DISSERTATIO PHYSICA

CONTINENS

EXPLICATIONEM PHÆNOMENI OPTICI, QUO OBJECTA AQUÆ SUBMERSA DUPLICATA CONSPICIUNTUR.

QUAM

VENIA AMPL. FACULT. PHILOS. ABOëNS.

PUBLICO EXAMINI SUBMITTIT

AUCTOR

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RESPONDENTE

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Tavastense.

In Auditorio Majori die 14 Mart. 1801.

Horis a. m. consvetis.

PARS IV.

ABOÆ,

In Officina FRENCKELLIANA.

7.

*a*Q*g* pars LRSTUQPNL nigra mox animadvertebatur, & quando tandem laminæ pars FCY*e*F nigrum quoque ferebat colorem, imaginis pars RKMODUTSR nigra videbatur, ut quoque præterea simul imaginis partium LK*a* & *hfg* dimidium nigrum esse observabatur. Si vero laminæ extremitas ADF nigra pingebatur, manente reliqua parte AF*ed* alba, pars LRUV*f*QL imaginis laminæ nigra esse observabatur, reliquæ vero partes albæ.

Si similiter in Fig. 3 lamina parva AMND aquæ submersa ita pingebatur, ut partes ejus CKD*e* & CHMF nigræ essent, manentibus reliquis ejus partibus CHA*e* & CKNF albis, imaginis conspectæ EOMN*f*G*n*BPE pars OPBFMO nigra videbatur, pars vero *n*BFN*f* alba, cum simul dimidium imaginum OPE & *fn*G nigrum, & reliquum earum album conspiciebatur. Arcus vero semicircularis QTV eundem semper ac punctum C ferebat colorem.

Quando autem laminæ submersæ AFD pars AD, quæ a C circiter 8 lin. geom., existente distantia acuum 2 pollices, distabat, nigra pingebatur, manente reliqua acus parte alba; mucro B niger videbatur, reliquum vero totius conspectæ imaginis EOMN*f*G album, exceptis ejus partibus E & G.

Plura quidem hujusmodi experimenta institui, diversis partibus objectorum submersorum in Figuris 5, 6, 7 & 8 diverse pictis. Semper autem veros

E ob-

objectorum colores ferre imagines observavi. Hinc concludo, nec subrubram cuspidem D (Fig. 2), neque arcum LQ*f* discolorem, de quibus loquitur D:nus Klügel, ex refractione luminis, quæ imagines descriptas gignit, derivari debere. Numquam enim ita coloratas imagines animadvertere potui, nisi rubra fuit objecti submersi pars *dye*, & nisi diversos etiam extra aquam ostendit colores illud objectum. Cum vero orichalcum politum, quando radiis solaribus immediate exponitur, rubrum aliosque colores ostendat, etiam imago acus orichalceæ ita colorata conspici debet, quando a radiis solaribus immediate illuminatur, alio vero casu, quando flava videtur acus, etiam imago flava erit; quæ omnia experimentis exacte conveniunt.

§. XIII.

Ex allatis jam apparere putaverim, ex sola refractione luminis phænomenon nostrum totum explicari non solum facillime posse, verum etiam debere. Concedere itaque non possum Celeberrimo D:no Gilbert, ex occasione versionis Germanicæ Dissertationis præsentis particulæ primæ urgenti, extra omnem esse controversiam, imaginis curvedinem LKD*hf*QL (Fig. 2) ab inflexione luminis pendere, quum putet, lumen ex extremitate acus submersæ egressum adeo prope extremitatem acus superioperioris præterire, ut inflecti omnino queat (*). Mihi enim iis nixo ratiociniis, quæ jam §. III. pag. 6 & 7 attuli, perſuaſum eſt, radios luminis extremitatem acus ſuperioris prætereuntes ab illa adeo diſtare, ut inflecti adhuc non poſſint. Docente NEWTONO ſcimus, radiorum inflexorum maximam diſtantiam a corpore inflectente eſſe $= \frac{1}{800}$ unciæ londinenſis ſeu 0,00106837 pollicis geometr. ſuecani (**). Poſito itaque oculo in diſtantia viſionis diſtinctæ ſeu 8 pollicum ab acu ſuperiore, erit Tangens anguli, quem in oculo conſtituunt radius remotiſſimus inflexus & alius extremitatem acus contingens, æqualis $\frac{0,00106837}{8} = 0,00133354$, unde concluditur, eſſe hunc angulum minorem quam 28″. Cumque ex experimentis TOBIÆ MAYER notum

E 2 ſit,

(*) Vide: *Annalen der Phyſik, herausgegeben von* LUDW. WILH. GILBERT, Halle 1799, 3 B. 2 St. p. 248.

