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ON CALCULATING
WELFARE LOSSES
OF TAXATION AND
PUBLIC PROVISION
ON THE MARGIN AND
TO A DEGREE

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ABSTRACT: In the paper an index number measuring inefficiency of taxation is introduced which is based on an economy-wide generalisation of the input distance function. Related concepts are an input based quantity index, the index of real endowment, and a productivity index indicating effectiveness of the public sector and the society's welfare index which offer a convenient summary way of analyzing the extent and nature of government intervention from the perspective of welfare and standard-of-living in the society. These indices should have wide applicability since they are based on a reinterpretation of similar concepts used in production theory. Hedonic scales seem particularly promising in offering a practical way of measuring the extent of intervention through the change in the marginal rates of substitution of the private goods.

KEY WORDS: Welfare loss of taxation, Malmquist indices, Hedonic scales.

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TIIVISTELMÄ: Tutkimuksessa esitetään indeksi verotuksen hyvinvointitappioiden arvioimiseksi. Tämä perustuu talouden panoskäytön etäisyysfunktioon ja on Debreun resurssien käyttökertoimen johdannainen. Lisäksi esitellään panoskäytön volyyymi-indeksi, ns. reaaliarvojen indeksi, ja tuottavuusindeksi, joka kuvaa julkisen sektorin tuloksellisuutta sekä hyvinvointi-indeksi, joka antaa yleiskuvan julkisten toimien luonteesta ja laajuudesta talouden hyvinvoinnin kannalta. Indeksien käyttökelpoisuutta puoltaa se, että ne perustuvat tuotantoteorian käsitteiden uudelleen tulkintaan. Hedonisten hintojen menetelmä soveltuu julkisen sektorin toimien arviointiin, sillä vaikutuksia voidaan mitata epäsuorasti hyödykkeiden rajasubstituutiosuhteiden muutoksina.

ASIASANAT: Verotuksen hyvinvointitappio, Malmquistin indeksit, Hedoniset hinnat.

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1 Introduction

Tax policy, public provision and other forms of public intervention are concerned with choices between far from perfect instruments. The familiar social excess burden arises from the inability to employ optimal lump-sum taxes in pursuit of both revenue and distributional objectives. Calculation of deadweight losses has progressed from traditional constructs of consumer and producer surplus to the Hicksian concepts of equivalent and compensated variation.¹ The latter concepts are based on compensated demand functions holding utility level constant.

In the case of a single consumer the equivalent variation resulting from the transition from state 0, with prices and income p, m_0 to state 1, with prices and income q, m_1 is defined as the increase

$$EV(p, m_0, q, m_1) = e(p, u_1) - e(p, u_0), \quad (1)$$

where e is the expenditure function for the consumer, and the ex post level of utility $u_1 = V(q, m_1)$, and the ex ante level of utility $u_0 = V(p, m_0)$. On the other hand, (1) may be seen as a change in the money-metric utility, with the reference price vector p .

One can write (1) in an alternative form,

$$EV(p, m_0, q, m_1) = (m_1 - m_0) - (e(q, u_1) - e(p, u_1)). \quad (2)$$

Therefore equivalent variation is equal to the change in nominal income corrected for the corresponding change in the cost of living while prices change from p to q . The cost of living is defined as the expenditure needed to obtain the ex post utility level.

Consider a formula where the components on the R.H.S. of (1) are logarithms of the original terms. Similarly, one may go on to form a logarithmic counterpart to (2). Therefore, one finds that (2) which is based on absolute changes is in direct analogy with an expression formulated in terms of relative changes

$$\Delta(\log Q) = \Delta(\log m) - \Delta(\log P), \quad (3)$$

where $\Delta(\log m)$ is the relative increase in nominal income, P denotes for a Konüs cost-of-living index taken at the ex post utility level and Q is the corresponding implicit volume index.

Alternatively, one may arrive at a continuous (differential) formulation of (3) by simply differentiating the logarithm of the budget condition, with the Divisia price index $d(\log P) = \sum_k \hat{w}_k d(\log p_k)$ and the implicit Divisia volume index $d(\log Q) = \sum_k \hat{w}_k d(\log x_k)$, where w_k are the budget shares, $w_k = \partial e(p, u) / \partial \log p_k$. One substitutes for discrete approximation to arrive at (3) with the Törnqvist price index in differential form, $\Delta(\log P) = \sum_k \hat{w}_k \Delta(\log p_k)$, and the corresponding Törnqvist volume index, $\Delta(\log Q) = \sum_k \hat{w}_k \Delta(\log x_k)$, where $\hat{w}_k = 0.5(w_{k0} + w_{k1})$.

¹There is little reason to accept what may be a poor approximation when the proper calculation of Hicksian concepts is scarcely more difficult from observable demand data, for example using the algorithms in Vartia [1983].

Consider a tax structure which is defined by the difference between consumer and producer prices $q - p$. A partial equilibrium measure of the deadweight loss of such a tax structure can be defined as the negative of the equivalent variation from the first-best state 0 which is characterized by prices p and lump-sum taxes which are set equal to the tax revenue $(q - p)x(q, u_1)$ collected in the after-tax state. In this case $m_1 - m_0 = (q - p)x(q, u_1)$, and

$$-EV(p, m_0, q, m_1) = (e(q, u_1) - e(p, u_1)) - (q - p)x(q, u_1), \quad (4)$$

which corresponds to the triangle of loss under the compensated demand curve. Alternatively, it gives directly the smallest amount that the income in the first-best situation could be reduced to achieve at the same utility level as the second-best offers. In the single consumer case it provides an exact welfare indicator for three way comparisons, Pauwels (1978) which is minimized for optimal second-best taxes, Kay (1980).

The above type of indicators can generally be seen as belonging to an input-based (Debreu-Allais) approach to measure inefficiency as opposed to belonging to an output-based (Hicks-Boiteux) approach Diewert [1981]. The latter focuses on answering to the question: "By how much could output be increased?" and the former on the question "By how much could input be reduced?" In tax problems the former approach has more appeal because the "output" is measured in terms of utility levels which have no natural metric while the inputs are endowed with the canonical metric of the physical commodity space, Kay and Keen [1988].

Furthermore, selecting an output-based approach is tantamount to specifying a set of distributional weights in the form of the need to selecting a reference price vector at which to calculate the equivalent gains (King [1983] and Roberts [1980]). It has been noted in the literature that generally aggregate welfare indicators based on weighted sums of equivalent variation across consumers are the preferred choice but they have to be corrected for the cost of attainability of allocations in order to make Pareto consistent welfare comparisons, see Ebert [1985] and [1987].

Kay and Keen (1988) present a unified approach to the calculation of welfare losses in a many-person, general equilibrium framework. Their approach has wide applicability since they are utilizing the input-based method, *Debreu's* [1951] *coefficient of resource utilization* which identifies the input vector with the aggregate endowment of the economy. The 'output' is measured in terms of the utility levels that are achieved in the after-tax situation. The production process of the economy consists of the production possibility set and the second best tax technology. Furthermore, they show that this measure simultaneously generalizes formula (4) and gives it an exact general equilibrium interpretation, see also Ebert [1985]. Section 2 gives a terse treatment of their approach and introduces the relevant concepts and some notation.

In the following sections of the paper a complementary approach to Kay and Keen [1988] is introduced and its connections with previous work explored. Specifically, an alternative input-based measure of inefficiency is presented in form of an index number which provides equivalent amount of information on inefficiency in a general equilibrium framework. In the construction one utilizes the analogy of (3) with (2). As opposed to

Debreu's coefficient of resource utilization the index formula turns out to correspond to the *endowment distance function* of the economy, a generalization of (input) distance function introduced by Shephard [1953] in production theory and by Malmquist [1953] in consumer theory.

Section 3 presents some formulas for the endowment distance function that are applicable either as exact expressions or as providing useful local approximations using the *society's cost-of-living index* which is defined as a weighted sum of individual cost-of-living indices. Some restrictive classes of preference structures which have been used in the applied work provide examples where the calculations are simplified to a significant degree.

In section 4 the connections between the approach of the present paper and index number theory are explored further to evaluate nonmarginal changes in taxes which are accompanied by public intervention. The production interpretation of our input-based measure of inefficiency is exploited to present the Malmquist *index of real endowment*, a society level generalization of the Malmquist *standard-of-living index* in consumer theory and the Malmquist index of real input in production theory. Similarly, one may form the input-based Malmquist productivity indices, and *society's welfare index*, to evaluate two states of the economy. The productivity index compares differences in minimum endowment requirements conditional on the attainment of given utility profiles. The index is greater than one if the *efficiency of the public sector* is increased when the second best taxes underlying the utility producing technology are employed. Since the above indices build directly on the input distance function they offer potentially useful additional tools of assessing the multitude of the effects that public intervention may produce. The approach may be seen as a index number based generalization of the traditional CBA. The *society's welfare index* provides an economy-wide input based measure to evaluate the change in welfare resulting from the transition from state 0 characterized with prices and income and level of public provision to state 1 with change in all the above attributes of the economy. Considering interventions which have more far-reaching effects than just reforming an existing tax-subsidy system has hopefully more relevance and appeal to application.

Section 5 takes up an empirically implementable class of public interventions. Here, a "nonseparable" government intervention affects the marginal rates of substitution (or transformation) in the economy. As a concrete case one may consider e.g. the provision of public goods and services. In this case the effects are in principle estimable from the data by applying the well-known approach of hedonic prices. Explicit expressions and calculations are presented to assist in the application of the index number approach. In particular it turns out that the change in welfare as measured by the *society's welfare index* may be decomposed into a product two components with the first component, *index of real endowment* taking account of the real resource cost in the economy needed to finance the public intervention and the other component the productivity index measuring the change in the "effectiveness" of the economy after a benevolent intervention. The calculations are illustrated by examples in section 6. The last section concludes the study.

