DISSERTATIO
DE
ÆQUATIONE

$$y = A \, Sin \, (a + \alpha x) + B \, Sin \, (b + \beta x) + \&c.$$

AD INVENIENDAM LEGEM PHÆNOMENORUM
OBSERVATORUM APTA.

QUAM
CONSENTIENTE AMPL. FAC. PHIL. REG. ACAD. ABOËNSIS,
PRÆSIDE
MAG. *GUST. GABR. HÅLLSTRÖM,*
PHYSICES PROFESS. REG. ET ORDIN. ATQUE REG. SOCIET.
OECON. FENN. MEMBRO.

PRO GRADU PHILOSOPHICO
P. P.
ANDREAS LUDOV. BROBERG,
SUDERM. SVECUS.

In Auditorio Majori die 15. Maji MDCCCII.
Horis p. m. consuetis.

ABOÆ, TYPIS FRENCKELLIANIS.

17.

VIRO

PLURIMUM REVERENDO ATQUE PRÆCLARISSIMO,

DOMINO MAGISTRO

ADOLPHO LUNDGREN,

ECCLESIÆ IN DIOECESI STRENGNÆSENSI HUSBY-OPPUNDA ET CONTRACTUS ADJACENTIS PRÆPOSITO MERITISSMO,

FAUTORI SUO ET PATRONO PROPENSISSIMO,

Non quantum debet, ſed exiguum quod valet

offert

ANDREAS LUDOV. EROBERG.

Etiamſi laudabili induſtria Phyſicorum recentioris temporis factum ſit, ut inſtrumenta, quibus experimenta in legibus naturæ detegendis neceſſaria inſtituantur, admodum jam perfecta elaborari poſſint; tamen illis adeo non eſt fidendum, ut indolem rei omnia præciſe ſemper indicare credatur, hocque vel inde apparet, quod repetitis pluries iisdem experimentis aliquam non raro oſtendant diverſitatem. In his caſibus, ſi accuratiores deſiderantur obſervationes, plures ſunt de eadem re inſtituendæ, ut medium earum arithmeticum inveniatur. Quando vero magnus eſſe debet obſervationum numerus pro rebus parum mutatis, deeſt ſæpe occaſio omnes illas accuratius determinandi, quare ex datis quibusdam majori cautione factis reliquas ope calculi quærere ſolent Phyſici.

Inter æquationes diverſi generis, quibus in tali calculo commode uti poſſunt, hanc quoque:
$y = A\,Sin\,(a + \alpha x) + B\,Sin\,(b + \beta x) + C\,Sin\,(c + \gamma x)$ + &c. cujus quantitates A, a, α; B, b, β; C, c, γ; &c. ſunt conſtantes, ſed y & x variabiles, numerandam eſſe judicamus, & quidem ideo commendandam, quod ope Logarithmorum ſatis compendioſo calculo poſ-

ſint computari valores y, qui certis valoribus x reſpondent, nec non quod adhiberi queat ſi x vel numeros aliquos vel angulos deſignet. Non itaque inutilem nos acturos eſſe laborem putamus, ſi valores determinemus quantitatum conſtantium, quando ope experimentorum cogniti ſunt valores aliqui quantitatum x & y.

Datis valoribus $x = 0;\ x_{\prime};\ 2x_{\prime};\ 3x_{\prime};\ 4x_{\prime};$ &c. reſpondeant valores $y = y_0;\ y_{\prime};\ y_{\prime\prime};\ y_{\prime\prime\prime};\ y_{iv};$ &c. quibus in æquatione propoſita ſubſtitutis tot obtinentur diverſæ æquationes, quot habentur quantitates determinandæ. Ut itaque a ſimplicioribus ordiamur, conſideremus æquationem