(**) Cfr. *Tabeller, ſom föreſtålla förhållandet emellan Sveriges och andra Länders mynt, vigt och mått, utarbet. af* CHRIST. LUDV. JÖRANSSON, Stockh. 1777, p. 43, 44; unde conſtat, eſſe pedem anglicum ad ſuecanum ut 1000: 975, hoc eſt 12 unc. angl. ad 10 pollic. geom. ſuec. ut 1000: 975, adeoque haberi $\frac{1}{800}$ unc. angl. $= \frac{10.\ 1000}{800.\ 12.\ 975} = 0,00106837$ poll. ſuecan.

ſit, pro objectis lumine aut diurno aut candelæ unum pedem diſtantis illuſtratis minimum angulum opticum eſſe aut 30″ aut 51″ reſpective (*); patet, radios inflexos ab illis, qui acum ſuperiorem tangunt, nudis oculis diſtingui non poſſe, adeoque imagines per inflexionem lucis conſpectas acui huic contiguas neceſſario apparere. Maxime autem diverſis adhibitis illuminationis gradibus, in diſtantia viſionis diſtinctæ omnes reliquæ partes imaginis in Fig. 2, præter punctum extremum D cuſpidis ODZ, ab acu ſuperiore notabiliter ſemper diſtare obſervantur; quare tuto concluditur, imaginem hanc totam, excepto puncto D, radiis non inflexis ſed refractis conſpici poſſe. Quod autem punctum extremum D attinet, illud quidem radiis inflexis videri poteſt, quia acui ſuperiori contigua obſervatur; non autem ab illa inflexione figuram cuſpidis pendere, ex præcedentibus patet. Talis enim videtur, qualis conſpici debet ſi ſola refractio luminis hic ageret.

§. XIV.

Ut & locus & magnitudo imaginum apparens ſecundum legem refractionis lucis calculo determinentur, nota eſſe debet curvatura ſuperficiei aquæ eleva-

(*) *Experim. circa viſus aciem* in *Commentar. Societat. reg. ſcientiar. Gottingenſis*, Tom. IV. ad annum 1754, pag. 112, Tabul. collat. cum lin. 8, pag. 110.

elevatæ. Sit in A (Fig. 9) acus superior horizontalis, cui adhærens aqua superficiem curvam constituit. Sint quoque BSE & AHE intersectiones plani alicujus verticalis cum superficie aquæ horizontalis ante elevationem & ipsius elevatæ. Particulæ aquæ acum immediate contingentes huic quidem adhærent; attractionem autem ejus in particulas reliquas omnes vel minime remotas evanescentem considerari debere, docent experimenta (*). Adhærentibus acui particulis illam contingentibus atque hoc modo elevatis, reliquæ quoque particulæ illis proximæ, aliæ vero his, simul elevantur ob cohæsionem ipsius aquæ; atque sic circa acum in A coacervantur particulæ aquæ, quarum superficiei curvaturam ostendit linea AHE. Si inter se tantum, non autem cum subjacentibus aquæ particulis, cohærerent particulæ in curva AHE sitæ, catenaria simplex esset hæc. Patet autem, quamvis particulam, ut in H, cum aliis proxime sitis etiam in linea verticali HS, quæ ex H ad libellam aquæ demittitur, cohærere; quare curva AHE consideranda est ut curvatura catenæ, in qua cujusvis particulæ, ut in H, pondus proportionale est altitudini suæ, ut HS, supra aquæ superficiem horizontalem EB. Existente itaque E vertice Catenariæ, ubi horizonti est parallela, atque sumtis $ES = y$ & $SH = x$, erit

E 3 pondus

(*) Cf. *Inledning til Naturläran, af* L. REGNÉR, Upsala 1785, 1 Del. sid. 120, §. 188 &c.

pondus catenæ EH proportionale areæ EHS, hoc eſt, integrali $\int xdy$. Eſt autem in omnibus curvis catenariis ut dx ad dy ita pondus catenæ ad potentiam conſtantem $= a^2$, quæ catenam in E horizontaliter tendet (*); quare erit $dx:dy::\int xdy:a^2$, unde habetur æquatio $\int xdy = \frac{a^2dx}{dy}$. Differentiata hac provenit $xdy = \frac{a^2(dyddx - dxddy)}{dy^2}$, unde, ductis terminis in $\frac{dx}{dy}$, invenitur $xdx = \frac{a^2(dydxddx - dx^2ddy)}{dy^3}$. Ut hæc integretur fiat dy conſtans, ut ſit $ddy = o$, adeoque $xdx = \frac{a^2dxddx}{dy^2}$, atque integrando (denotante A conſtantem aliquam) obtinebitur $A^2 + x^2 = \frac{a^2dx^2}{dy^2}$. Hinc autem, terminis rite translatis & extracta radice quadrata obtinetur $dy = \frac{adx}{\sqrt{A^2+x^2}}$, quæ ita integrata, ut pro $x = 0$ ſit quoque $y = 0$, dat

(*) Cfr. Joh. Bernoulli *Opp.* Lauſ. & Genev. 1742, Tom. III, pag. 501.

dat pro curvatura aquæ elevatæ hanc æquationem:

$$1.)\ y = a \text{ Log. hyp. } \frac{x + \sqrt{A^2 + x^2}}{A}.$$

§. XV.

