{X}$.

Lemma 3.5.10. *Let v be a radial weight, w be any weight and $T: \mathcal{H}(\mathbb{D}) \rightarrow \mathcal{H}(\mathbb{D})$ be a *co-co* continuous linear operator. If $T: H_v^0 \rightarrow H_w^\infty$ is weakly compact, then for every $f \in H_v^\infty$ it holds that*

$$T^{**}(\Lambda(f)) = \widehat{T(f)}^{H_w^\infty},$$

where $\Lambda: H_v^\infty \rightarrow (H_v^0)^{**}$ is given by (3.5.3).

Proof. Assume that $T: H_v^0 \rightarrow H_w^\infty$ is weakly compact, choose a function $f \in H_v^\infty$ not identically zero and define $g := \frac{f}{\|f\|_{H_v^\infty}} \in B_{H_v^\infty}$. Note that the operator $T: H_v^\infty \rightarrow H_w^\infty$ is bounded by Lemma 3.5.8 (see Remark 3.5.9), so the equality about to be proven makes sense. Since the mapping $\Lambda: H_v^\infty \rightarrow (H_v^0)^{**}$ is an isometric isomorphism, we have by Goldstine's theorem that

$$\Lambda(g) \in B_{(H_v^0)^{**}} = \overline{B_{H_v^0}}^{w^*}.$$

Hence, there exists a net $\{\widehat{g}_\gamma^{H_v^0}\} \subset B_{\widehat{H}_v^0}$, where every $g_\gamma \in B_{H_v^0}$, such that $\widehat{g}_\gamma^{H_v^0} \xrightarrow{w^*} \Lambda(g)$ in $(H_v^0)^{**}$. Moreover, since $\rho^{-1}(\ell)|_{H_v^0} = \ell$ for every $\ell \in (H_v^0)^*$, we have that

$$\Lambda(g_\gamma) = \left(\widehat{g}_\gamma^{H_v^0} \Big|_{*H_v^\infty} \right) \circ \rho^{-1} = \widehat{g}_\gamma^{H_v^0},$$

and thus $\Lambda(g_\gamma) \xrightarrow{w^*} \Lambda(g)$ in $(H_v^0)^{**}$. For any $z \in \mathbb{D}$, $\delta_z \in {}^*H_v^\infty$ and $\rho(\delta_z) \in (H_v^0)^*$, which gives that

$$g_\gamma(z) = \Lambda(g_\gamma)(\rho(\delta_z)) \rightarrow \Lambda(g)(\rho(\delta_z)) = g(z).$$

Furthermore, $\{g_\gamma\}$ is an equicontinuous family because $\|g_\gamma\|_{H_v^\infty} \leq 1$ for every γ , so we actually have that $g_\gamma \xrightarrow{co} g$, and since T is co - co continuous we get $T(g_\gamma) \xrightarrow{co} T(g)$.

On the other hand, since $T: H_v^0 \rightarrow H_w^\infty$ is weakly compact, $T^*: (H_w^\infty)^* \rightarrow (H_v^0)^*$ is also weakly compact, and hence $T^{**}: (H_v^0)^{**} \rightarrow (H_w^\infty)^{**}$ is w^* - w continuous by Theorem 3.1.8. We thus have that

$$\widehat{T(g_\gamma)}^{H_w^\infty} = T^{**}(\widehat{g}_\gamma^{H_v^0}) \xrightarrow{w} T^{**}(\Lambda(g)), \quad (3.5.4)$$

where the first equality holds because

$$\widehat{T(g_\gamma)}^{H_w^\infty}(\ell) = \ell(T(g_\gamma)) = T^*(\ell)(g_\gamma) = \widehat{g}_\gamma^{H_v^0}(T^*(\ell)) = T^{**}(\widehat{g}_\gamma^{H_v^0})(\ell)$$

for any $\ell \in (H_w^\infty)^*$. The left-hand side of (3.5.4) is contained in $\widehat{H_w^\infty}$ for every γ , which gives that

$$T^{**}(\Lambda(g)) \in \overline{\widehat{H_w^\infty}}^w = \overline{\widehat{H_w^\infty}}^{\|\cdot\|_{(H_w^\infty)^{**}}} = \widehat{H_w^\infty},$$