$$y = A\ Sin\ \alpha x$$

cujus quantitates conſtantes A & α ope duorum experimentorum determinari queunt. Factis nimirum ſucceſſive $x = x_{\prime}$, & $x = 2x_{\prime}$, nec non $y = y_{\prime}$, & $y = y_{\prime\prime}$; obtinetur $y_{\prime} = A\ Sin\ \alpha x_{\prime}$, & $y_{\prime\prime} = A\ Sin\ 2\alpha x_{\prime} = 2A\ Sin\ \alpha x_{\prime}\ Coſ\ \alpha x_{\prime} = 2y_{\prime}\ Coſ\ \alpha x_{\prime}$, adeoque $Coſ\ \alpha x_{\prime} = \frac{y_{\prime\prime}}{2y_{\prime}}$, atque facto illo angulo $= v$, cujus coſinus $= \frac{y_{\prime\prime}}{2y_{\prime}}$, erit $\alpha = \frac{v}{x_{\prime}}$. De cetero eſt $A = \frac{y_{\prime}}{Sin\ \alpha x_{\prime}}$ $= \frac{y_{\prime}}{\sqrt{1 - Coſ\ \alpha x_{\prime}^2}} = \frac{y_{\prime}}{\sqrt{1 - \frac{y_{\prime\prime}^2}{4y_{\prime}^2}}} = \frac{2y_{\prime}^2}{\sqrt{4y_{\prime}^2 - y_{\prime\prime}^2}}$. Hinc apparet, æquationem $y = A\ Sin\ \alpha x$ non poſſe adhiberi niſi ſit $2y_{\prime} > y_{\prime\prime}$.

Tri-

Tribus obſervationibus ſatisfacit æquatio

$$y = A\ Sin\ (a + \alpha x).$$

Tres enim æquationes auxiliares adſunt:

$y_0 = A\ Sin\ a;$

$y_1 = A\ Sin\ (a + \alpha x_1) = A\ Sin\ a\ Cof\ \alpha x_1 + A\ Cof\ a\ Sin\ \alpha x_1;$

$y_{11} = A\ Sin\ (a + 2\alpha x_1) = A\ Sin\ a\ Cof\ 2\alpha x_1 + A\ Cof\ a\ Sin\ 2\alpha x_1.$

Subſtitutis autem valoribus $A\ Sin\ a = y_0$, & $A\ Cof\ a = \sqrt{A^2 - y_0^2}$; obtinentur

$y_1 = y_0\ Cof\ \alpha x_1 + \sqrt{A^2 - y_0^2}.\ Sin\ \alpha x_1;$

$y_{11} = y_0\ Cof\ 2\alpha x_1 + \sqrt{A^2 - y_0^2}.\ Sin\ 2\alpha x_1;$

atque ſi quantitas radicalis $\sqrt{A^2 - y_0^2}$ exterminatur,

$$\frac{y_1 - y_0\ Cof\ \alpha x_1}{Sin\ \alpha x_1} = \frac{y_{11} - y_0\ Cof\ 2\alpha x_1}{Sin\ 2\alpha x_1} = \frac{y_{11} + y_0 - 2y_0\ Cof\ \alpha x_1^2}{2 Sin\ \alpha x_1\ Cof\ \alpha x_1}.$$

Hinc invenitur $Cof\ \alpha x_1 = \frac{y_0 + y_{11}}{2y_1}$, adeoque facto $\frac{y_0 + y_{11}}{2y_1} = Cof\ v$, erit $\alpha = \frac{v}{x_1}$. Habetur quoque $Sin\ \alpha x_1 = \frac{\sqrt{4y_1^2 - (y_0 + y_{11})^2}}{2y_1}$. Subſtitutis vero hisce $Cof\ \alpha x_1$ & $Sin\ \alpha x_1$ valoribus in æquatione $y_1 = y_0\ Cof\ \alpha x_1 + Sin\ \alpha x_1 \sqrt{A^2 - y_0^2}$, obtinetur $y_1 = \frac{y_0 (y_0 + y_{11})}{2y_1} + \frac{\sqrt{A^2 - y_0^2} \cdot \sqrt{4y_1^2 - (y_0 + y_{11})^2}}{2y_1}$, unde eruitur

$A =$

$A = \frac{2y_{,} \sqrt{(y_{,}^2 - y_{0} y_{,,})}}{\sqrt{(4y_{,}^2 - (y_{0} + y_{,,})^2)}}$. Apparet itaque A esse realem, adeoque æquationem propositam adhiberi posse si simul sit $y_{,} > \sqrt{y_{0} y_{,,}}$ & $y_{,} > \frac{1}{2}(y_{0} + y_{,,})$, vel etiam simul $y_{,} < \sqrt{y_{0} y_{,,}}$ & $y_{,} < \frac{1}{2}(y_{0} + y_{,,})$.