2 Input based measures of efficiency

The production process concerns with transformation of inputs $z \in \mathbb{R}_+^k$ into outputs $x \in \mathbb{R}_+^n$ which is modelled by a closed and convex input requirement correspondence $Z(x) = \{z : (-z, x) \in Y\}$, where Y is the (net) production possibility set. Free disposal of net output is assumed for simplicity. Let α be a boundary point, $\alpha \in \partial Z(x)$. By the support theorem for convex sets, $\exists p(\alpha) \in \mathbb{R}^k, p(\alpha) > 0$ such that $z \in Z(x) \implies p(\alpha)(z - \alpha) \geq 0$, (Figure 1).

Consider a pair (x, \bar{z}) , $\bar{z} \in Z(x)$, and define a measure of distance of \bar{z} from the boundary of $Z(x)$ in form of a price based index of value (see Kay and Keen [1988, p. 260–262]).

$$D(x, \bar{z}) = \inf\{p(\alpha)(\bar{z} - \alpha)/p(\alpha)\bar{z} : \alpha \in \partial Z(x)\} = 1 - \sup_{\alpha} \{p(\alpha)\alpha/p(\alpha)\bar{z}\}. \quad (5)$$

It is easy to see that $D(x, \bar{z}) = 1 - \rho$ where the infimum of (5) is obtained in $\tilde{\alpha} = \rho\bar{z}$. Above ρ is the celebrated Debreu's [1951] *coefficient of resource utilization*. Kay and Keen (1988) refer to $D(x, \bar{z})$ as the *Debreu measure of inefficiency* as the value index (5) gives directly the amount of input wasted in comparison to an efficient combination of inputs, see Figure 1, where $D(x, z)$ is given by the ratio BA/BO . Under standard conditions $\rho = 1/\lambda(x, \bar{z})$, where $\lambda(x, \bar{z})$ is the *input distance function* introduced independently by Shephard [1953] and Malmquist [1953].

Assume in the following k commodities and a competitive market in each. Consumer and producer prices are denoted by q and p , respectively.² There are F firms in the economy that behave as price takers with net production vectors $y_f \in Y_f$, and profits $\pi_f = py_f$. There are no externalities between the firms so that the total production set of the economy $Y = \sum_f Y_f$.

There are n households in the economy. Their preferences are characterized by expenditure functions $e_h(q, u_h)$. The household h faces a budget constraint

$$qx_h = e_h(q, u_h) = b_h + p\omega_h + \theta_h\pi, \quad (6)$$

where b_h is the lump-sum grant from the government, ω_h is the factor endowment of the household,³ and θ_h is the household's vector of shares, $\sum_h \theta_h = \mathbf{1}$ in the profits of firms.

The total endowment of the economy $\omega = \sum_h \omega_h$ corresponds to the input vector \bar{z} . The output vector of the economy is the utility profile of the economy in the taxed state

²Introduction of nonlinear consumer prices (for example tax schedules w.r.t. the consumption of some portion of the endowment of leisure) presents no unsurmountable difficulties, in principle. However, one must be careful in defining and proving the existence of the corresponding equilibrium concepts in the relevant markets. Furthermore, the structure of tax equilibria may be of complicated nature, see Fuchs and Guesnerie [1983].

³Kay and Keen make an important distinction as actually measuring ω as traded units. The motivation is that if true endowments were observable one could directly introduce first-best taxes. But if commodity taxes are levied only on transactions one should introduce lump-sum terms corresponding to the consumption of that part which is not transacted in markets, e.g. leisure and labour input used in household production. These points are illustrated in an example in section 6.

$\mathbf{u} = (u_1, \dots, u_n)$. The production process of the economy consists of the production possibility set and the second best tax technology.

Following Kay and Keen [1988] define

$$\Psi(\mathbf{u}) = \sum_h \Psi_h(u_h), \quad (7)$$

where $\Psi_h(u_h) = \{x_h : u_h(x_h) \geq u_h\}$. Above $\Psi(\mathbf{u})$ is the set of aggregate consumption bundles that can be distributed among the consumers so that no one is less well-off than in the taxed state. Furthermore,

$$Z(\mathbf{u}) = \Psi(\mathbf{u}) - \mathbf{Y}, \quad (8)$$

corresponds to the input requirement set $Z(\mathbf{x})$.

Define the aggregate expenditure function

$$E(q, \mathbf{u}) = \sum_h e_h(q, u_h), \quad (9)$$

for the utility vector $\mathbf{u} = (u_1, \dots, u_n)$ and similarly, aggregate compensated demand function

$$X(q, \mathbf{u}) = \sum_h x_h(q, u_h). \quad (10)$$

Equilibrium condition for the economy is

$$X(q, \mathbf{u}) - \omega = y + g, \quad (11)$$

where $y = \sum_f y_f$ and g is the vector of government net production.

Summing over the households budget constraint (6) one finds by the Walras' law that

$$(q - p)X(q, \mathbf{u}) = \sum_h b_h + pg. \quad (12)$$

so that the government's budget constraint is satisfied in an equilibrium. Assume below for simplicity that $g = 0$.⁴

Kay and Keen show that under standard regularity conditions (e.g. Debreu [1959, p. 83–88]) the Debreu measure of inefficiency due to introduction of a tax system and lump-sum grants gives precise and general formulations of the familiar concepts of consumer and producer surplus.

$$\begin{aligned} D(\mathbf{u}, \omega) &= \tilde{p}X(q, \mathbf{u}) - E(\tilde{p}, \mathbf{u}) + \tilde{p}(\tilde{y} - y) \\ &= E(q, \mathbf{u}) - E(\tilde{p}, \mathbf{u}) - (q - p)X(q, \mathbf{u}) \\ &\quad + \tilde{\pi} + \tilde{p}\omega - (\pi + p\omega). \end{aligned} \quad (13)$$

⁴See, however, section 6. Possible costs and distortions associated with the existing second best transfer system, the case considered by Wildasin [1984], could be subsumed under g .

The second equation is obtained by adding the L.H.S. and subtracting the R.H.S. of (11) multiplied with prices p . In addition, Kay and Keen [1988] define \tilde{p} using a convenient normalization of the first-best prices. The index number approach that is developed below is not dependent on the particular normalization.

The expression on the second row of (13) is the sum of negative equivalent variations over the households, see (4) and corresponds to the triangle of loss under the compensated aggregate demand curve. The bottom row corresponds to the general equilibrium measure of lost producer surplus. It consists of the distortion of profit income and the change in the value of aggregate endowment where the latter is interpreted as the rental payment earned by the factors of production in fixed supply.⁵

3 Distance function as an efficiency index

The Debreu measure of inefficiency has the following properties which should according to Kay and Keen [1988] be considered as necessary conditions on any candidate for measure of inefficiency:

P1: $D(\mathbf{u}, \omega) \geq 0$.

P2: $D(\mathbf{u}, \omega) = 0 \iff \mathbf{u}$ is a Pareto efficient utility profile.

P3: $\mathbf{u}_a > \mathbf{u}_b \implies D(\mathbf{u}_b, \omega) > D(\mathbf{u}_a, \omega)$.

Note that output oriented efficiency measures generally suffer from failure to satisfy P3. It may happen that for two initial equilibria of the economy which generate the same utility profiles the associated measures of inefficiency are not equal. Since the Debreu measure is input oriented it identifies an increase in efficiency as an endowment reducing tax change rather than as an Pareto improving tax change as the output oriented approach does.

The condition P3 implies that for two aggregate endowments, $\omega_a > \omega_b \implies D(\mathbf{u}, \omega_a) > D(\mathbf{u}, \omega_b)$. The converse to P3 is generally not valid. However, if output is one-dimensional the converse statement holds. In the above framework this is achieved by introducing a specific concave social welfare function which has as arguments the utility profile $\mathbf{u} = (u_1, \dots, u_n)$, for details see Kay and Keen [1988].

Recall that $D(\mathbf{u}, \omega) = 1 - \rho$ where the utility profile \mathbf{u} is under the first-best obtainable by the endowment $\tilde{\alpha} = \rho\omega$. It is immediately obvious that any monotone transformation of $1 - \rho$ say ψ that is normalized in the sense that $\psi(0) = 0$ satisfies the properties P1-P3. In the sequel one considers the monotone function

$$\psi : (1 - \rho) \longmapsto (1/\rho) = \lambda(\mathbf{u}, \omega), \quad (14)$$

⁵In principle one could allow for productive inefficiency at the initial state in (13), i.e. y is not necessarily profit maximizing at prices p .

where λ is the Malmquist-Shephard input, or here, endowment distance function. This is an index number measuring inefficiency so that $\lambda(\mathbf{u}, \omega) = 1$ is equivalent with \mathbf{u} corresponding to a Pareto efficient utility profile.