Ut conſtantes quantitates a & A nondum determinatæ innoteſcerent, experimenta quædam inſtitui. Primum per parietes vaſis vitrei, in quo continebatur aqua temperiei 16° Thermometri Celſiani, & gravitatis ſpecificæ 1,001, elevationem acu chalybea diametri 0,3 lin. geom. factam inſpexi, & ducta per A linea verticali OAD altitudinem BA menſuravi, quam proxime 1,2 lin. geom. inveni. Deinde exiſtente D in linea OAD loco objecti cujusdam parvi aquæ immerſi, ex O objectum D ope radiorum lucis DH & HO intuitus ſum, ductaque ex H recta HT normali ad OD, pro quatuor asſumtis valoribus diverſis diſtantiæ AO & duobus diſtantiæ AD menſuravi diſtantiam HT & angulum AOH. Sic ſequens orta eſt tabula, ubi quantitatum AO, AD & HT unitas eſt linea geometrica ſvecana, Tangentium vero Sinus totus = 1.

AO

	AO	AD = 20	
		HT	Tg AOH
I.	20	0,68	0,0330
II.	30	0,73	0,0238
		AD = 50	
III.	50	1,03	0,0202
IV.	60	1,05	0,0172

Facta jam AB $=b$, BE $=c$, AD $=e$, & AO $=f$, erit OT $=f+b-x$, TH $=c-y$, & *Tg* AOH $=\frac{HT}{TO}=\frac{c-y}{f+b-x}$, adeoque $x=b+f-\frac{c-y}{Tg\,AOH}$, unde videtur, pro assumto f datis ope experimentorum quantitatibus b, $c-y$ & *Tg* AOH, respondentem inveniri posse abscissam x. Est ulterius *Sin* AOH $=\frac{c-y}{\sqrt{(f+b-x)^2+(c-y)^2}}$ & *Cos* AOH $=\frac{f+b-x}{\sqrt{(f+b-x)^2+(c-y)^2}}$. Ducta itaque per punctum H, ubi radius lucis DH in superficie aquæ refringitur, recta NHM ad curvam AHE in H normali, quæ rectæ OD in M occurrat, erit *Sin* HMO $=\frac{dx}{\sqrt{dx^2+dy^2}}=\frac{\sqrt{A^2+x^2}}{\sqrt{a^2+A^2+x^2}}$, & *Cos* HMO $=\frac{a}{\sqrt{a^2+A^2+x^2}}$; quare ob ang. NHO $=$

HMO

HMO + AOH, habetur Sin NHO $= Sin$ (HMO + AOH) $=$

$$\frac{(f+b-x)\sqrt{A^2+x^2}+a(c-y)}{\sqrt{(a^2+A^2+x^2)\left((f+b-x)^2+(c-y)^2\right)}}.$$ Est quoque

DT $= e-b+x$, adeoque Tg HDA $= \frac{HT}{DT} = \frac{c-y}{e-b+x}$,

Sin HDA $= \frac{c-y}{\sqrt{(e-b+x)^2+(c-y)^2}}$, & Cos HDA $=$

$\frac{e-b+x}{\sqrt{(e-b+x)^2+(c-y)^2}}$. Ob ang MHD $=$ HMO $-$ HDA

erit itaque Sin MHD $= Sin$ (HMO $-$ HDA) $=$

$$\frac{(e-b+x)\sqrt{A^2+x^2}-a(c-y)}{\sqrt{(a^2+A^2+x^2)\left((e-b+x)^2+(c-y)^2\right)}}.$$

Facta vero ratione Sinus anguli incidentiæ ad Sinum anguli refractionis, quando ex aëre in aquam transmittitur lux, ut $1:m$, erit $m.\,Sin$ NHO $= Sin$ MHD, seu

$$\text{II.)}\quad \frac{m(f+b-x)\sqrt{A^2+x^2}+ma(c-y)}{\sqrt{(f+b-x)^2+(c-y)^2}} =$$

$\frac{(e-b+x)\sqrt{A^2+x^2}-a(c-y)}{\sqrt{(e-b+x)^2+(c-y)^2}}$. Positis itaque $e-b+x$ $=$ B, $f+b-x =$ C, & $c-y =$ D, si ope experimentorum (pag 36 tab.) determinantur quantitates B, C & D, invenitur