and hence there exists $h \in H_w^\infty$ such that $T^{**}(\Lambda(g)) = \widehat{h}^{H_w^\infty}$. Now since obviously

$$\widehat{T(g_\gamma)}^{H_w^\infty} \xrightarrow{w^*} \widehat{h}^{H_w^\infty}$$

and $\delta_z \in (H_w^\infty)^*$ for any $z \in \mathbb{D}$, it follows that $T(g_\gamma)$ converges pointwise to h on the unit disc. Hence, we must have that $h = T(g)$, showing that

$$T^{**}(\Lambda(g)) = \widehat{T(g)}^{H_w^\infty},$$

which also holds when g is replaced by $f = \|f\|_{H_v^\infty} g$ due to linearity, and the proof is complete. \square

Corollary 3.5.11. [24, Corollary 3.12] *Let v and w be normal weights and assume that v is equivalent to its associated weight \bar{v} . If $\varphi(0) = 0$ then the following statements are equivalent:*

- (1) $T_g^\varphi: H_v^\infty \rightarrow H_w^\infty$ is compact.
- (2) $T_g^\varphi: H_v^\infty \rightarrow H_w^\infty$ is weakly compact.
- (3) $T_g^\varphi: H_v^\infty \rightarrow H_w^0$ is compact.
- (4) $T_g^\varphi: H_v^\infty \rightarrow H_w^0$ is weakly compact or equivalently bounded.

- (5) $T_g^\varphi : H_v^0 \rightarrow H_w^0$ is compact.
- (6) $T_g^\varphi : H_v^0 \rightarrow H_w^0$ is weakly compact.
- (7) $T_g^\varphi : H_v^0 \rightarrow H_w^\infty$ is compact.
- (8) $T_g^\varphi : H_v^0 \rightarrow H_w^\infty$ is weakly compact.
- (9) $\lim_{|\varphi(z)| \rightarrow 1} (1 - |z|) |(g \circ \varphi)'(z)| \frac{w(z)}{v(\varphi(z))} = 0$.

Proof. We begin by justifying the equivalence within statement (4). If $T : H_v^\infty \rightarrow H_w^0$ is any bounded operator and $\{\ell_n\}_{n=1}^\infty \subset B_{(H_w^0)^*}$, then there is a w^* -convergent subsequence $\{\ell_{n_k}\}_{k=1}^\infty$ with some limit $\ell \in B_{(H_w^0)^*}$. This follows from the Banach-Alaoglu theorem, since the topology $(B_{(H_w^0)^*}, \sigma((H_w^0)^*, H_w^0))$ is metrizable due to the separability of H_w^0 . This gives that $T^*(\ell_{n_k}) \xrightarrow{w^*} T^*(\ell)$, but $H_v^\infty \approx \ell^\infty$ is a Grothendieck space meaning that we actually have weak convergence of $T^*(\ell_{n_k})$ to $T^*(\ell)$, and hence both $T^* : (H_w^0)^* \rightarrow (H_v^\infty)^*$ and $T : H_v^\infty \rightarrow H_w^0$ are weakly compact.

The statements (1) and (2) are equivalent by Theorem 3.5.7 and the equivalence between (1) and (3) follows from Theorem 3.5.6. If the operator $T_g^\varphi : H_v^0 \rightarrow H_w^\infty$ is weakly compact, then $(T_g^\varphi)^{**} : (H_v^0)^{**} \rightarrow \widehat{H_w^\infty}$ is weakly compact, and hence, if $Q : H_w^\infty \rightarrow \widehat{H_w^\infty}$ is the canonical embedding $Q(f) = \widehat{f}$, we have by Lemma 3.5.10 that

$$Q^{-1} \circ (T_g^\varphi)^{**} \circ \Lambda = T_g^\varphi : H_v^\infty \rightarrow H_w^\infty$$

is weakly compact, and therefore (8) implies (2). Finally, (1) is equivalent to (9) by Theorem 3.5.5, and the rest of the implications are obvious. \square

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