Assume below for simplicity that the vector of government production $g = 0$. In analogy with Kay and Keen [1988], our inefficiency index can now be expressed as

$$\frac{1}{\rho} = \lambda(\mathbf{u}, \omega) = \frac{E(q, \mathbf{u}) - (q - p)X(q, \mathbf{u}) - \pi + (\tilde{p} - p)\omega}{E(\tilde{p}, \mathbf{u}) - \tilde{\pi}}. \quad (15)$$

If the production set is a convex cone the total profits are zero and the rental payments earned by the factors in fixed supply are equal to total expenditures valued at producer prices $E(\tilde{p}, \mathbf{u}) = \rho\tilde{p}\omega$,⁶ and

$$\begin{aligned} \frac{1}{\rho} = \lambda(\mathbf{u}, \omega) &= \frac{E(q, \mathbf{u}) - (q - p)X(q, \mathbf{u})}{E(\tilde{p}, \mathbf{u})} + \frac{(\tilde{p} - p)\omega}{\rho\tilde{p}\omega} \\ &= \frac{pX(q, \mathbf{u})}{E(q, \mathbf{u})} \frac{E(q, \mathbf{u})}{E(\tilde{p}, \mathbf{u})} \frac{\tilde{p}\omega}{p\omega}. \end{aligned} \quad (16)$$

Above the input distance function is expressed as a product of three numbers, where the first is the value share of goods and services net of taxes out of total value and the second is *society's cost-of-living index* $E(q, \mathbf{u})/E(\tilde{p}, \mathbf{u})$, a generalized cost-of-living index. The third term consists of the relative value of the total endowment using first best prices in comparison to the income earned by factors of production in the second best situation.

The above analysis shows that the assumption of constant producer prices is in general unnecessary for the measurement of inefficiency. However, it is a common simplification in practical empirical work particularly in the case of assessing marginal changes in taxes with micro data.⁷ Let $\tilde{p} = p$, and write the society's expenditure function in the form $E(q, \mathbf{u}) = \sum_h \exp\{\log e_h(q, u_h)\}$, and introduce budget shares using Shephard's lemma

$$w_{hk}(q, u_h) = \frac{q_k x_{hk}(q, u_h)}{e_h(q, u_h)} = \frac{\partial \log e_h(q, u_h)}{\partial \log q_k}. \quad (17)$$

Now one may write R.H.S. of (16) in the form of a weighted sum

$$\begin{aligned} \lambda(\mathbf{u}, \omega) &= \sum_h s_h \left(1 - \sum_k \frac{(q_k - p_k)}{q_k} w_{hk}(q, u_h) \right) e^{\log e_h(q, u_h) - \log e_h(p, u_h)} \\ &= \sum_h s_h \left(\sum_k \frac{1}{1 + \tau_k} \frac{\partial \log e_h(q, u_h)}{\partial \log q_k} \right) e^{\log e_h(q, u_h) - \log e_h(p, u_h)} \\ &= \sum_k \frac{1}{1 + \tau_k} \frac{\sum_h s_h \partial(e_h(q, u_h)/e_h(p, u_h))}{\partial \log q_k}. \end{aligned} \quad (18)$$

⁶Recall that ρ is the largest radial contraction of the aggregate endowment that could be done without making anybody worse off than in the taxed state.

⁷Changes in producer prices would normally act to lessen the deadweight loss caused by a tax increase.

where the weight function s_h is the individual's h share of the society's total expenditure at producer prices $s_h = e_h(p, u_h)/E(p, \mathbf{u})$, and implicit tax rates τ_k are defined in terms of producer's prices $q_k = (1 + \tau_k)p_k$. The formula (18) is perhaps best interpreted in conjunction with the existence of at least one nontaxable good, say leisure, cf. Kay and Keen interpretation given in footnote 3. The last member on the R.H.S. of (18) involves $e_h(q, u_h)/e_h(p, u_h)$ which is the individual cost-of-living index evaluated at utility level u_h .

The above inefficiency index is in fact based on a covariance between a vector of producer's shares $1/(1 + \tau_k)$, and the price gradient of the society's cost-of-living index at the point with taxes. The latter index is obtained by a weighted sum of individually defined cost-of-living indices. Therefore, inefficiency is decreased as the angle between these two positive vectors is increased. In order to minimize inefficiency one should have low producer's shares in those commodities that have large weight when we calculate the change of the society's cost-of-living index, a version of the Ramsey rule. On the other hand, inefficiency index is maximized if the vector of producer's shares and the gradient of the society's cost-of-living index are collinear.

An additional interesting feature is that since the above measure of inefficiency is calculated as a index number it can easily be written using budget shares and tax-rates instead of using unit taxes and compensated demands. This has some appeal in empirical applications where some data are typically available in the form of indices, with no natural units of measurement. In addition, numerical calculation may work better since value shares are usually more slowly changing variates than quantities, and there are available excellent local approximations, see Vartia [1983].

Similarly as in the case of considering *Debreu measure of inefficiency* one may derive the necessary conditions for the inefficiency index to be minimized and to arrive at the well-known necessary first-order conditions on the optimal commodity taxes. However, this avenue is not pursued in the present paper. Consider instead two interesting preference structures:

1: homothetic preferences.

In this case the expenditure function has the form $e_h(q, u_h) = \beta(q)u_h$. Therefore utility is produced under constant returns. Furthermore, $w_{hk}(q, u_h) = \beta_k^*(q)$ is independent of the utility level u_h or real expenditure. Furthermore,

$$\begin{aligned}\lambda(\mathbf{u}, \omega) &= \frac{\beta(q)}{\beta(p)} \sum_k \frac{w_k(q)}{1 + \tau_k} \\ &= \frac{\beta(q)}{\beta(p)} \left(1 - \sum_k \frac{\tau_k w_k(q)}{1 + \tau_k} \right),\end{aligned}\tag{19}$$

and is independent of any distributional effects.⁸ The coefficient in the front of the sum is simply the true cost-of-living index. In the last expression the sum is taken

⁸Homothetic functional forms are often employed in numerical general equilibrium modelling work, e.g. CES-forms and linear expenditure systems in consumer analysis.

over the taxable commodities and the tax rates $\tau_k/(1 + \tau_k)$ are now calculated taking the after tax situation (ad valorem) as the reference point.

2: budget shares are (locally) independent of relative prices.

The example refers to a generalization of a Stone-Geary utility function with the share parameters smoothly deforming the indifference curves as the utility level changes. In this case the expenditure function must be of the form $\log e_h(q, u_h) = \sum_k \alpha_{hk} \log q_k$, leaving out the constant of integration which depends only on the utility level, giving $w_h(q, u_h) = \alpha_{hk}$, where $\sum_k \alpha_{hk} = 1$. Constant budget shares α_{hk} depend only on utility levels, i.e. on index h .⁹ Similarly,

$$\begin{aligned} \lambda(\mathbf{u}, \omega) &= \sum_h s_h \exp\left\{\sum_i \alpha_{hi} \log(1 + \tau_i)\right\} \sum_k \frac{\alpha_{hk}}{1 + \tau_k} \\ &= \sum_h s_h \frac{\sum_k \partial (\prod_i (1 + \tau_i)^{\alpha_{hi}})}{\partial \tau_k}. \end{aligned} \quad (20)$$

To arrive at an interesting interpretation of the inefficiency index define a probability distribution where a random vector α takes values α_h on unit simplex, $\sum_k \alpha_{hk} = 1$ with point probabilities s_h . Consider the following nonstandard form of the probability generating function $\mathcal{P}(\nu) = \mathcal{E}_\alpha \prod_i (1 + \nu_i)^{\alpha_i}$, which is defined by a multidimensional power series with non-integer powers. Then (20) can be expressed as the sum of the gradient components of the probability generating function at the point $\nu_k = \tau_k$. A Second order Taylor approximation at the point $\nu = (0, 0, \dots, 0)$ gives

$$\begin{aligned} \left[\frac{\partial \mathcal{E}_\alpha (\prod_i (1 + \nu_i)^{\alpha_i})}{\partial \nu_k} \right]_\tau &\approx \mathcal{E}_\alpha \alpha_k - \mathcal{E}_\alpha \alpha_k \tau_k + \sum_i \text{Cov}(\alpha_k, \alpha_i) \tau_i + \mathcal{E}_\alpha \alpha_k \sum_i \mathcal{E}_\alpha \alpha_i \tau_i \\ &+ \frac{1}{2} \left(2\mathcal{E}_\alpha \alpha_k \tau_k^2 - 2\mathcal{E}_\alpha \alpha_k \sum_i \alpha_i \tau_k \tau_i - \mathcal{E}_\alpha \alpha_k \sum_i \alpha_i \tau_i^2 + \mathcal{E}_\alpha \alpha_k \sum_i \sum_j \alpha_i \alpha_j \tau_i \tau_j \right). \end{aligned}$$

Summing over k gives by noting, $\sum_i \alpha_i = 1 \implies \sum_i \text{Cov}(\alpha_k, \alpha_i) = 0$,

$$\begin{aligned} \lambda(\mathbf{u}, \omega) &\approx 1 + \sum_k \sum_i \text{Cov}(\alpha_k, \alpha_i) \tau_i + \sum_k \tau_k^2 \mathcal{E}_\alpha \alpha_k - \sum_i \sum_j \tau_i \tau_j \mathcal{E}_\alpha \alpha_i \alpha_j \\ &= 1 + \frac{1}{2} \sum_k \tau_k^2 \mathcal{E}_\alpha \alpha_k - \frac{1}{2} \sum_k \sum_i \tau_k \tau_i (\text{Cov}(\alpha_k, \alpha_i) + \mathcal{E}_\alpha \alpha_k \mathcal{E}_\alpha \alpha_i). \end{aligned} \quad (21)$$

The leading term in the approximation refers to the inefficiency loss of the mean consumer.

⁹This expenditure function should only be considered as an approximation in a subset of the price space since it cannot be globally increasing in the utility level if the prices are free to take any values. This rather specialized example is included here because in some microsimulation work of tax reforms that neglect a proper modelling of demand responses customarily use a convention of keeping individual budget shares constant to maintain consistent aggregates of the after reform figures.