F $a=$

$a = \frac{(B\sqrt{C^2 + D^2} - mC\sqrt{B^2 + D^2})\sqrt{A^2 + x^2}}{D(\sqrt{C^2 + D^2} + m\sqrt{B^2 + D^2})}$. Similiter pro datis aliis $e' - b + x' = B', f' + b - x' = C'$, & $e \cdot y' = D'$ habetur $a = \frac{(B'\sqrt{C'^2 + D'^2} - mC'\sqrt{B'^2 + D'^2})\sqrt{A^2 + x'^2}}{D'(\sqrt{C'^2 + D'^2} + m\sqrt{B'^2 + D'^2})}$, adeoque comparando ambos valores quantitatis a, obtinetur

$$\frac{(B\sqrt{C^2 + D^2} - mC\sqrt{B^2 + D^2})\sqrt{A^2 + x^2}}{D(\sqrt{C^2 + D^2} + m\sqrt{B^2 + D^2}} =$$

$$\frac{(B'\sqrt{C'^2 + D'^2} - mC'\sqrt{B'^2 + D'^2})\sqrt{A^2 + x'^2}}{D'(\sqrt{C'^2 + D'^2} + m\sqrt{B'^2 + D'^2}}.$$

Fiat $B\sqrt{C^2 + D^2} - mC\sqrt{B^2 + D^2} = E, \sqrt{C^2 + D^2} + m\sqrt{B^2 + D^2} = F, B'\sqrt{C'^2 + D'^2} - mC'\sqrt{B'^2 + D'^2} = E'$, & $\sqrt{C'^2 + D'^2} + m\sqrt{B'^2 + D'^2} = F'$, ut sit $\frac{E\sqrt{A^2 + x^2}}{DF} = \frac{E'\sqrt{A^2 + x'^2}}{D'F'}$; & habebitur $A^2 = \frac{D'^2E^2F'^2x^2 - D^2E'^2F^2x'^2}{D^2E'^2F^2 - D'^2E^2F'^2}$. Cum itaque sit proxime $m = \frac{3}{4}$, & $b = 1,2$; erit ex experimento I & II:

$D = 0,68$	$D' = 0,73$
$x = 0,6$	$x' = 0,5$
$B = 19,4$	$B' = 19,3$
$C = 20,6$	$C' = 30,7$
$E = 99,95$	$E' = 148,61$
$F = 35,17$	$F' = 45,19$

Deno-

Denotante itaque L Logarithmum vulgarem, erit

2L.D′=0,7266458—1	2L.D=0,6650178—1
2L.E=3,9995656	2L.E′=4,3410514
2L.F′=3,3100846	2L.F=3,0923448
2L.x=0,5563025—1	2L.x′=0,3979400—1
6,5925985=	6,4963540=
=L.3913798.	=L.3135840.

3913798 — 3135840 = 777958.

2L.D=0,6650178—1	2L.D′=0,7266458—1
2L.E′=4.3410514	2L.E=3,9995656
2L.F=3,0923448	2L.F′=3,3100846
7,0984140=	7,0362960=
=L.12543360.	=L.10871600.

12543360 — 10871600 = 1671760.

L. 777958=6,8909562 — 1

L.1671760=6,2231738

L.A′=0,6677824— 1=L.0,4653528.

L.A =0,8338912— 1=L.0,682168.

A =0,6822

Inventa sic qvantitate A, facile quoque invenitur *a*. Est enim $a = \frac{E\sqrt{A^2 + x^2}}{DF}$, adeoque

$L.\sqrt{A^2+x^2}$=0.9583198—1	L.D=0,8325089—1
L.E=1,9997828	L.F=1,5461724
1,9581026	1,3786813.
1,3786813	

L.*a*=0,5794213=L.3,7968, atque *a*=3,7968.

Experimenta III & IV dant

$D = 1,03$	$D' = 1,05$
$x = 0,3$	$x' = 0,2$
$B = 49,1$	$B' = 49$
$C = 50,9$	$C' = 61$
$E = 625,03$	$E' = 747,08$
$F = 87,74$	$F' = 97,77$

$$\begin{array}{ll} 2L.D' = 0,0423786 & 2L.D = 0,0256744 \\ 2L.E = 5,5918018 & 2L.E' = 5,7467342 \\ 2L.F' = 3,9804112 & 2L.F = 3,8863952; \\ 2L.x = 0,9542425 - 2 & 2L.x' = 0,6020600 - 2 \\ \hline 8,5688341 = & 8,2608638 = \\ = L.370539100. & = L.182332400. \end{array}$$

$$370539100 - 182332400 = 188206700.$$

$$\begin{array}{ll} 2L.D = 0,0256744 & 2L.D' = 0,0423786 \\ 2L.E' = 5,7467342 & 2L.E = 5,5918018 \\ 2L.F = 3,8863952 & 2L.F' = 3,9804112 \\ \hline 9,6588038 = & 9,6145916 = \\ = L.4558309000. & = L.4117102000. \end{array}$$

$$4558309000 - 4117102000 = 441207000$$

$$\begin{array}{l} L.188206700 = 9,2746350 - 1 \\ L.441207000 = 8,6446424 \\ \hline L.A^2 = 0,6299926 - 1 = L.0,426572. \\ A = 0,6531. \end{array}$$

