

SPECIMEN MATHEMATICUM

DE

*USU INTERPOLATIONIS IN LOGARITHMIS LINEARUM TRIGONOMETRICARUM INVENIENDIS.*

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QUOD

CONS. AMPL. FAC. PHIL. REG. ACAD. ABOENS.

*Publico examini subjiciunt*

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In Audit. Mathematico die xi Junii MDCCCVI.

b. a. m. c.

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*ABOÆ, Typis FRENCKELLIANIS.*

3.

À  
MONSIEUR  
ALBERT JULES SEGERSTEDT  
DOCTEUR EN PHILOSOPHIE ET EN MéDECINE,  
LECTEUR DE MéDECINE ET D'HISTOIRE  
NATURELLE,  
& à  
MADAME  
GUSTAVE SEGERSTEDT,  
SON ÈPOUSE, NÉE DE BERGIUS,  
MONSIEUR & MADAME!

*Daignez, je Vous prie, agréer ce petit ouvrage, comme un  
faible temoignage de la reconnaissance, que je dois à vos bon-  
tés, & des sentimens respectueux, avec lesquels je suis*

*MONSIEUR & MADAME*

VOTRE

très-humble & très-  
obéissant serviteur  
J. HOLMLIN.



### §. I.

**Q**uam diffusa, molestaque, qui primum Tabulas construxerint Logarithmicas in eo usi sunt Geometræ methodo, media scil. quærendo limitum arithmeticæ & Geometricæ, facili patebit cuicunque negotio, qui vel parum infinitas fere istas multiplicationes, divisiones, extractionesque radicum unici tantum logarithmi inveniendi causa peragendas tentaverit. Ut vero in tota Geometria, ita præfertim hac in parte, doctrina fluxionum viam aperuit longe facilioriem, ostendendo functionem quamcunque transcendentem & in seriem infinitam algebraicam resolvi posse, &, si quis fuerit cognitus valor, quantum augmentum sive decrementum respondeat augmento sive decremento functionis datae; quæ si vehementer convergat series, nonnulli tantum computati termini satis exactum præbebunt valorem functionis numericum. Nec jam nisi in tabulis minoribus Logarithmus cujuscunque numeri sive lineæ Trigonometricæ

tricæ desideratur. Sed quoniam hæ frequentissimæ sunt, operæ pretium duximus Opusculo hoc Academico investigare, quomodo interpolatione ad cognitionem Logarithmi lineæ trigonomitricæ in tabulis non occurrentis perveniri possit, nec non, si requiratur logarithmus in pluribus, quam quæ in tabulis existent, locis decimalibus, quæ præcipue adhibendæ sint series genuinæ, sperantes tentamina nostra mitem Tui, L. B. expertura esse censuram.

## §. II.

Quum Logarithmi linearum trigonometricarum pro quounque minuto primo in laudatis occurrant tabulis, primo nobis investigandum est, quantum dati cuiusdam logarithmi fiat vel augmentum vel decrementum accendentibus arcui correspondenti minutis secundis, (quorum tamen numerus 30 superare non debet, quoniam proximo semper utendum est Logarithmo).

Sit ergo  $x$  Arcus gradibus & minutis primis integris expressus, & exploretur quantum adferant Logarithmo  $\sin x$ ,  $\cos x$  e. s. p. augmentum sive decrementum differentia  $n$  minutorum secundorum.

Cum jam, denotante  $f$  functionem quamcunque quantitatis  $x + n$ , posita  $dx$  constante, vi Theorematis notissimi Tayloriani, generaliter sit:

$$f(x +$$

$$f(x+n) = f(x) + \frac{n^1 d f(x)}{dx} + \frac{n^2 d^2 f(x)}{2 \cdot dx^2} + \frac{n^3 d^3 f(x)}{2 \cdot 3 \cdot dx^3} + \frac{n^4 d^4 f(x)}{2 \cdot 3 \cdot 4 \cdot dx^4} + \dots + \frac{n^i d^i f(x)}{2 \cdot 3 \cdot 4 \cdots i dx^i}, \text{ erit facta } f(x+n) = \log \sin(x+n),$$

$$\log \sin(x+n) = \log \sin x + M \left( \frac{n d \log \sin x}{dx} + \frac{n^2 d^2 \log \sin x}{2 \cdot dx^2} + \frac{n^3 d^3 \log \sin x}{2 \cdot 3 \cdot dx^3} + \dots + \frac{n^i d^i \log \sin x}{2 \cdot 3 \cdot 4 \cdots i dx^i} + \dots \right)$$