Si quatuor observationibus satisfiat, æquatio

$$y = A \, Sin\,(a + \alpha x) + B$$

adhibeatur, in qua methodo determinandi constantes quantitates sequente uti volumus. Sit N numerus, cujus Logarithmus hyperbolicus = 1, ut habeatur $Sin\,(a + \alpha x) = \frac{N^{(a + \alpha x)\sqrt{-1}} - N^{-(a + \alpha x)\sqrt{-1}}}{2\sqrt{-1}}$, seu factis $\frac{N^{a\sqrt{-1}}}{2\sqrt{-1}} = H_{,}$; $N^{\alpha\sqrt{-1}} = a_{,}$; $\frac{N^{-a\sqrt{-1}}}{2\sqrt{-1}} = H_{,,}$ & $N^{-\alpha\sqrt{-1}} = a_{,,}$; $Sin\,(a + \alpha x) = H_{,} a_{,}^{x} - H_{,,} a_{,,}^{x}$. In hanc itaque transmutatur æquatio proposita: $y = AH_{,} a_{,}^{x} - AH_{,,} a_{,,}^{x} + B$, quare sequentes habentur æquationes auxiliares:

$$y_{0} = AH_{,} - AH_{,,} + B.$$
$$y_{,} = AH_{,} a_{,}^{x} - AH_{,,} a_{,,}^{x} + B.$$
$$y_{,,} = AH_{,} a_{,}^{2x_{,}} - AH_{,,} a_{,,}^{2x_{,}} + B.$$
$$y_{,,,} = AH_{,} a_{,}^{3x_{,}} - AH_{,,} a_{,,}^{3x_{,}} + B.$$

Sum-

Sumtis itaque differentiis, ut exterminetur B, erit
$\Delta y_0 = AH_{,} (a_{,}^{x_{,}} - 1) - AH_{,,} (a_{,,}^{x_{,}} - 1)$;
$\Delta y_{,} = AH_{,} (a_{,}^{x_{,}} - 1)\, a_{,}^{x_{,}} - AH_{,,} (a_{,,}^{x_{,}} - 1)\, a_{,,}^{x_{,}}$;
$\Delta y_{,,} = AH_{,} (a_{,}^{x_{,}} - 1)\, a_{,}^{2x_{,}} - AH_{,,} (a_{,,}^{x_{,}} - 1)\, a_{,}^{2x_{,}}$; adeoque
$\Delta y_{,} - a_{,}^{x_{,}} \Delta y_0 = - AH_{,,} (a_{,,}^{x_{,}} - 1)(a_{,,}^{x_{,}} - a_{,}^{x_{,}})$; &
$\Delta y_{,,} - a_{,}^{x_{,}} \Delta y_{,} = - AH_{,,} (a_{,,}^{x_{,}} - 1)(a_{,,}^{x_{,}} - a_{,}^{x_{,}})\, a_{,,}^{x_{,}}$.
Hinc autem invenitur $(\Delta y_{,} - a_{,}^{x_{,}} \Delta y_0)\, a_{,,}^{x_{,}} = \Delta y_{,,} - a_{,}^{x_{,}} \Delta y_{,}$ seu $\Delta y_{,,} - (a_{,}^{x_{,}} + a_{,,}^{x_{,}}) \Delta y_{,} + a_{,}^{x_{,}} a_{,,}^{x_{,}} \Delta y_0 = 0$. Cum vero sit $a_{,}^{x_{,}} = N^{\alpha x_{,} \sqrt{-1}}$ atque $a_{,,}^{x_{,}} = N^{-\alpha x_{,} \sqrt{-1}}$; erit $a_{,}^{x_{,}} + a_{,,}^{x_{,}} = N^{\alpha x_{,} \sqrt{-1}} + N^{-\alpha x_{,} \sqrt{-1}} = 2\, Cos\, \alpha x_{,}$; nec non $a_{,}^{x_{,}} a_{,,}^{x_{,}} = N^{\alpha x_{,} \sqrt{-1}} N^{-\alpha x_{,} \sqrt{-1}} = 1$, adeoque
$\Delta y_{,,} - 2 \Delta y_{,}\, Cos\, \alpha x_{,} + \Delta y_0 = 0$, &