Est itaque

$L.\sqrt{A^2 + x^2} =$

$L.\sqrt{A^2+x^2}=0,8565655-1$	$L.D=0,0128372$
$L.F=2,7959009$	$L.F=1,9431976$
$2,6524664$	$1,9560348$
$1,9560348$	

$L.a=0,6964316=L.4,9708$, & $a=4,9708$.

Simili calculo eruuntur ex experimentis

I & III,	I & IV,	II & III,	II & IV,
$A=0,2555$;	$0,3427$;	$0,1627$;	$0,2983$;
$a=2,7255$;	$2,8877$;	$2,3604$;	$2,6137$.

Si itaque omnium ſic inventorum valorum medium arithmeticum quæritur, invenitur quam proxime $A=0,4$ atque $a=3,2$; quibus in æqv. I ſubſtitutis, obtinetur pro curvatura aquæ elevatæ hæc æquatio

$$\text{III.)}\ y=3,2\ \text{Log. hyp.}\ \frac{x+\sqrt{0,16+x^2}}{0,4}.$$

Facta in hac æquatione $x=b$, evadit $y=c$; quare ob $b=1,2$ lin., erit $c=3,2$ Log. hyp. $\frac{1,2+\sqrt{0,16+1,44}}{0,4}=$ $3,2$ Log. hyp. $6,16=3,2.\ 2,302585.\ L.\ 6,16.$ Eſt autem $L.\ 6,16=0,7885807$, atque

$$L.\ 3,2=0,5051500$$
$$L.\ 2,302585=0,3622156$$
$$L.\ 0,7885807=0,8968462-1$$

adeoque $L.\ c=0,7642118=L.\ 5,81$
& proxime $c=5,8$.

 §. XVI.

§. XVI.

Cognitis jam quantitatibus A, a & c, ope æquationum I & II determinari poteſt magnitudo apparens objectorum aquæ ſubmerſorum. Denotante primo N numerum cujus Log. hyp. $=1$, erit ex æqv. I:

$$x = \frac{A(N^{2y:a} - 1)}{2N^{y:a}}, \ \& \ \sqrt{A^2 + x^2} = \frac{A(N^{2y:a} + 1)}{2N^{y:a}}.$$

Eſt deinde $\left((e-b+x)^2 + (c-y)^2\right)^{-\frac{1}{2}} = \frac{1}{e-b+x} -$

$$-\frac{1}{2} \cdot \frac{(c-y)^2}{(e-b+x)^3} + \frac{1.3}{1.2.2^2} \cdot \frac{(c-y)^4}{(e-b+x)^5} -$$

$\frac{1.3.5}{1.2.3.2^3} \cdot \frac{(c-y)^6}{(e-b+x)^7} +$ &c., quare cum ſit quantitas

$\frac{TH}{TD} = \frac{c-y}{e-b+x}$ admodum parva, erit ſatis exacte

$\left((e-b+x)^2 + (c-y)^2\right)^{-\frac{1}{2}} = \frac{1}{e-b+x}$, adeoque

$$\frac{(e-b+x)\sqrt{A^2+x^2} - a(c-y)}{\sqrt{(e-b+x)^2 + (c-y)^2}} = \sqrt{A^2+x^2} - \frac{a(c-y)}{e-b+x}.$$

Similiter ob qvantitatem $\frac{TH}{TO} = \frac{c-y}{f+b-x}$ admodum par-

yam

vam erit satis exacte $\frac{m(f+b-x)\sqrt{A^2+x^2}+ma(c-y)}{\sqrt{(f+b-x)^2+(c-y)^2}} = m\sqrt{A^2+x^2}+\frac{ma(c-y)}{f+b-x}$. His in æqv. II factis substitutionibus habetur $\sqrt{A^2+x^2}-\frac{a(c-y)}{e-b+x} = m\sqrt{A^2+x^2}+\frac{ma(c-y)}{f+b-x}$, seu ob $m=\frac{3}{4}$,

IV.) $\sqrt{A^2+x^2} = a(c-y)\left(\frac{4}{e-b+x}+\frac{3}{f+b-x}\right)$.