Return to the general case and invoke the mean value theorem to expand the price gradient of the cost-of-living indices around the no tax point. Now one can write (18) in the form

$$\begin{aligned}
\lambda(\mathbf{u}, \omega) &= \sum_h s_h \left(\sum_k \frac{1}{1 + \tau_k} \frac{\partial \log e_h(p, u_h)}{\partial \log q_k} \right) \\
&- \sum_h s_h \exp\{\log e_h(q^*, u_h) - \log e_h(p, u_h)\} \\
&\times \sum_k \frac{1}{1 + \tau_k} \sum_l \left(\frac{\partial^2 \log e_h(q^*, u_h)}{\partial \log q_k \partial \log q_l} + \frac{\partial \log e_h(q^*, u_h)}{\partial \log q_k} \frac{\partial \log e_h(q^*, u_h)}{\partial \log q_l} \right) \frac{\tau_l}{1 + \tau_l} \\
&= \sum_h s_h \left(\sum_k \frac{w_{hk}(p, u_h)}{1 + \tau_k} \right) \\
&- \sum_h s_h \left(\frac{e_h(q^*, u_h)}{e_h(p, u_h)} \right) \sum_{k,l} \left(\frac{\partial w_{hk}(q^*, u_h)}{\partial \log q_l} + w_{hk}(q^*, u_h) w_{hl}(q^*, u_h) \right) \frac{\tau_l}{(1 + \tau_l)(1 + \tau_k)},
\end{aligned} \tag{22}$$

where $q^* = p + \lambda(q - p)$, i.e. $q_k^* = (1 + \lambda\tau_k)p_k$, with the scalar $\lambda \in [0, 1]$.

The above formula provides an alternative approximation to those previously appearing in the literature. A well-known result due to Diewert shows that the Törnqvist index provides a second-order approximation to the true cost-of-living index and thus the use of it provides a first-order approximation to the change in (18). Furthermore, the second-order approximation is exact if the corresponding linearly homogeneous expenditure function is translog. Similarly, the index by Vartia can be shown to be exact for a CES-aggregator function, see Diewert [1976] and Sato [1981].

In contrast to using index numbers as a tool of approximation the formula (13) due to Kay and Keen [1988] and index form (18) are exact expressions. Next we illustrate the use of distance function (18) in the case of a popular functional form used in the econometric analysis of micro expenditure data to provide an alternative approach to empirical application.

Example 1. Consider a quadratic extension of the logarithmic polar form which has been discovered by Deaton [1986] and is somewhat more flexible than the form used by Blundell et.al. [1993],

$$\log e_h(q, u_h) = \log \alpha(q) - \frac{1}{\beta(q)u + \delta(q)}, \tag{23}$$

where

$$\begin{aligned}
\log \alpha(q) &= \sum_k a_k \log q_k + \frac{1}{2} \sum_k \sum_l \gamma_{kl} \log q_k \log q_l, \\
\log \beta(q) &= \sum_k \beta_k \log q_k, \quad \log \delta(q) = \sum_k \delta_k \log q_k.
\end{aligned}$$

Homogeneity properties of the expenditure function imply that α is a degree one homogenous function in prices q and β and δ are homogenous of degree zero. This

implies that the following restrictions hold for the parameters

$$\begin{aligned}\sum_k a_k &= 1, \\ \sum_l \gamma_{kl} &= 0, \quad \forall k = 1, \dots, m, \\ \sum_k \beta_k &= 0 = \sum_k \delta_k.\end{aligned}$$

In the special case of $\delta(q) = \beta(q)$, the formula reduces to the AIDS-form of the expenditure function as originally introduced by Deaton and Muellbauer [1980]. In applications the parameters a_k and β_k may depend on household characteristics, see Blundell et.al. [1993]. By Shephard's lemma

$$w_k(q, u) = a_k + \sum_l \gamma_{kl} \log q_l + \frac{\beta_k}{\beta(q)u + \delta(q)} + \frac{(\delta_k - \beta_k)\delta(q)}{(\beta(q)u + \delta(q))^2}, \quad (24)$$

$$w_k(q, \bar{m}) = a_k + \sum_l \gamma_{kl} \log q_l + \beta_k \bar{m} + (\delta_k - \beta_k)\delta(q)\bar{m}^2, \quad (25)$$

where $\bar{m} = \log m - \log \alpha(q)$, a kind of real expenditure term deflated by the function $\alpha(q)$.

The logarithmic cost-of-living index is given by

$$\begin{aligned}\log \left(\frac{e_h(q, u_h)}{e_h(p, u_h)} \right) &= \sum_k a_k \tau_k + \frac{1}{2} \sum_k \sum_l \gamma_{kl} (2 \log q_k - \tau_k) \tau_l + (m_h - \log \alpha(q)) \\ &\quad - \frac{\exp\{\sum_k \beta_k \tau_k\} (m_h - \log \alpha(q))}{1 + (m_h - \log \alpha(q)) \exp\{\sum_k \delta_k \log q_k\} (1 - \exp\{\sum_k (\beta_k - \delta_k) \tau_k\})},\end{aligned} \quad (26)$$

where $\tau_k = \log q_k - \log p_k$, and m_h is the total consumption outlay of consumer h after introduction of second best taxes, given by (6). Now (25) and (26) give the needed components of (18).¹⁰

4 Malmquist indexes and evaluation of nonmarginal public intervention

In the following, the analogies with the index number theory and our approach to calculate welfare losses of taxation are explored further. The production interpretation of our input-based measure of inefficiency is exploited to present the Malmquist index of real endowment, a society level generalization of the Malmquist *standard-of-living index* in consumer theory and the Malmquist index of real input in production theory. Similarly, one may form the input-based Malmquist productivity index, and *society's welfare*

¹⁰Because the above measure of inefficiency is calculated as an index number involving budget shares and tax-rates, calculation of its confidence limits may work better since value shares and parameters in (23) are usually better behaved than corresponding quantities.

index. Since these indices build directly on the input distance function they offer potentially useful tools of assessing the multitude of the effects that public intervention may produce, see Caves, et.al. [1982] for analogous results in the production theory.

Consider comparing two states of the economy, (\mathbf{u}^j, ω^j) , $j = 1, 2$, characterized by the utility profiles and total endowments of the economy. Similarly as before the production technology consists of the production possibility set and the second best tax technologies. The Malmquist index of real endowment with respect to the technology in state two and utility profile \mathbf{u}^2 is given by

$$Q^2(\omega^2, \omega^1) = \frac{\lambda^2(\mathbf{u}^2, \omega^2)}{\lambda^2(\mathbf{u}^2, \omega^1)}. \quad (27)$$

In consumer theory the index (27) is the standard-of-living index introduced by Malmquist [1953] which is dual to the Konüs cost-of-living index. Note that an increase in the aggregate endowment $\omega^2 > \omega^1 \implies Q^2(\omega^2, \omega^1) > 1$. On the other hand, $Q^2(\omega^2, \omega^1) > 1$ implies that endowment ω^2 is greater than ω^1 from the point of view of attaining the utility profile \mathbf{u}^2 .

Suppose that \mathbf{u}^2 is associated with consumer prices q^2 , second best tax rates $1 + \tau_{2k} = q_k^2/p_k^2$, and lump-sum transfers as in (6). In analogy with (15) one can produce the formula

$$Q^2(\omega^2, \omega^1) = \frac{(E(q^2, \mathbf{u}^2) - (q^2 - p^2)X(q^2, \mathbf{u}^2) - \pi^2 + (\tilde{p}^2 - p^2)\omega^2)(E(\tilde{p}^1, \mathbf{u}^2) - \tilde{\pi}^1)}{(E(\tilde{p}^2, \mathbf{u}^2) - \tilde{\pi}^2)(E(q^1, \mathbf{u}^2) - (q^1 - p^1)X(q^1, \mathbf{u}^2) - \pi^1 + (\tilde{p}^1 - p^1)\omega^1)}. \quad (28)$$

Above p^i is the vector of producer prices associated with the equilibrium $(q^i, \mathbf{u}^i, \omega^i, \tau^i)$, and similarly \tilde{p}^i supports the efficient outcomes, $i = 1, 2$. Assume for simplicity that profits are zero. Now the formula (16) corresponds to

$$Q^2(\omega^2, \omega^1) = \frac{\sum_h s_{2h} \frac{e_h(q^2, u_h^2)}{e_h(\tilde{p}^2, u_h^2)} \sum_k \frac{w_{hk}(q^2, u_h^2)}{1 + \tau_{2k}} \frac{\tilde{p}^2 \omega^2}{p^2 \omega^2}}{\sum_h s_{1h} \frac{e_h(q^1, u_h^2)}{e_h(\tilde{p}^1, u_h^2)} \sum_k \frac{w_{hk}(q^1, u_h^2)}{1 + \tau_{1k}} \frac{\tilde{p}^1 \omega^1}{p^1 \omega^1}}, \quad (29)$$

where the weight function s_{ih} is the individual's h share of the society's total expenditure at producer prices \tilde{p}^i , $s_{ih} = e_h(\tilde{p}^i, u_h^i)/E(\tilde{p}^i, \mathbf{u}^i)$, $i = 1, 2$. Notice that the index (29) is a ratio where both the denominator and the nominator are formed by multiplying the covariance between the producer's shares and the price gradient of the society's cost-of-living index with the value of the total endowment in relation to the earnings of the factors of production in the second best.