existente  $M = 0,4342944819032518\dots$ , qua logarithmi nativi sunt multiplicandi, ut in Tabulares abeant. Revolutis vero fluxionibus  $\log \sin x$ , inveniuntur:

$$d \log \sin x = \frac{dx \cos x}{\sin x}$$

$$d^2 \log \sin x = - \frac{dx^2}{\sin x^2}$$

$$d^3 \log \sin x = \frac{2 dx^3 \cos x}{\sin x^3}$$

$$d^4 \log \sin x = - \frac{2 dx^4 (3 - 2 \sin x^2)}{\sin x^4}$$

$$d^5 \log \sin x = \frac{8 dx^5 (3 - \sin x^2) \cos x}{\sin x^5}$$

$$d^6 \log \sin x = - \frac{8 dx^6 (15 - 15 \sin x^2 + 2 \sin x^4)}{\sin x^6}$$

$$d^7 \log \sin x = \frac{16 dx^7 (45 - 30 \sin x^2 + 2 \sin x^4) \cos x}{\sin x^7}$$

$$d^8 \log \sin x = - \frac{16 dx^8 (315 - 420 \sin x^2 + 126 \sin x^4 - 4 \sin x^6)}{\sin x^8}$$

unde jam concludere licet, esse, pro  $i$  numero pari,  
 $d^i \log \sin x = -\frac{dx^i}{\sin x^i} (a - b \sin x^2 + c \sin x^4 - e \sin x^6 + \dots \pm p \sin x^{i-4} \mp q \sin x^{i-2})$ , respondente ultima coëfficiente  $q$  termino ordinis  $\frac{i}{2}$ . Sumta vero hujus fluxione emergit:

$$d^{i+1} \log \sin x = \frac{dx^{i+1} \cos x}{\sin x^{i+1}} (ia - (i-2)b \sin x^2 + (i-4)c \sin x^4 - (i-6)e \sin x^6 + \dots \pm 4p \sin x^{i-4} \mp 2q \sin x^{i-2}).$$

Si vero sit  $i$  numerus impar, erit

$$d^i \log \sin x = \frac{dx^i \cos x}{\sin x^i} (a - b \sin x^2 + c \sin x^4 - \dots \pm p \sin x^{i-5} \mp q \sin x^{i-3}),$$

qua differentiata nascitur

$$d^{i+1} \log \sin x = -\frac{dx^{i+1}}{\sin x^{i+1}} (ia - ((i-1)a + (i-2)b) \sin x^2 + ((i-3)b + (i-4)c) \sin x^4 - ((i-5)c + (i-6)e) \sin x^6 + \dots \mp (4p + 3q) \sin x^{i-3} \pm 2q \sin x^{i-1}),$$

ubi  $q$  respondet termino ordinis  $\frac{i-1}{2}$ , quarum formularum ope, cum terminus generalis non pateat, series fluxionum, quo-usque requiratur, continuari potest.

Insertis ergo valoribus inventis, habebitur:

$$(A) \log \sin(x+n) = \log \sin x + M \left( \frac{n \cos x}{\sin x} - \frac{n^2}{2 \sin x^2} + \frac{2n^3 \cos x}{2 \cdot 3 \cdot \sin x^3} - \frac{2n^4(3 - 2 \sin x^2)}{2 \cdot 3 \cdot 4 \cdot \sin x^4} + \frac{8n^5(3 - \sin x^2)\cos x}{2 \cdot 3 \cdot 4 \cdot 5 \cdot \sin x^5} - \frac{8n^6(15 - 15 \sin x^2 + 2 \sin x^4)}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot \sin x^6} + \dots \right),$$

quæ formula  
ut collustretur exemplo:

$$\text{Sit } x = 29^\circ 45' \text{ & } n = 30'' = \frac{\pi}{21600} \approx 0,000145444.$$

$$\text{invenitur } \log \sin x = 0,6956712 - 1$$

$$\log n = 0,1626961 - 4$$

$$\log \frac{\cos x}{\sin x} = 0,2429480$$

$$\log \frac{n \cos x}{\sin x} = 0,4056441 - 4 = \log 0,000254474. \text{ Est porro:}$$

$$2 \log n = 0,3253922 - 8$$

$$2 \text{ Compl. Log } \sin x = 0,6086576$$

$$\text{Compl. Log } 2 = 0,6989700 - 1$$

$$\log \frac{n^2}{2 \sin x^2} = 0,6330198 - 8 = \log 0,000000043$$

qui valor in septimo loco decimalium nullum adfert errorem, & cum reliqui termini adhue sint minores, tuto hi negligi posunt.