Repræsentet linea recta PQ, lineæ AD normalis, objectum aquæ submersum, quod radiis luminis ultimis PHO & QH'O conspicitur. Occurrat HP producta lineæ OAD in D, quo fit, ut in præcedentibus, $AD=e$, $TD=e-b+x$, & $TH=c-y$. Facta itaque $AL=g$, ut sit $LD=e-g$, & $LP=k$, erit ob similitudinem triangulorum TDH & LDP, $e-b+x : c-y :: e-g : k$, unde invenitur $e=\frac{g(c-y)-k(b-x)}{-k+c-y}$, adeoque $e-b+x=\frac{(g-b+x)(c-y)}{-k+c-y}$. Hoc valore in æqv. IV. substituto obtinetur $\sqrt{A^2+x^2}=\frac{4a(-k+c-y)}{g-b+x}+\frac{3a(c-y)}{f+b-x}$, unde adhibendo valores quantitatum $\sqrt{A^2+x^2}$

& x

& x antea inventos habetur V.) $\frac{A(N^{2y:a}+1)}{2N^{y:a}}=$

$$\frac{4a(-k+c-y)}{g-b+\frac{A(N^{2y:a}-1)}{2N^{y:a}}}+\frac{3a(c-y)}{f+b-\frac{A(N^{2y:a}-1)}{2N^{y:a}}}.$$

Ut autem hinc inveniatur incognita y, positis $\alpha=2A^2(f-g+2b)$; $\beta=A(A^2+4\overline{g-b}.\overline{f+b}+4a.\overline{c-4k})$; $\gamma=4aA$, $\delta=8a(4\overline{f+b}.\overline{c-k}+3c.\overline{g-b})$, $\varepsilon=8a(4\overline{f+b}+3\overline{g-b})$, & $\zeta=A(A^2+4\overline{g-b}.\overline{f+b}-4a.\overline{c-4k})$, æquatio V, terminis rite multiplicatis atque translatis, mutatur in hanc: $-A^3N^{6y:a}+\alpha N^{5y:a}+\beta N^{4y:a}-\gamma yN^{4y:a}-\delta N^{3y:a}+\varepsilon yN^{3y:a}+\zeta N^{2y:a}+\gamma yN^{2y:a}-\alpha N^{y:a}-A^3=0$, cujus termini singuli exponentiales in series convertantur. Est autem

$-A^3N^{6y:a}$

$$-A' N^{6y:a} = -A'\left(1+\frac{6}{1}\cdot\frac{y}{a}+\frac{6^2}{1.2}\cdot\frac{y^2}{a^2}+\frac{6^3}{1.2.3}\cdot\frac{y^3}{a^3}+\frac{6^4}{1.2.3.4}\cdot\frac{y^4}{a^4}+\&c.\right),$$

$$\alpha N^{5y:a} = \alpha\left(1+\frac{5}{1}\cdot\frac{y}{a}+\frac{5^2}{1.2}\cdot\frac{y^2}{a^2}+\frac{5^3}{1.2.3}\cdot\frac{y^3}{a^3}+\frac{5^4}{1.2.3.4}\cdot\frac{y^4}{a^4}+\&c.\right),$$

$$\beta N^{4y:a} = \beta\left(1+\frac{4}{1}\cdot\frac{y}{a}+\frac{4^2}{1.2}\cdot\frac{y^2}{a^2}+\frac{4^3}{1.2.3}\cdot\frac{y^3}{a^3}+\frac{4^4}{1.2.3.4}\cdot\frac{y^4}{a^4}+\&c.\right),$$

$$-\gamma y N^{4y:a} = -\gamma\left(\frac{a}{1}\cdot\frac{y}{a}+\frac{4a}{1}\cdot\frac{y^2}{a^2}+\frac{4^2 a}{1.2}\cdot\frac{y^3}{a^3}+\frac{4^3 a}{1.2.3}\cdot\frac{y^4}{a^4}+\&c.\right),$$

$$-\delta N^{3y:a} = -\delta\left(1+\frac{3}{1}\cdot\frac{y}{a}+\frac{3^2}{1.2}\cdot\frac{y^2}{a^2}+\frac{3^3}{1.2.3}\cdot\frac{y^3}{a^3}+\frac{3^4}{1.2.3.4}\cdot\frac{y^4}{a^4}+\&c.\right),$$

$$\varepsilon y N^{3y:a} = \varepsilon\left(\frac{a}{1}\cdot\frac{y}{a}+\frac{3a}{1}\cdot\frac{y^2}{a^2}+\frac{3^2 a}{1.2}\cdot\frac{y^3}{a^3}+\frac{3^3 a}{1.2.3}\cdot\frac{y^4}{a^4}+\&c.\right),$$

$$\zeta N^{2y:a} = \zeta\left(1+\frac{2}{1}\cdot\frac{y}{a}+\frac{2^2}{1.2}\cdot\frac{y^2}{a^2}+\frac{2^3}{1.2.3}\cdot\frac{y^3}{a^3}+\frac{2^4}{1.2.3.4}\cdot\frac{y^4}{a^4}+\&c.\right),$$