In the special case of homothetic preferences

$$Q^2(\omega^2, \omega^1) = \frac{\beta(\tilde{p}^1)\beta(q^2) \left(\sum_k \frac{w_k(q^2)}{1 + \tau_{2k}} \right) \tilde{p}^2 \omega^2 p^1 \omega^1}{\beta(\tilde{p}^2)\beta(q^1) \left(\sum_k \frac{w_k(q^1)}{1 + \tau_{1k}} \right) \tilde{p}^1 \omega^1 p^2 \omega^2}. \quad (30)$$

The above formula is a product of three ratios. The first consists of price indices taken at the consumer prices and at prices supporting the corresponding first-best. The

second ratio is taken between expressions that are sums of the by now familiar covariance between second-best producer's shares and the budget shares. The third ratio consists of the income earned by fixed factors, and the value of total endowment at first best prices.

Furthermore, one may go on to construct the input based Malmquist productivity index using the implicit utility technology in state 2, see Caves, et.al. [1982].

$$M^2(\omega^2, \omega^1, \mathbf{u}^2, \mathbf{u}^1) = \frac{\lambda^2(\mathbf{u}^1, \omega^1)}{\lambda^2(\mathbf{u}^2, \omega^2)}. \quad (31)$$

The index compares differences in minimum endowment requirements conditional on the attainment of the given utility profiles. The index is greater than one if the endowment ω^2 and the second best taxes underlying the utility producing technology in state 2 are more productive than in the alternative state, i.e. the efficiency in the economy is increased.

The above equation offers a convenient tool to evaluate the effectiveness of government intervention. Specifically the index calculates the relative change in the inefficiency measure of taxation induced by the intervention. Government intervention may in this formulation affect both initial endowments and utility profiles in the economy. The change in the former may happen through the provision of some public goods and public services, say free of charge, and the latter in the form of introducing second-best taxes and lump-sum transfers.

The following index which is dual to the Malmquist index of real endowment provides a society level generalisation of the original Malmquist welfare index which has the aggregate endowment level, say ω^2 as the point of reference,

$$U^2(\mathbf{u}^2, \mathbf{u}^1) = \frac{\lambda^2(\mathbf{u}^1, \omega^2)}{\lambda^2(\mathbf{u}^2, \omega^2)}. \quad (32)$$

By the condition P3, Pareto improvement in the economy implies that the above welfare index is increased, but the converse implies only potential Pareto improvement.¹¹ On the other hand,

$$M^2(\omega^2, \omega^1, \mathbf{u}^2, \mathbf{u}^1)Q^2(\omega^2, \omega^1) = \frac{\lambda^2(\mathbf{u}^1, \omega^1)}{\lambda^2(\mathbf{u}^2, \omega^1)}, \quad (33)$$

which is the Malmquist welfare index which has in turn the aggregate endowment level ω^1 as a point of reference.

There exists a complementary Malmquist index of real endowment which has the utility profile \mathbf{u}^1 and the corresponding technology as a point of reference,

$$Q^1(\omega^2, \omega^1) = \frac{\lambda^1(\mathbf{u}^1, \omega^2)}{\lambda^1(\mathbf{u}^1, \omega^1)}. \quad (34)$$

Following Caves, et.al. [1982] in the production context one may form the geometric mean of the indices (27) and (34). However, under the utility producing framework of

¹¹Using the Kaldor criterion that the gainers in state two can fully compensate the losers.

the present paper the geometric mean has far less relevance.¹²

$$[Q^1(\omega^2, \omega^1)Q^2(\omega^2, \omega^1)]^{\frac{1}{2}} = \left(\frac{\lambda^1(\mathbf{u}^1, \omega^2)}{\lambda^1(\mathbf{u}^1, \omega^1)} \frac{\lambda^2(\mathbf{u}^2, \omega^2)}{\lambda^2(\mathbf{u}^2, \omega^1)} \right)^{\frac{1}{2}} = \left(\frac{\rho_1^1 \rho_1^2}{\rho_2^1 \rho_2^2} \right)^{\frac{1}{2}}, \quad (35)$$

which is expressed in terms of the corresponding *Debreu's coefficients of resource utilization* with some self-explaining notation.

Similarly, one may define productivity index using the implicit utility technology in state 1

$$M^1(\omega^2, \omega^1, \mathbf{u}^2, \mathbf{u}^1) = \frac{\lambda^1(\mathbf{u}^1, \omega^1)}{\lambda^1(\mathbf{u}^2, \omega^2)}, \quad (36)$$

and $U^1(\mathbf{u}^2, \mathbf{u}^1)$, and potential Pareto improvement using the Hicks criterion that the losers in state two cannot bribe the gainers to oppose the change.

Consider circumstances where government intervention has a nonseparable effect on the utility levels of consumers, i.e. the marginal rates of substitution (MRS) are not invariant but change after the intervention. Now one may in principle measure the effect through the visible effect in the consumption patterns of households. In the alternative case where the marginal rates of substitution are invariant, the possible efficiency increase cannot be estimated from the observed behaviour of the MRS. In this case possible economies or diseconomies created by the the intervention are impossible to identify even if they exist. Similar observations hold with respect to production technology since scale effects may be difficult to distinguish from nonseparable effects of government intervention, see Sato (1981). However, due to the direct observability of output one has correspondingly somewhat less strict conditions for identifiability.

A specific case of public intervention could be setting emission standards to limit air pollution. Standards operate as restrictions in production technology, e.g. in power generation. On the other hand, polluting may hurt production in other sectors of the economy. Laundry business offers a classic example. Therefore, after setting some environmental standards it may be possible to use less inputs to produce the same output as before in some sectors of the economy and possibly the converse holds in some other sectors. In the following public intervention is examined from this perspective and the well-known hedonic scales are employed to the above purpose.

¹²In the case of a translog distance function the Törnqvist index is consistent with a flexible representation of technology. In addition, if λ^1 and λ^2 are translog and have identical second order coefficients it is exact for this geometric mean, if the corresponding society level utility profile used as a reference is the mean of original profiles, see the analogous result by Caves, et.al. [1982]. On the other hand, Blac-korby et.al [1978] show that once separability is imposed (i.e. additive separability over the members of the society in the "utility production" framework considered in this paper) translog function is no longer capable of providing a second order approximation to an arbitrary separable endowment distance function. Therefore, in a many person economy with nonhomothetic preferences the translog function must be interpreted as an exact functional form and not as an approximation.

5 Hedonic prices in uncovering the effects of public provision

Hedonic prices were originally motivated by the need to adjust cost of living indices for quality improvements (the classic reference is Fisher and Shell [1972]). This approach views that the quality of a good or a service is related to measurable characteristics such as performance ratings. In the process of estimating a demand function of the consumer (or the production technology) the shadow prices of the quality characteristics of a commodity (or an input) are also estimated. The existence of a quality-adjusted price index implies the existence of a corresponding quality-adjusted quantity index. Multiplication of the two adjusted indices yields the same value index as would the unadjusted indices. The standard results of neoclassical theory, especially those related to various forms of duality, hold in the quality-adjusted price and quantity space once we base the analysis on quality-adjusted indices.¹³

The above approach offers a convenient way to conceptually separate the effects of taxes and subsidies from public intervention of a more general nature. Furthermore, the introduction of public goods and publicly provided non-transferable services may in several instances be seen as changing the marginal rates of substitution between private goods. Similar effects may occur as public intervention of a more general nature takes place.

To be more specific, consider a public facility of size g . Assume that after its introduction consumers may use ordinary commodities more effectively, say x_k units of the commodity k in the new situation are equally effective as $\theta_k(g)x_k$ units before the introduction of the facility. Equivalently, one may think of this innovation as corresponding to the price of the commodity k , q_k being lowered to $q_k^\theta = q_k/\theta_k(g)$ in the initial state. Below one considers transformed quantities and prices

$$\begin{aligned} x^\theta &= \Theta(g)x, \quad y^\theta = \Theta(g)y, \quad q^\theta = \Theta(g)^{-1}q, \quad p^\theta = \Theta(g)^{-1}p \\ e_h(q^\theta, u_h^\theta) &= q^\theta x^\theta(q^\theta, u_h^\theta) = qx^*(q, u_h^\theta) = e_h^\theta(q, u_h^\theta) \\ p^\theta y^\theta(p^\theta) &= py^*(p), \end{aligned} \tag{37}$$

where $\Theta(g)$ is the diagonal matrix with constants $\theta_k(g)$ on the diagonal, $\Theta(g) = \text{diag}(\theta_1, \dots, \theta_n)$.¹⁴

The utility level of the consumer is changed to u_h^θ , $u_h^\theta(x_h) = u_h(x_h^\theta) = u_h(\Theta x_h)$, after the introduction of the public facility. In Figure 2 the initial choice of the consumer facing

¹³Hedonic prices have been used in numerous cases other than mere adjustment of price indices for quality change, e.g. in the construction of equivalence scales to adjust the consumption levels of households for differences in household composition, in several industry-level production functions and in evaluating the effects of a major new urban transit system or the siting of a waste dump.

¹⁴Gorman in his 1976 article proposes an immediate extension of the simple structure in (37) by allowing $\Theta(g)$ to be an arbitrary positive matrix to provide a change of coordinates (with presumably a positive inverse, Gorman leaves to the reader the details of the model) in conjunction with a translation of coordinates. The extension may be interpreted as allowing for spillover effects in household production. Most of the results given below can be extended to cover the more general affine case.

prices $p = (1, 1)$ is given by the point x corresponding to the utility level U . Consumer goods are enhanced with $\Theta = \text{diag}(2, 0)$ and effectively the price of commodity 1 is lowered by 50 per cent. The budget line (in efficiency units) is subsequently shifted to the right giving a choice x^θ with a higher utility level U^θ . In original (physical) units of measurement x^θ corresponds to x^* and the indifference curve indexed by U^θ is deformed to the curve U^* drawn with a dotted curve in Figure 2.