$$\text{Hinc } \frac{n \cos x}{\sin x} - \frac{n^2}{2 \sin x^2} = 0,000254431, \text{ &}$$

$$\log 0,000254431 = 0,4055700 - 4$$

$$\log M = 0,6377843 - 1 \quad \text{unde}$$

$$\log M, 0,000254431 = 0,0433543 - 4 = \log 0,0001105.$$

ad-

addito ergo invento hoc valore 0,0001105  
ad cognitum  $\text{Log Sin } 29^\circ 45' = 0,6956712 \dots$   
reperitur  $\text{Log Sin } 29^\circ 45' 30'' = 0,6957817 \dots$

Cum manifestum sit, seriem allatam eo vehementius esse convergentem, quo propius  $x$  ad  $90^\circ$  seu  $\frac{\pi}{2}$  accedit & quo minor fuerit  $n$ , patet, Logarithmos Sinuum angulorum aliquot gradus excedentium non-nullis tantum computatis terminis inveniri. Quod si fiat  $x$  admodum parvus, permulti requirerentur termini, quo valor logarithmi quæsiti vel in septem locis decimalibus errore vacuis inveniatur; huic vero casui infra medela afferetur optima ope alias seriei eo vehementius convergentis, quo minor asfumatur angulus.

*Coroll I.* Posito  $x = \frac{\pi}{2}$ , sive  $\text{Sin } x = 1$ ,  $\text{Cos } x = 0$ , facilissimo e formula (A) derivatur negotio series motatu dignissimæ:

$$\text{Log Sin} \left( \frac{\pi}{2} + n \right) = \text{Log Sin} \left( \frac{\pi}{2} - n \right) = \text{Log Cos} n = -M \left( \frac{n^2}{2} + \frac{n^4}{12} + \frac{n^6}{45} + \frac{17n^8}{2520} + \frac{31n^{10}}{14175} + \frac{691n^{12}}{935550} + \frac{10922n^{14}}{42567525} + \dots \right)$$

(B), quæ pro  $n$  exiguo fatis commodam præbet methodum etiam in pluribus, quam quæ in tabulis inveniuntur decimalibus, inveniendo  $\text{Log Cos } n$ .

Co-

*Coroll. II.* Si in formula (B) loco  $n$  substituatur  $\frac{\pi}{2} - n$ , efficitur esse:

$$\begin{aligned} \text{Log Cof}\left(\frac{\pi}{2} - n\right) = & \text{Log Sin } n - M\left(\frac{1}{2}\left(\frac{\pi}{2} - n\right)^2 + \frac{1}{12}\left(\frac{\pi}{2} - n\right)^4\right. \\ & \left. + \frac{1}{45}\left(\frac{\pi}{2} - n\right)^6 + \frac{17}{2520}\left(\frac{\pi}{2} - n\right)^8 + \dots\right), \text{ quæ accedente } n \\ & \text{ad } 90^\circ \text{ maxime convergit.} \end{aligned}$$

### §, III.

Methodo plane consimili, ac supra facta interpolatione eruimus valorem  $\text{Log Sin}(x+n)$ , derivari potest  $\text{Log Cof}(x+n) = \text{Log Cof } x - M\left(\frac{n \text{Sin } x}{\text{Cof } x} + \frac{n^2}{2 \text{Cof } x^2} + \frac{2n^3 \text{Sin } x}{2 \cdot 3 \cdot \text{Cof } x^3} + \frac{2n^4(3-2 \text{Cof } x^2)}{2 \cdot 3 \cdot 4 \cdot \text{Cof } x^4} + \frac{8n^5(3-\text{Cof } x^2)\text{Sin } x}{2 \cdot 3 \cdot 4 \cdot 5 \cdot \text{Cof } x^5} + \dots\right)$  (C), quæ pro  $x$  exquo veliementer convergit.