$$\gamma y N^{2y:a} = \gamma\left(\frac{a}{1}\cdot\frac{y}{a}+\frac{2a}{1}\cdot\frac{y^2}{a^2}+\frac{2^2 a}{1.2}\cdot\frac{y^3}{a^3}+\frac{2^3 a}{1.2.3}\cdot\frac{y^4}{a^4}+\&c.\right),$$

$$-\alpha N^{y:a} = -\alpha\left(1+\frac{1}{1}\cdot\frac{y}{a}+\frac{1}{1.2}\cdot\frac{y^2}{a^2}+\frac{1}{1.2.3}\cdot\frac{y^3}{a^3}+\frac{1}{1.2.3.4}\cdot\frac{y^4}{a^4}+\&c.\right),$$

quibus omnibus terminis in ſummam collectis, poſitisque

$\alpha' = -2A' + \beta - \delta + \varepsilon + \zeta,$

$\beta' = -6A' + \frac{5-1}{1}\cdot\alpha + 4\beta - 3\delta + a\varepsilon + 2\zeta,$

 $\gamma' =$

$$\gamma' = -\frac{6^2 A^2}{1.2} + \frac{\overline{5^2-1}.\alpha}{1.2.} + \frac{4^2\beta}{1.2} - \frac{\overline{4-2}.a\gamma}{1} - \frac{3^2\delta}{1.2} + \frac{3a\varepsilon}{1} + \frac{2^2\zeta}{1.2},$$

$$\delta' = -\frac{6^3 A^3}{1.2.3} + \frac{\overline{5^3-1}.\alpha}{1.2.3} + \frac{4^3\beta}{1.2.3} - \frac{\overline{4^2-2^2}.a\gamma}{1.2} - \frac{3^3\delta}{1.2.3} + \frac{3^2 a\varepsilon}{1.2} + \frac{2^3\zeta}{1.2.3},$$

$$\varepsilon' = -\frac{6^4 A^4}{1.2.3.4} + \frac{\overline{5^4-1}.\alpha}{1.2.3.4} + \frac{4^4\beta}{1.2.3.4} - \frac{\overline{4^3-2^3}.a\gamma}{1.2.3} - \frac{3^4\delta}{1.2.3.4} + \frac{3^3 a\varepsilon}{1.2.3} + \frac{2^4\zeta}{1.2.3.4}.$$

&c = &c.
obtinetur

$$\text{VI.)}\quad 0 = \alpha' + \beta'.\frac{y}{a} + \gamma'.\frac{y^2}{a^2} + \delta'.\frac{y^3}{a^3} + \varepsilon'.\frac{y^4}{a^4} + \&c.$$

Ut hinc eruatur $\frac{y}{a}$, revertenda eſt hæcce ſeries, quare ponatur

$$\text{VII.)}\quad \frac{y}{a} = \mathfrak{A}\alpha' + \mathfrak{B}\alpha'^2 + \mathfrak{C}\alpha'^3 + \mathfrak{D}'^4 + \&c.$$

Factis hinc ſubſtitutionibus in æqv. VI, obtinetur

$$0 = \left.\begin{matrix} +1 \\ +\beta'\mathfrak{A} \end{matrix}\right\}\alpha' + \left.\begin{matrix} \beta'\mathfrak{B} \\ +\gamma'\mathfrak{A}^2 \end{matrix}\right\}\alpha'^2 + \left.\begin{matrix} \beta'\mathfrak{C} \\ +2\gamma'\mathfrak{A}\mathfrak{B} \\ +\delta'\mathfrak{A}^3 \end{matrix}\right\}\alpha'^3 + \left.\begin{matrix} \beta'\mathfrak{D} \\ +\gamma'\mathfrak{B}^2 \\ +2\gamma'\mathfrak{A}\mathfrak{C} \\ +3\delta'\mathfrak{A}^2\mathfrak{B} \\ +\varepsilon'\mathfrak{A}^4 \end{matrix}\right\}\alpha'^4 + \&c.$$

Hujus autem ſeriei coëfficientes evaneſcentes ſumendo pro determinandis coëfficientibus $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, &c. ſequentes obtinentur æquationes: $\mathfrak{A} = -\frac{1}{\beta'}$, $\mathfrak{B} = -\frac{\gamma'}{\beta'^3}$, $\mathfrak{C} = -\frac{2\gamma'^2}{\beta'^5} + \frac{\delta'}{\beta'^4}$, $\mathfrak{D} = -\frac{4\gamma'^3}{\beta'^8} - \frac{\gamma'^3}{\beta'^7} + \frac{5\gamma'\delta'}{\beta'^6} - \frac{\varepsilon'}{\beta'^5}$, &c. quibus cognitis, pro datis f, g & k ope pe æquationis VII invenitur quantitas $\frac{y}{a}$. Hoc autem valore adhibito in æquatione $x = \frac{A(N^{2y:a} - 1)}{2N^{y:a}}$, hinc facile pro dato casu eruitur valor abscissæ x. Quando itaque sic cognoscuntur x & y, innotescit quoque $Tg\,AOH = \frac{c-y}{f+b-x}$, adeoque angulus AOH.