Consider the equilibrium of the economy after the introduction of lump-sum transfers b_h , consumer prices q , and a facility with a hedonic effect characterized by Θ , with the corresponding attainable utility profile u^θ . The facility is "financed" by resource cost $-g$ which is now interpreted as the government net demand for final goods rather than government production as in the earlier case. The individual (6), and the government budget condition (12) are modified accordingly with the lump-sum transfers adjusted for the total costs of the facility, $-pg$, valued at producer prices, see (11).

Start with the equilibrium condition (11) in the final state. By (37) the expenditure functions corresponding to the initial state $e_h(q^\theta, u_h^\theta)$, and the final state $e_h^\theta(q, u_h^\theta)$ are identical. Therefore one may write

$$y(p) + \omega - g = X(q, u^\theta) = \Theta^{-1} X^\theta(q^\theta, u^\theta). \quad (38)$$

Next multiply (38) from the left by the matrix Θ . The aggregate supply $y^\theta(p) + \Theta(\omega - g)$ corresponds to an equilibrium in a modified economy with the net production set $\sum \Theta Y_f$ and endowment $\Theta(\omega - g)$ since y_f maximizes the profits py_f in Y_f is equivalent with the statement y_f^θ maximizes the profits in ΘY_f if the prices are p^θ .

Furthermore, the reference point $p(\alpha)$ corresponding to the utility profile u^θ in the final state changes to the supporting hyperplane characterized by $\Theta^{-1}p(\alpha)$ corresponding to the utility profile u^θ in the modified initial state.

Therefore, the "after tax and facility" equilibrium with prices p and q and resources ω corresponds to an equilibrium in a modified initial state with prices p^θ and q^θ and where the net production set is ΘY , and the resources available to the economy are simultaneously enhanced and adjusted for the resource costs $-g$, giving $\omega^\theta - g^\theta = \Theta(g)(\omega - g)$. The welfare cost of the subsidy and tax system $q^\theta - p^\theta$ associated with the last equilibrium corresponds to $\lambda(u^\theta, \omega^\theta - g^\theta)$. This is equal to $\lambda^\theta(u^\theta, \omega - g)$, the welfare cost of the tax system $q - p$ in the final state.¹⁵

The Malmquist indices of real endowment (29), with respect to the utility profile u^θ , at the final state and u at the initial state, respectively are given by

$$Q^\theta(\omega - g, \omega) = \frac{\lambda^\theta(u^\theta, \omega - g)}{\lambda^\theta(u^\theta, \omega)} = \frac{\lambda(u^\theta, \Theta(\omega - g))}{\lambda(u^\theta, \Theta\omega)}, \quad (39)$$

$$Q(\omega - g, \omega) = \frac{\lambda(u, \omega - g)}{\lambda(u, \omega)} = \frac{\lambda^\theta(u, \Theta^{-1}(\omega - g))}{\lambda^\theta(u, \Theta^{-1}\omega)}, \quad (40)$$

where the distance functions without index θ refer to the initial state with the modified net production set. These indices measure the volume of total resources available after

¹⁵If the welfare cost is measured in terms of the original economy one must assume $\Theta Y \subset Y + \mathbb{R}_+\omega$.

deducting for the input requirements due to providing the public facility. If $g = \nu\omega$, i.e. the share of these resource costs in the value of total resources is independent of the producer prices used as the reference price vector in the economy, then

$$Q^\theta(\omega - g, \omega) = 1 - \nu = Q(\omega - g, \omega), \quad (41)$$

i.e. the indices of real input are in this case independent on the choice of the reference utility profile.

The endowment-based Malmquist productivity index (31) with respect to the second best tax technology available after intervention, is given by

$$\begin{aligned} M^\theta(\omega - g, \omega, \mathbf{u}^\theta, \mathbf{u}) &= \frac{\lambda^\theta(\mathbf{u}, \omega)}{\lambda^\theta(\mathbf{u}^\theta, \omega - g)} = \frac{\lambda(\mathbf{u}, \Theta\omega)}{\lambda(\mathbf{u}^\theta, \Theta(\omega - g))} \\ &= M(\Theta(\omega - g), \Theta\omega, \mathbf{u}^\theta, \mathbf{u}). \end{aligned} \quad (42)$$

The society's welfare index with the endowment $\omega^\theta - g^\theta$ as a point of reference is given by

$$\begin{aligned} U^\theta(\mathbf{u}^\theta, \mathbf{u}) &= \frac{\lambda^\theta(\mathbf{u}, \omega - g)}{\lambda^\theta(\mathbf{u}^\theta, \omega - g)} = \frac{\lambda(\mathbf{u}, \Theta(\omega - g))}{\lambda(\mathbf{u}^\theta, \Theta(\omega - g))} \\ &= \frac{\lambda(\mathbf{u}, \Theta\omega)/\lambda(\mathbf{u}^\theta, \Theta(\omega - g))}{\lambda(\mathbf{u}, \Theta\omega)/\lambda(\mathbf{u}, \Theta(\omega - g))} \\ &= M(\Theta(\omega - g), \Theta\omega, \mathbf{u}^\theta, \mathbf{u}) \times Q(\Theta(\omega - g), \Theta\omega). \end{aligned} \quad (43)$$

Note that $U^\theta > 1$ does not necessarily imply a Pareto improvement in the economy. The welfare index is a product of two indices. The gain from the introduction of the public facility consists of the impact in the "effectiveness" of the public sector as measured by the productivity index M after a benevolent public sector intervention which takes account of the potential increase in the virtual endowment in the economy ω , and the potential net change in the welfare loss of taxation which is due to the tax-subsidy system considered. On the other hand, the Malmquist index of real endowment, Q directly measures the volume of resources available after deducting for the real resource costs of providing the public facility.

An interesting ratio of the inefficiency indices in before and after intervention situations is representable using the welfare index, see (40) and (42), as

$$\begin{aligned} \frac{\lambda(\mathbf{u}, \omega)}{\lambda^\theta(\mathbf{u}^\theta, \omega - g)} &= \frac{M(\Theta(\omega - g), \Theta\omega, \mathbf{u}^\theta, \mathbf{u})}{Q(\Theta\omega, \omega)} \\ &= \frac{U^\theta(\mathbf{u}^\theta, \mathbf{u})}{Q(\Theta(\omega - g), \Theta\omega)Q(\Theta\omega, \omega)} = \frac{\rho_\theta}{\rho}, \end{aligned} \quad (44)$$

where in the first-best the modified economy can achieve utility profile \mathbf{u}^θ with total endowment $\alpha_\theta = \rho_\theta\Theta(\omega - g)$. Similarly, the economy can achieve utility profile \mathbf{u} with total endowment $\alpha = \rho\omega$. A strong sufficient condition which allows for \mathbf{u}^θ to be a

potential Pareto improvement P3 relative to the initial state \mathbf{u} using redistribution in each commodity, is that $\alpha_\theta > \alpha$. In the special case where $g = \nu\omega$, i.e. resource costs are independent of the reference producer price vector, the inequality is transformed into

$$\frac{U^\theta(\mathbf{u}^\theta, \mathbf{u})}{Q(\Theta\omega, \omega)} \Theta\omega > \omega. \quad (45)$$

To get an example of the calculations involved suppose again for simplicity that profits are zero. Furthermore, assume the case of homothetic preferences to further simplify the calculations, see (30). In this case, one gets

$$Q^\theta(\omega - g, \omega) = \frac{\beta(\tilde{p}_\theta^0)\beta(q_\theta^g) \left(\sum_k \frac{w_k(q_\theta^g)}{1+\tau_k^g} \right) \frac{\tilde{p}^g(\omega-g)}{p^g(\omega-g)}}{\beta(\tilde{p}_\theta^g)\beta(q_\theta^0) \left(\sum_k \frac{w_k(q_\theta^0)}{1+\tau_k^0} \right) \frac{\tilde{p}^0\omega}{p^0\omega}}, \quad (46)$$

and

$$\begin{aligned} M(\Theta(\omega - g), \Theta\omega, \mathbf{u}^\theta, \mathbf{u}) &= \frac{\lambda(\mathbf{u}, \Theta\omega)}{\lambda(\mathbf{u}^\theta, \Theta(\omega - g))} \\ &= \frac{\beta(\tilde{p}_\theta^g)\beta(\tilde{q}^0) \left(\sum_k \frac{w_k(\tilde{q}^0)}{1+\tilde{\tau}_k} \right) \frac{\tilde{p}^0\Theta\omega}{p^0\Theta\omega}}{\beta(\tilde{p}^0)\beta(q_\theta^g) \left(\sum_k \frac{w_k(q_\theta^g)}{1+\tau_k^g} \right) \frac{\tilde{p}^g(\omega-g)}{p^g(\omega-g)}}. \end{aligned} \quad (47)$$

Above efficiency units $\tilde{p}_\theta^0 = \Theta^{-1}\tilde{p}^0$ are used and the superscript g indicates instances where the tax system, $q^g - p^g$ has to finance the facility with a resource cost $-p^g g$, (in physical units of measurement) and lump sum transfers b_h . In general, there are now changes both in the total endowment of the economy and in the effectiveness of the public sector. Since the equilibrium prices are generally changed, the owners of resources with a "virtual increase" in endowment may be additionally affected by a decrease in the producer price and their total earnings may go even down giving respectively less scope for taxation of owners. These somewhat unexpected effects are illustrated in the next section.