Præterea cum sit  $\frac{\text{Sin } \varphi}{\text{Cof } \varphi} = \text{Tang } \varphi$  & hinc  $\text{Log Sin } \varphi - \text{Log Cof } \varphi = \text{Log Tang } \varphi$ , exorietur valor  $\text{Log Tang}(x+n)$ , si a formula (A) subtrahatur formula (C), quo facto obtinetur:

$$\text{Log Tang}(x+n) = \text{Log Sin } x - \text{Log Cof } x + M\left(\frac{n}{\text{Sin } x \text{Cof } x}\right)$$

$$-\frac{n^2 (\Cos x^2 - \Sin x^2)}{2 \Sin x^2 \Cos x^2} + \frac{2 n^3 (\Sin x^4 + \Cos x^4)}{2 \cdot 3 \cdot \Sin x^3 \Cos x^3} -$$

$$\frac{2 n^4 ((3 - 2 \Sin x^2) \Cos x^4 - (3 - 2 \Cos x^2) \Sin x^4)}{2 \cdot 3 \cdot 4 \cdot \Sin x^4 \Cos x^4} + \dots,$$

quæ formula ad hanc reduci potest:

$$\Log \Tang x + M \left( \frac{2 n}{\Sin 2x} - \frac{4 n^2 \Cos 2x}{2 (\Sin 2x)^2} + \frac{8 n^2 (2 - \Sin 2x^2)}{2 \cdot 3 \cdot (\Sin 2x)^3} \right.$$

$$\left. - \frac{16 n^4 (6 - \Sin 2x^2) \Cos 2x}{2 \cdot 3 \cdot 4 \cdot (\Sin 2x)^4} + \dots \right) \quad (D),$$

quæ, si  $x$  fuerit prope  $45^\circ$ , maxime convergit.

#### §. IV.

Quamquam allatæ formulæ, existente  $x > n$  semper convergunt, attamen, si plures quam tres aut quatuor primi adhibendi fuerint termini, calculus nimis evadit prolixus. Interest ergo scire, quam parvum in formula (A) assumere liceat angulum  $x$ , ne pro  $n = 30^\circ$  in septem locis decimalibus quis inveniatur error, si tres tantum primi adhibeantur termini. Sit ergo error = 0,0000005 quem pariant termini ultra tertium. Idem vero error esse debet =  $\Log \Sin(x + 30^\circ)$  —  $\Log \Sin x - M \left( \frac{30^\circ \Cos x}{\Sin x} - \frac{30^\circ^3}{2 \Sin x^2} \right)$ , sed quoniam

hoc

hoc in casu  $x$  evadit admodum parvus, assumi potest  
 $\sin(x+30^\circ) = x + 30''$ ,  $\sin x = x$  &  $\cos x = 1$ , quo facto  
habebitur:  $\log\left(\frac{x+30''}{x}\right) = \frac{30'' M}{x} + \frac{30''^2 M}{2x^2} = 0,00000005$ .

Est vero  $\log(1+y) = M(y - \frac{1}{2}y^2 + \frac{1}{3}y^3 - \frac{1}{4}y^4 + \dots)$   
quare si ponatur  $y = \frac{30''}{x}$ , erit:

$$\log\left(\frac{x+30''}{x}\right) = \log\left(1 + \frac{30''}{x}\right) = M\left(\frac{30''}{x} - \frac{30''^2}{2x^2} + \frac{30''^3}{3x^3} - \dots\right)$$

quo inferto valore exsurgit:

$$M\left(\frac{30''}{x} - \frac{30''^2}{2x^2} + \frac{30''^3}{3x^3} - \frac{30''}{x} + \frac{30''^2}{2x^2}\right) = 0,00000005, \text{ unde}$$

de invenitur  $x = 30'' \sqrt[3]{\frac{M}{3,0,000005}}$  qua formula,  
ope Logarithmorum, investigata reperitur:

$$\log M = 0,6377843 - 1$$

Compl.  $\log 0,00000015 = 6,8239087$

$$\log \frac{M}{0,00000015} = 6,4616930$$

$$\log \frac{M}{0,00000015} = 2,1538977, \text{ unde}$$

$$\sqrt{\frac{M}{0,00000015}} = 142,52\dots \& x = 1^\circ 11' 26'' \text{ proxime.}$$

Apparet ergo quoties fuerit  $x$  aut  $=$  aut  $< 1^{\circ} 11'$   
septem loca decimalia erroribus vacua tribus tantum  
computatis terminis non inveniri.