$$Cos\, \alpha x_{,} = \frac{\Delta y_0 + \Delta y_{,,}}{2 \Delta y_{,}} = \frac{y_0 - y_{,} + y_{,,} - y_{,,,}}{2(y_{,} - y_{,,})},$$ nec non

$$Sin\, \alpha x_{,} = \frac{\sqrt{4 \Delta y_{,}^2 - (\Delta y_0 + \Delta y_{,,})^2}}{2 \Delta y_{,}} = \frac{\sqrt{4(y_{,} - y_{,,})^2 - (y_0 - y_{,} + y_{,,} - y_{,,,})^2}}{2(y_{,} - y_{,,})}$$

His vero cognitis, cognoscuntur quoque $Sin\, 2\alpha x_{,}$ & $Sin\, 3\alpha x_{,}$, atque $Cos\, 2\alpha x_{,}$, & $Cos\, 3\alpha x_{,}$. Porro in ipsa aequ. $y = A\, Sin\, (a + \alpha x) + B$ substituantur quantitates per experimenta cognitae, ut sit $y_0 = A\, Sin\, a + B$;
$y_{,} = A\, Sin (a + \alpha x_{,}) + B = A\, Sin\, a\, Cos\, \alpha x_{,} + A\, Cos\, a\, Sin\, \alpha x_{,} + B$;
$y_{,,} = A\, Sin (a + 2\alpha x_{,}) + B = A\, Sin\, a\, Cos\, 2\alpha x_{,} + A\, Cos\, a\, Sin\, 2\alpha x_{,} + B$;
$y_{,,,} = A\, Sin (a + 3\alpha x_{,}) + B = A\, Sin\, a\, Cos\, 3\alpha x_{,} + A\, Cos\, a\, Sin\, 3\alpha x_{,} + B$.
Harum prima dat $A\, Sin\, a = y_0 - B$, & $A\, Cos\, a$

$= \sqrt{A^2 - (y_0 - B)^2}$; quibus valoribus ſubſtitutis, reliquæ in ſequentes mutantur:

$y_{,} = (y_0 - B)\, Cof\, \alpha x_{,} + \sqrt{A^2 - (y_0 - B)^2}.\, Sin\, \alpha x_{,} + B;$

$y_{,,} = (y_0 - B)\, Cof\, 2\alpha x_{,} + \sqrt{A^2 - (y_0 - B)^2}.\, Sin\, 2\alpha x_{,} + B;$

$y_{,,,} = (y_0 - B)\, Cof\, 3\alpha x_{,} + \sqrt{A^2 - (y_0 - B)^2}.\, Sin\, 3\alpha x_{,} + B;$

ſeu inſtituta ſubtractione

$\Delta y_{,} = (y_0 - B)(Cof 2\alpha x_{,} - Cof \alpha x_{,}) + (Sin 2\alpha x_{,} - Sin \alpha x_{,})\sqrt{A^2 - (y_0 - B)^2};$

$\Delta y_{,,} = (y_0 - B)(Cof 3\alpha x_{,} - Cof 2\alpha x_{,}) + (Sin 3\alpha x_{,} - Sin 2\alpha x_{,})\sqrt{A^2 - (y_0 - B)^2}.$