6 Examples

Example 2.¹⁶ Consider an exchange economy with two commodities, so that $Y = \{(0, 0)\}$. The traders, l and c initially hold $(1, 0)$ and $(0, 1)$, respectively. Both traders have identical preferences which are homothetic. The indifference curve has slope -1, at $(1, 1)$ and on the line $x = 2y$ it has slope $-1/8$ (Figure 3).

Since the preferences are homothetic, the budget line corresponding to equilibrium prices is tangent to the indifference curve at the total endowment, i.e. it has slope -1

¹⁶This example has been previously used by Aumann and Peleg (1974) to show that a competitive equilibrium can be manipulated by discarding endowments. Details are available on request.

at $\omega = (1, 1)$. Therefore, the equilibrium prices are in the ratio 1:1 and the competitive bundles for both traders lie on the line $x = y$, and are equal to $x = (1/2, 1/2)$ (Figure 3).

Introduce government to the economy. Government provides a public good (education) which doubles the (leisure) endowment of trader l as measured in efficiency units $\omega^\theta = \Theta\omega$, with $\Theta = \text{diag}(2, 0)$. The resource cost of providing the public good g^θ is $1/5$ of the total endowment.

Consider the first-best case where the public good is financed by a tax on the *consumption* of the second commodity, production. The tax rate is set to $\tau_c^\theta = (q_2^\theta - p_2^\theta)/p_2^\theta = 1/3$.¹⁷ The society's real resource cost of providing the public good is measured by standard of living indices, see (39) and (40). These are independent of the reference utility level and are both equal to $4/5$. Since a first best situation is considered, $\lambda(\mathbf{u}^\theta, \Theta(\omega - g)) = 1$.

By Walras' law the government's budget condition is always satisfied in an equilibrium in the commodity markets, see (12). The endowment available after reduction due to the resource costs of providing the public good is $\omega^\theta - g^\theta = (8/5, 4/5)$.¹⁸ By homotheticity, the budget line associated with the equilibrium prices, q^θ , is supporting the indifference curve at the above point. The prices are $q^\theta = (1, 8)$, giving producer income 2 to trader l and 6 to trader c , as measured in units of leisure. The equilibrium bundles lie on the line $2y = x$, and are $x_l^\theta = (2/5, 1/5)$, $x_c^\theta = (6/5, 3/5)$ (Figure 3). The tax burden is proportional to income $T_l = 2/5$, and $T_c = 6/5$, in units of leisure. In this case the Malmquist productivity index (42) which indicates the effectiveness of the public sector is by construction of the example

$$M^\theta(\omega - g, \omega, \mathbf{u}^\theta, \mathbf{u}) = \frac{\lambda(\mathbf{u}, \Theta\omega)}{\lambda(\mathbf{u}^\theta, \Theta(\omega - g))} = \frac{\|\omega^\theta\|/\|\omega^\theta - g^\theta\|}{1} = \frac{5}{4}. \quad (48)$$

On the other hand, the society's welfare index

$$U^\theta(\mathbf{u}^\theta, \mathbf{u}) = \frac{\lambda(\mathbf{u}, \Theta(\omega - g))}{\lambda(\mathbf{u}^\theta, \Theta(\omega - g))} = 1, \quad (49)$$

and shows no change. Note that after introducing the public good the trader l is considerably worse-off than before whereas the trader c has a substantial increase in utility.

In the above case the government was able to *tax the consumption* of the second commodity and no inefficiencies were created by taxation. Consider next the second-best case where the government can only set a *tax on market transactions* of the commodity, production. The example is otherwise the same as before but the resource cost due to the provision of the public good are increased to $g^\theta = (4/5, 1/5)$, and the tax rate is correspondingly raised to $\tau_c^\theta = (q_2^\theta - p_2^\theta)/p_2^\theta = 3$ (Figure 4). The equilibrium is obtained at the same consumer prices $q^\theta = (1, 8)$, as before with bundle on the line $2y = x$,

¹⁷All calculations are made in terms of the initial state, in efficiency units $(x^\theta, y^\theta) = (\Theta x, \Theta y)$, and $(p^\theta, q^\theta) = (\Theta^{-1}p, \Theta^{-1}q)$, cf. Figure 2.

¹⁸The example is constructed so as to keep the points ω and $\omega^\theta - g^\theta$ on the same indifference curve.

for trader l , $x_l^\theta = (2/5, 1/5)$, and income of one unit of leisure. Since tax is levied only on transactions the budget line for trader c is defined by 'producer prices' which are in equilibrium at $p^\theta = (1, 2)$, giving a bundle $x_c^\theta = (4/5, 3/5)$, with income of two units of leisure (Figure 4). The entire tax burden is seemingly on the trader l , with $T_l = 6/5$, and $T_c = 0$. However, the terms of trade are less favourable to c than if the first best taxes were used so the incidence of the tax lies more heavily on him. The first best equilibrium of the economy corresponding to the endowment $\omega^\theta - g^\theta$ is supported by prices which correspond to allocations on the line from the origo to $\omega^\theta - g^\theta$. The utility levels corresponding to the second best allocations, $\{x_l^\theta, x_c^\theta\}$ are obtainable by an endowment ω^* lying just below the point $\omega^\theta - g^\theta$ (Figure 4). In the first best case the tax burden is proportional to the income of traders as in the case considered above.

The inefficiency index corresponding to the use of second best taxes relative to the first best is equal to

$$\lambda^\theta(\mathbf{u}^\theta, \omega - g) = \frac{\|\omega^\theta - g^\theta\|}{\|\omega^*\|}. \quad (50)$$

The society's real resource cost of providing the public good is measured by the standard-of-living indices with respect to the after intervention utility profiles \mathbf{u}^θ and the initial utility profiles \mathbf{u} , respectively. These are given by

$$\begin{aligned} Q^\theta(\omega - g, \omega) &= \frac{\lambda^\theta(\mathbf{u}^\theta, \omega - g)}{\lambda^\theta(\mathbf{u}^\theta, \omega)} = \frac{\|\omega^\theta - g^\theta\|/\|\omega^*\|}{\|\omega^\theta - g^\theta\|/\|\alpha^\theta\|} < 1 \\ Q(\omega - g, \omega) &= \frac{\lambda(\mathbf{u}, \omega - g)}{\lambda(\mathbf{u}, \omega)} = \frac{\|\omega - g\|/\|\beta\|}{\|\omega\|/\|\omega\|} < 1. \end{aligned} \quad (51)$$

The dual concepts that correspond to those above are the society's welfare indices

$$\begin{aligned} U^\theta(u^\theta, u) &= \frac{\lambda^\theta(\mathbf{u}, \omega - g)}{\lambda^\theta(\mathbf{u}^\theta, \omega - g)} = \frac{\|\omega^\theta - g^\theta\|/\|\mu(\omega^\theta - g^\theta)\|}{\|\omega^\theta - g^\theta\|/\|\omega^*\|} = \frac{\|\omega^*\|}{\mu\|\omega^\theta - g^\theta\|} < 1 \\ U(u^\theta, u) &= \frac{\lambda(\mathbf{u}, \omega - g)}{\lambda(\mathbf{u}^\theta, \omega - g)} = \frac{\|\omega - g\|/\|\beta\|}{\|\omega - g\|/(\|\omega^*\|\|\beta\|/(\mu\|\omega^\theta - g^\theta\|))} = U^\theta(u^\theta, u), \end{aligned} \quad (52)$$

where μ is the radial expansion of the vector $\omega^\theta - g^\theta$ needed in order to just reach the indifference curve corresponding to the initial endowment ω (dotted curve in Figure 4). Similarly, $\|\omega^*\|/(\mu\|\omega^\theta - g^\theta\|)$ gives the radial contraction of the vector β needed in order to just reach the indifference curve corresponding to the endowment ω^* .¹⁹

Now the society's welfare index is decreased, implying that potential Pareto improvement has not taken place. This is due to the increased resource cost and the effect involves both an improvement in the situation of the trader c and a substantial worsening in the situation of the trader l , relative to the initial situation. But somewhat surprisingly, the increased cost is not affecting the utility level of the trader l if the comparison is made in reference to the first best situation considered above.

¹⁹To see this, note that with homothetic preferences the utility function may be assumed to be linearly homogenous with no loss of generality. Then $U(\beta) = u_\omega \implies U((1/\mu)\beta) = (1/\mu)u_\omega$.

In this case the Malmquist productivity indices (31) (47) that correspond to the effectiveness public sector intervention are given by

$$\begin{aligned}
 M^\theta(\omega - g, \omega, \mathbf{u}^\theta, \mathbf{u}) &= \frac{\lambda(\mathbf{u}, \Theta\omega)}{\lambda(\mathbf{u}^\theta, \Theta(\omega - g))} = \frac{\|\omega^\theta\|/\|\alpha\|}{\|\omega^\theta - g^\theta\|/\|\omega^*\|} > 1 \\
 M(\omega - g, \omega, \mathbf{u}^\theta, \mathbf{u}) &= \frac{\lambda(\mathbf{u}, \omega)}{\lambda(\mathbf{u}^\theta, \omega - g)} = \frac{\|\omega\|/\|\omega\|}{\|\omega - g\|/(\|\omega^*\|/\|\beta\|/(\mu\|\omega^\theta - g^\theta\|))} \\
 &= \frac{\|\omega^*\|/\|\beta\|}{\mu\|\omega - g\|\|\omega^\theta - g^\theta\|} > 1.
 \end{aligned} \tag{53}$$

They both show an increase in the effectiveness of the public sector. On the other hand, this is more than counterbalanced by the heavy cost of providing the public good (52).