Occurrat ulterius radius luminis QH′ lineæ OAL in D′, & factis $AD' = e'$, $T'D' = e' - b + x'$, $T'H' = c - y'$, & $LQ = k'$, erit $D'L = g - e'$, & $e' - b + x' : c - y' :: g - e' : k'$, adeoque $e' = \frac{g(c-y') + k'(b-x')}{k'+c-y'}$, $e' - b + x' = \frac{(g-b+x')(c-y')}{k'+c-y'}$

$$\& \; \frac{A(N^{2y:a} + 1)}{2N^{y:a}} = \frac{4a(k'+c-y')}{g - b + \frac{A(N^{2y':a} - 1)}{2N^{y':a}}} + \frac{3a(c-y')}{f + b - \frac{A(N^{2y':a} - 1)}{2N^{y':a}}},$$

unde simili modo, quo ex æqv. V inventus est valor $\frac{y}{a}$, eruitur $\frac{y'}{a}$, ponendo

tantum

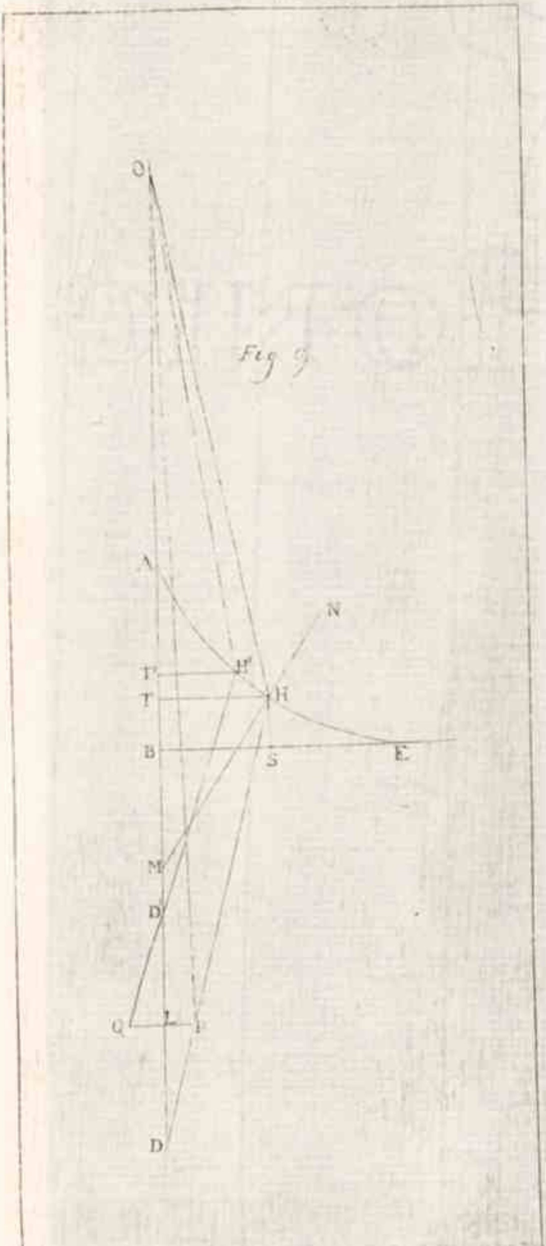

Fig. 9

tantum ubique $+k'$ pro $-k$. Data autem $\frac{y'}{a}$ invenitur

$$x' = \frac{A.(N^{2y:a} - 1)}{2.N^{y':a}}$$ adeoque etiam $Tg\,\text{AOH}' = \frac{c-y'}{f+b-x'}$,

& angulus AOH'. Eſt autem objecti PQ, ex O per HH' conſpecti, apparens magnitudo = AOH — AOH', atque locus apparens ipſis angulis AOH & AOH' determinatur. Ducta recta OP, eſt $Tg\,\text{LOP} = \frac{k}{f+g}$, atque ſimiliter

$Tg\,\text{LOQ} = \frac{k'}{f+g}$, unde invenitur angulus LOP, ut etiam angulus LOQ, qui conjuncti præbent objecti PQ magnitudinem apparentem ſi abeſt aqua, ut radiis rectilineis conſpiciatur objectum. Si itaque parvum eſt objectum, erunt ejus magnitudines apparentes, quando abeſt aqua & quando per ſuperficiem aquæ AE conſpicitur, ut

$$\frac{k+k'}{f+g} : \frac{c-y}{f+g-x} - \frac{c-y'}{f+g-x'}.$$