Hinc autem exterminando $\sqrt{A^2 - (y_0 - B)^2}$ obtinetur $\Delta y_{,,}(Sin\, 2\alpha x_{,} - Sin\, \alpha x_{,}) - \Delta y_{,}(Sin\, 3\alpha x_{,} - Sin\, 2\alpha x_{,}) = (y_0 - B)[(Cof\, 3\alpha x_{,} - Cof\, 2\alpha x_{,})(Sin 2\alpha x_{,} - Sin\, \alpha x_{,}) - (Cof\, 2\alpha x_{,} - Cof\, \alpha x_{,})(Sin\, 3\alpha x_{,} - Sin\, 2\alpha x_{,})]$, unde, factis $S_{,} = Sin\, \alpha x_{,}$; $S_{,,} = Sin\, 2\alpha x_{,}$; & $S_{,,,} = Sin\, 3\alpha x_{,}$; nec non $C_{,} = Cof\, \alpha x_{,}$; $C_{,,} = Cof\, 2\alpha x_{,}$; $C_{,,,} = Cof\, 3\alpha x_{,}$; eruitur

$$y_0 - B = \frac{\Delta y_{,,}(S_{,,} - S_{,}) - \Delta y_{,}(S_{,,,} - S_{,,})}{(C_{,,,} - C_{,,})(S_{,,} - S_{,}) - (C_{,,} - C_{,})(S_{,,,} - S_{,,})}$$

$$\& \; B = y_0 - \frac{\Delta y_{,,}(S_{,,} - S_{,}) - \Delta y_{,}(S_{,,,} - S_{,,})}{(C_{,,,} - C_{,,})(S_{,,} - S_{,}) - (C_{,,} - C_{,})(S_{,,,} - S_{,,})}.$$

His vero cognitis invenitur

$$A = \sqrt{\left(\frac{(\Delta y_{,} - (y_0 - B)(C_{,,} - C_{,}))^2}{(S_{,,} - S_{,})^2} + y_0 - B\right)},$$

atque $Sin\, a = \frac{y_0 - B}{A}$.

Alia,

Alia, ab hac parum diverſa, methodus conſtantes quantitates determinandi adhiberi quoque poteſt, ut oſtendemus in illo caſu, ſi invenienda eſt lex ſex phænomenorum obſervatorum, quando uti convenit æquatione

$$y = A\ Sin\ (a + \alpha x) + B\ Sin\ (b + \beta x).$$

Retentis hic præcedentibus denominationibus & poſitis

$k_{,} = \frac{N^{b\sqrt{-1}}}{2\sqrt{-1}}$; $k_{,,} = \frac{N^{-b\sqrt{-1}}}{2\sqrt{-1}}$; $N^{\beta\sqrt{-1}} = b_{,}$; atque $N^{-\beta\sqrt{-1}} = b_{,,}$;

habentur æquationes auxiliares ſequentes:

$y_0 = AH_{,} - AH_{,,} + Bk_{,} - Bk_{,,}$;

$y_{,} = AH_{,}\, a_{,}^{x_{,}} - AH_{,,}\, a_{,,}^{x_{,}} + Bk_{,}\, b_{,}^{x_{,}} - Bk_{,,}\, b_{,,}^{x_{,}}$;

$y_{,,} = AH_{,}\, a_{,}^{2x_{,}} - AH_{,,}\, a_{,,}^{2x_{,}} + Bk_{,}\, b_{,}^{2x_{,}} - Bk_{,,}\, b_{,,}^{2x_{,}}$;

$y_{,,,} = AH_{,}\, a_{,}^{3x_{,}} - AH_{,,}\, a_{,,}^{3x_{,}} + Bk\, b_{,}^{3x_{,}} - Bk_{,,}\, b_{,,}^{3x_{,}}$;

$y_{IV} = AH_{,}\, a_{,}^{4x_{,}} - AH_{,,}\, a_{,,}^{4x_{,}} + Bk_{,}\, b_{,}^{4x_{,}} - Bk_{,,}\, b_{,,}^{4x_{,}}$;