Example 3. To present an example for use in econometric estimation of the hedonic effects consider the quadratic extension of the AIDS-form (23) with hedonic price indices. These indices depend on the size of the facility through the function, say $\theta_k(g)$, defined by a suitable, e.g. loglinear parametrization. The parameters present in θ_k may be estimated from the equation system

$$\begin{aligned}
 w_k^\theta(q^\theta, u^\theta) &= \frac{\partial \log e_h^\theta(q^\theta, u_h^\theta)}{\partial \log q_k^\theta} \\
 &= a_k + \sum_l \gamma_{kl} \log q_l^\theta + \beta_k \bar{m}^\theta + (\delta_k - \beta_k) \delta(q^\theta) (\bar{m}^\theta)^2 \\
 &= a_k + \sum_l \gamma_{kl} (\log q_l + \log \theta_l) + \beta_k \bar{m}^\theta + (\delta_k - \beta_k) \delta(q^\theta) (\bar{m}^\theta)^2
 \end{aligned} \tag{54}$$

where $\bar{m}^\theta = \log m - \log \alpha(q^\theta)$, and

$$\log \alpha(q^\theta) = \sum_k a_k (\log q_k + \log \theta_k) + \frac{1}{2} \sum_k \sum_l \gamma_{kl} (\log q_k + \log \theta_k) (\log q_l + \log \theta_l). \tag{55}$$

In general, one is only able to identify relative hedonic scales, i.e. in the loglinear case one has to set $\log \theta_k = \nu_k g$.²⁰

7 Conclusion

In the present paper an inefficiency index which builds directly on an economy-wide generalisation of the input distance function has been introduced. Similarly as with the Debreu's coefficient of resource utilization it offers a measure of inefficiency that has wide applicability. The paper surveys through some important recent literature and presents economy-wide input based quantity index, the index of real endowment, and a

²⁰Conditions for identification of meaningful hedonic scales are similar to those concerning the equivalence scales of consumption and are considered in detail by Dickens, et. al (1993).

productivity index indicating the effectiveness of the public sector. Together with the society's welfare index these indices offer a convenient summary way of analyzing the nature and extent of government intervention from the perspective of society's welfare and standard-of-living. The empirical applicability of the method has been demonstrated by presenting some examples.

Hedonic prices seem particularly promising in offering a chance of incorporating the above analysis into the calculation of the society's welfare and standard-of-living indices. The hedonic model concentrates on the influences in virtual endowment and consumption of commodities as effectively allowing the production process to accommodate frictionlessly the enhancement of consumer goods. The public intervention considered by the above method may in principle range through widely differing spheres of economic activity. One may consider provision of public goods or public infrastructure, for example construction of roads, a bridge, railway link, port facilities, etc. Alternatively, one may examine public monitoring and regulation of privately provided services, or the introduction of legal restrictions and passing of laws for the protection of the consumer, e.g. cases of setting minimum standards on the provision of information on safety and hazards related to the use of certain products.

In some cases it may seem preferable to measure the hedonic effect involved solely in terms of producer prices and producer technology as opposed to a global effect considered above. First, the scope of the government intervention may be limited to a specific geographic area or to a special activity. Second, the introduction of public facilities such as infrastructure often have effects directly on the producer sector of the economy and the shadow price is best measured in that sector. Similarly, it may be practical over-simplification to assume that the hedonic effects are uniform across consumers and production processes. If the public intervention considered has effects only on the production side of the economy it could be dealt in an analogous way as an increase in productivity that is factor augmenting. A particularly important example concerns spending on education and labour training both of which enhance human capital. In empirical studies one could apply a specification of industry cost functions where the effects concern the input requirements or equivalently, their hedonic factor costs.

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Figure 1.

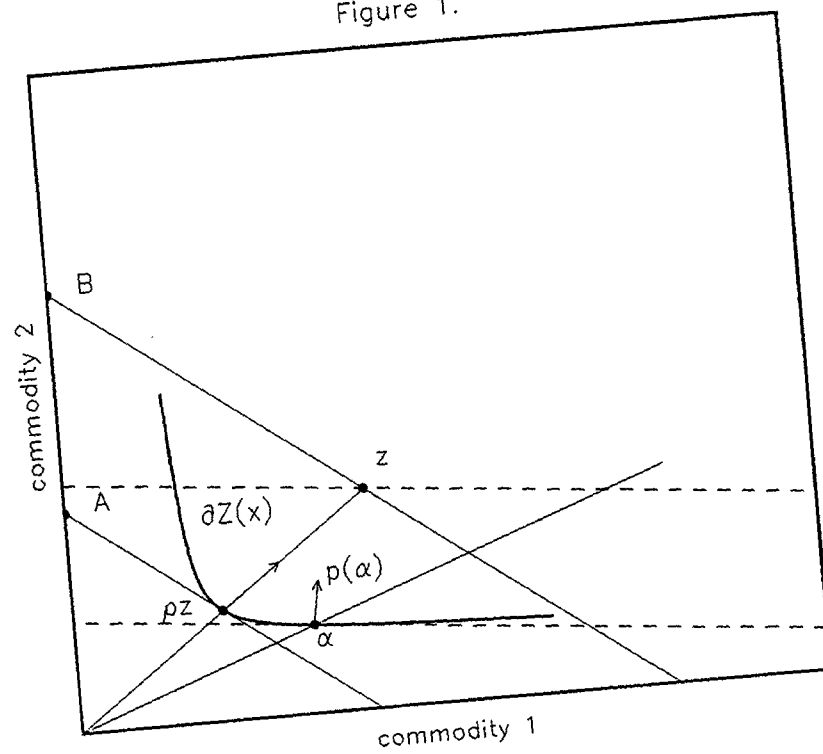


Figure 2.

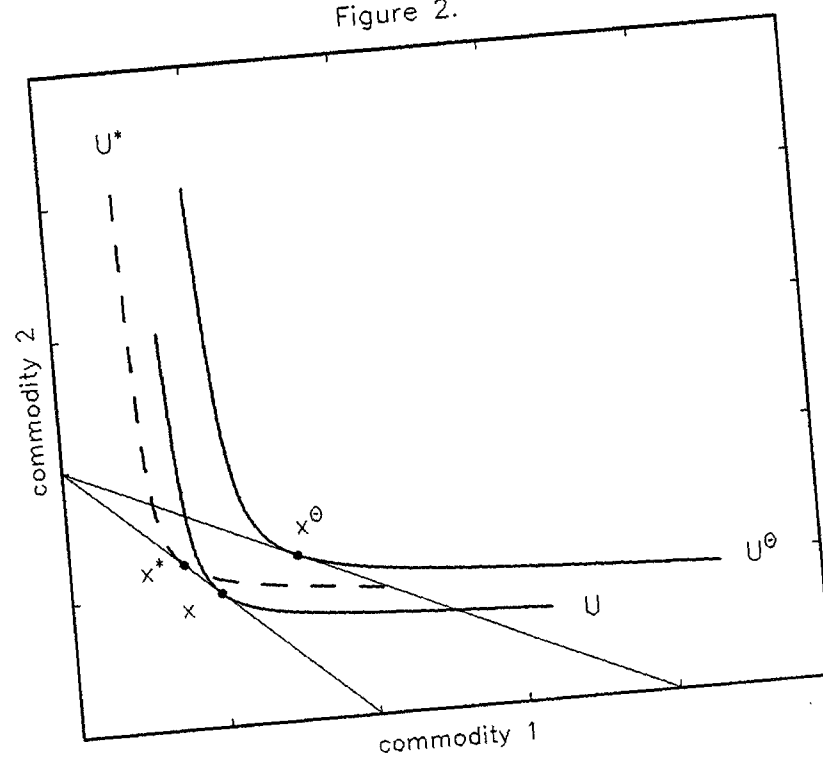


Figure 3.

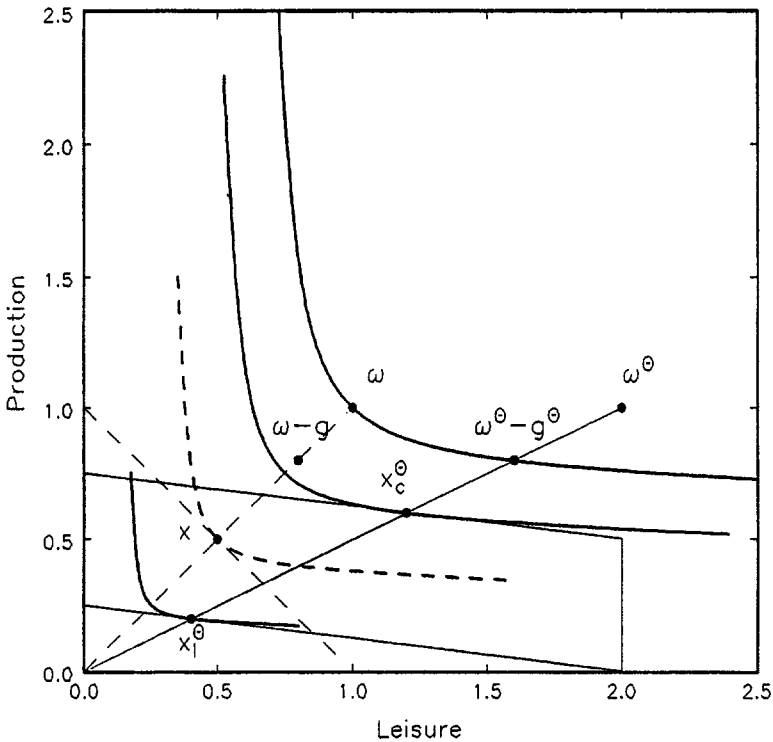


Figure 4.