$y_{V} = AH_{,}\, a_{,}^{5x_{,}} - AH_{,,}\, a_{,,}^{5x_{,}} + Bk_{,}\, b_{,}^{5x_{,}} - Bk_{,,}\, b_{,,}^{5x_{,}}$;

unde $y_{,} - y_0\, a_{,}^{x_{,}} = - AH_{,,}\, (a_{,,}^{x_{,}} - a_{,}^{x_{,}}) + Bk_{,}\, (b_{,}^{x_{,}} - a_{,}^{x_{,}})$
$- Bk_{,,}\, (b_{,,}^{x_{,}} - a_{,}^{x_{,}})$;

$y_{,,} - y_{,}\, a_{,}^{x_{,}} = - AH_{,,}\, (a_{,,}^{x_{,}} - a_{,}^{x_{,}})\, a_{,,}^{x_{,}} + Bk_{,}\, (b_{,}^{x_{,}} - a_{,}^{x_{,}})\, b_{,}^{x_{,}}$
$- Bk_{,,}\, (b_{,,}^{x_{,}} - a_{,}^{x_{,}})\, b_{,,}^{x_{,}}$;

$y_{,,,} - y_{,,}\, a_{,}^{x_{,}} = - AH_{,,}\, (a_{,,}^{x_{,}}\, a_{,}^{x_{,}})\, a_{,,}^{2x_{,}} + Bk_{,}\, (b_{,}^{x_{,}} - a_{,}^{x_{,}})\, b_{,}^{2x_{,}}$
$- Bk_{,,}\, (b_{,,}^{x_{,}} - a_{,}^{x_{,}})\, b_{,,}^{2x_{,}}$;

y_{IV} —

$$y_{IV} - y_{III}\, a_{I}^{x_I} = - AH_{II}\, (a_{II}^{x_I} - a_{I}^{x_I})\, a_{II}^{3x_I} + Bk_{I}\, (b_{I}^{x_I} - a_{I}^{x_I})\, b_{I}^{3x_I}$$
$$- Bk_{II}\, (b_{II}^{x_I} - a_{I}^{x_I})\, b_{II}^{3x_I};$$
$$y_{V} - y_{IV}\, a_{I}^{x_I} = - AH_{II}\, (a_{II}^{x_I} - a_{I}^{x_I})\, a_{II}^{4x_I} + Bk_{I}\, (b_{I}^{x_I} - a_{I}^{x_I})\, b_{I}^{4x_I}$$
$$- Bk_{II}^{x_I}\, (b_{II}^{x_I} - a_{II}^{x_I})\, b_{II}^{4x_I}.$$

Eliminata ulterius quantitate AH_{II} obtinentur æquationes

$$y_{II} - y_{I}\, (a_{I}^{x_I} + a_{II}^{x_I}) + y_{0}\, a_{I}^{x_I}\, a_{II}^{x_I} = Bk_{I}\, (b_{I}^{x_I} - a_{I}^{x_I})\, (b_{I}^{x_I} - a_{II}^{x_I})$$
$$- Bk_{II}\, (b_{II}^{x_I} - a_{I}^{x_I})\, (b_{II}^{x_I} - a_{II}^{x_I});$$
$$y_{III} - y_{II}\, (a_{I}^{x_I} + a_{II}^{x_I}) + y_{I}\, a_{I}^{x_I}\, a_{II}^{x_I} = Bk_{I}\, (b_{I}^{x_I} - a_{I}^{x_I})\, (b_{I}^{x_I} - a_{II}^{x_I})\, b^{x_I}$$
$$- Bk_{II}\, (b_{II}^{x_I} - a_{I}^{x_I})\, (b_{II}^{x_I} - a_{II}^{x_I})\, b_{II}^{x_I};$$
$$y_{IV} - y_{III}(a_{I}^{x_I} + a_{II}^{x_I}) + y_{II}\, a_{I}^{x_I}\, a_{II}^{x_I} = Bk_{I}(b_{I}^{x_I} - a_{I}^{x_I})(b_{I}^{x_I} - a_{II}^{x_I} b_{I})^{2x_I}$$
$$- Bk_{II}\, (b_{II}^{x_I} - a_{I}^{x_I})\, (b_{II}^{x_I} - a_{II}^{x_I})\, b_{II}^{2x_I};$$
$$y_{V} - y_{IV}(a_{I}^{x_I} + a_{II}^{x_I}) + y_{II}\, a_{I}^{x_I}\, a_{II}^{x_I} = Bk_{I}(b_{I}^{x_I} - a_{I}^{x_I})(b_{I}^{x_I} - a_{II}^{x_I})\, b_{I}^{3x_I}$$
$$- Bk_{II}\, (b_{II}^{x_I} - a_{I}^{x_I})\, (b_{II}^{x_I} - a_{II}^{x_I})\, b_{II}^{3x_I}.$$

Similiter erit

$$y_{III} - y_{II}(a_{I}^{x_I} + a_{II}^{x_I}) + y_{I} a_{I}^{x_I} a_{II}^{x_I} - (y_{II} - y_{I}(a_{I}^{x_I} + a_{II}^{x_I}) + y_{0} a_{I}^{x_I} a_{II})\, b_{I}^{x_I}$$
$$= - Bk_{II}\, (b_{II}^{x_I} - a_{I}^{x_I})\, (b_{II}^{x_I} - a_{II}^{x_I})\, (b_{II}^{x_I} - b_{I}^{x_I});$$
$$y_{IV} - y_{III}(a_{I}^{x_I} + a_{II}^{x_I}) + y_{II} a_{I}^{x_I} a_{II}^{x_I} - (y_{III} - y_{II}(a_{I}^{x_I} + a_{II}^{x_I}) + y_{I} a_{I}^{x_I} a_{II}^{x_I})\, b_{I}^{x_I}$$
$$= - Bk_{II}\, (b_{II}^{x_I} - a_{I}^{x_I})\, (b_{II}^{x_I} - a_{II}^{x_I})\, (b_{II} a^{x_I} - b_{I}^{x_I})\, b_{II}^{x_I};$$
$$y_{V} - y_{IV}(a_{I}^{x_I} + a_{II}^{x_I}) + y_{II} a_{I}^{x_I} a_{II}^{x_I} - (y_{IV} - y_{III}(a_{I}^{x_I} + a_{II}^{x_I}) + y_{II} a_{I}^{x_I} a_{II}^{x_I})\, b_{I}^{x_I}$$
$$= - Bk_{II}\, (b_{II}^{x_I} - a_{I}^{x_I})\, (b_{II}^{x_I} - a_{II}^{x_I})\, (b_{II}^{x_I} - b_{I}^{x_I})\, b_{II}^{2x_I};$$

adeo-

adeoque $[y_{,,,} - y_{,,}(a_{,}^{x,} + a_{,,}^{x,}) + y_{,}a_{,}^{x,}a_{,,}^{x,} - (y_{,,} - y_{,}(a_{,}^{x,} + a_{,,}^{x,}) + y_{0}a_{,}^{x,}a_{,,}^{x,})\,b_{,}^{x,}]\,b_{,,}^{x,} = y_{iv} - y_{,,,}(a_{,}^{x,} + a_{,,}^{x,}) + y_{,,}a_{,}^{x,}a_{,,}^{x,} - (y_{,,,} - y_{,,}(a_{,}^{x,} + a_{,,}^{x,}) + y_{,}a_{,}^{x,}a_{,,}^{x,})\,b_{,}^{x,}$; seu $y_{iv} - y_{,,,}(a_{,}^{x,} + a_{,,}^{x,} + b_{,}^{x,} + b_{,,}^{x,}) + y_{,,}(a_{,}^{x,}a_{,,}^{x,} + a_{,}^{x,}b_{,}^{x,} + a_{,}^{x,}b_{,,}^{x,} + a_{,,}^{x,}b_{,}^{x,} + a_{,,}^{x,}b_{,,}^{x,} + b_{,}^{x,}b_{,,}^{x,}) - y_{,}(a_{,}^{x,}a_{,,}^{x,}b_{,}^{x,} + a_{,}^{x,}a_{,,}^{x,}b_{,,}^{x,} + a_{,}^{x,}b_{,}^{x,}b_{,,}^{x,} + a_{,,}^{x,}b_{,}^{x,}b_{,,}^{x,}) + y_{0}a_{,}^{x,}a_{,,}^{x,}b_{,}^{x,}b_{,,}^{x,} = 0$. Cum autem sit $a_{,}^{x,} + a_{,,}^{x,} = 2Cos\,\alpha x_{,}$; $b_{,}^{x,} + b_{,,}^{x,} = 2Cos\,\beta x_{,}$; $a_{,}^{x,}a_{,,}^{x,} = 1$; $b_{,}^{x,}b_{,,}^{x,} = 1$; erit $a_{,}^{x,}a_{,,}^{x,} + a_{,}^{x,}b_{,}^{x,} + a_{,}^{x,}b_{,,}^{x,} + a_{,,}^{x,}b_{,}^{x,} + a_{,,}^{x,}b_{,,}^{x,} + b_{,}^{x,}b_{,,}^{x,} = 2 + 4Cos\,\alpha x_{,}Cos\,\beta x_{,}$; $a_{,}^{x,}a_{,,}^{x,}b_{,}^{x,} + a_{,}^{x,}a_{,,}^{x,}b_{,,}^{x,} + a_{,}^{x,}b_{,}^{x,}b_{,,}^{x,} + a_{,,}^{x,}b_{,}^{x,}b_{,,}^{x,} = 2Cos\,\alpha x_{,} + 2Cos\,\beta x_{,}$; atque $a_{,}^{x,}a_{,,}^{x,}b_{,}^{x,}b_{,,}^{x,} = 1$. Hisce valoribus substitutis habebitur æquatio:

$y_{0} + 2y_{,,} + y_{iv} - 2(y_{,} + y_{,,,})(Cos\,\alpha x_{,} + Cos\,\beta x) + 4y_{,,}Cos\,\alpha x_{,}Cos\,\beta x_{,} = 0$. Similiter invenitur $y_{,} + 2y_{,,,} + y_{v} - 2(y_{,,} + y_{iv})(Cos\,\alpha x_{,} + Cos\,\beta x_{,}) + 4y_{,,,}Cos\,\alpha x_{,}Cos\,\beta x_{,} = 0$; quæ æquationes, factis

$$M = \frac{y_{0} + 2y_{,,} + y_{iv}}{4y_{,,}};\ P = \frac{y_{,} + y_{,,,}}{4y_{,,}},$$

$$N = \frac{y_{,} + 2y_{,,,} + y_{v}}{4y_{,,,}};\ Q = \frac{y_{,,} + y_{iv}}{4y_{,,,}};$$

dant $Cos\,\alpha x_{,} = [M - N + \sqrt{((M-N)^2 - 4(MQ - NP)(P - Q))}] : 2(P - Q)$; atque $Cos\,\beta x_{,} = [M - N - \sqrt{((M-N)^2 - 4(MQ - NP)(P - Q))}] :$

$2(P -$

$2(P - Q)$. Cognitis autem $Cos\ \alpha x_{,}$ & $Cos\ \beta x_{,}$ innotescunt quoque $Sin\ \alpha x_{,}$ & $Sin\ \beta x_{,}$, adeoque $a_{,}^{x_{,}} = Cos\ \alpha x_{,} + Sin\ \alpha x_{,} \sqrt{-1}$; $a_{,,}^{x_{,}} = Cos\ \alpha x_{,} - Sin\ \alpha x_{,} \sqrt{-1}$; $b_{,}^{x_{,}} = Cos\ \beta x_{,} + Sin\ \beta x_{,} \sqrt{-1}$; $b_{,,}^{x_{,}} = Cos\ \beta x_{,} - Sin\ \beta x_{,} \sqrt{-1}$; quibus in valoribus y_{0}; $y_{,}$; $y_{,,}$; &c. substitutis determinari possunt constantes A; B; $H_{,}$; $H_{,,}$; $k_{,}$ & $k_{,,}$; nec non $Sin\ a$ & $Sin\ b$.