Idem quoque valet de  $\text{Log. Tang}(x+n)$ ; for-  
mula vero ( $C$ ) usque ad  $88^{\circ} 49'$  tribus absolvitur ad  
minimum terminis. Quod si angulus ultra hos limi-  
tes versaretur, sequentes addere voluimus formulas  
his in casibus apprime inservientes:

### §. V.

Vi Theorematis allati Tayloriani, prodit:

$$\begin{aligned}\text{Tang}(x+n) &= \text{Tang } x + \frac{n d \text{Tang } x}{dx} + \frac{n^2 d^2 \text{Tang } x}{2 d x^2} + \frac{n^3 d^3 \text{Tang } x}{2 \cdot 3 \cdot d x^3} + \dots \\ &= \text{Tang } x + \frac{n}{\text{Cos } x^2} + \frac{2n^2 \text{Sin } x}{2 \text{Cos } x^3} + \frac{2n^3(3-2\text{Cos } x^2)}{2 \cdot 3 \cdot \text{Cos } x^4} + \frac{8n^4(3-\text{Cos } x^2)\text{Sin } x}{2 \cdot 3 \cdot 4 \cdot \text{Cos } x^5} \\ &\quad + \frac{8n^5(15-15\text{Cos } x^2+2\text{Cos } x^4)}{2 \cdot 3 \cdot 4 \cdot 5 \cdot \text{Cos } x^6} + \dots\end{aligned}$$

Facto ergo  $x = 0$ , sive  $\text{Tang } x = 0$ ,  $\text{Sin } x = 0$ ,  $\text{Cos } x = 1$   
emergit:

$$\begin{aligned}\text{Tang } n &= n + \frac{n^3}{3} + \frac{2n^5}{15} + \frac{17n^7}{315} + \frac{62n^9}{2835} + \frac{1382n^{11}}{155925} + \\ &\quad \frac{21844n^{13}}{6081075} + \dots \text{ Est vero } \text{Cotg } n = \frac{1}{\text{Tang } n} = \\ &\quad \left(n + \frac{n^3}{3} + \frac{2n^5}{15} + \dots\right)^{-1}, \text{ qua secundum Theore-} \\ &\quad \text{ma binomiale evoluta, invenitur: } \text{Cotg}\end{aligned}$$

$$\operatorname{Cotg} n = \frac{1}{n} - \frac{n}{3} - \frac{n^3}{45} - \frac{2n^5}{945} - \frac{n^7}{4725} - \frac{2n^9}{93555} - \dots$$

sed cum sit  $d_n \operatorname{Log Sin} n = \frac{dn \operatorname{Cos} n}{\operatorname{Sin} n}$  =  $d_n \operatorname{Cotg} n$  erit  
adhibito valore invento  $\operatorname{Cotangentis} n$

$$d \operatorname{Log Sin} n = dn \left( \frac{1}{n} - \frac{n}{3} - \frac{n^3}{45} - \frac{2n^5}{945} - \frac{n^7}{4725} - \dots \right)$$

quare integrando habebitur:

$$(E) \operatorname{Log Sin} n = \operatorname{Log} n - M \left( \frac{n^2}{6} + \frac{n^4}{180} + \frac{n^6}{2835} + \frac{n^8}{37800} + \frac{n^{10}}{467775} + \dots \right). \text{ Et inserto pro } n \text{ valore } \frac{\pi}{2} - n:$$

$$(F) \operatorname{Log Sin} \left( \frac{\pi}{2} - n \right) = \operatorname{Log Cos} n = \operatorname{Log} \left( \frac{\pi}{2} - n \right) - M \left( \frac{1}{6} \left( \frac{\pi}{2} - n \right)^2 + \frac{1}{180} \left( \frac{\pi}{2} - n \right)^4 + \frac{1}{2835} \left( \frac{\pi}{2} - n \right)^6 + \dots \right),$$

quæ formulæ, si fuerit  $n < 12^\circ$  sive  $> 78^\circ$  respective,  
tribus tantum sumtis terminis, septem præbent loca  
decimalia vera.

Quod si differentias arcuum proxime in tabulis  
insequentium, eandem sequi supponere liceat rationem,  
ac differentias Logarithmorum Sinuum eorundem,  
fatis concinna nascitur vulgaris computandi metho-  
dus. Dictis enim his arenibus  $x$  &  $x+m$ , Logarith-  
misque eorum Sinuum  $a$  &  $a+b$ , quæratur  $\operatorname{Log}$   
 $\operatorname{Sin}(x+n)$  his interjacens; quapropter erit  $m:n::b:$   
quan-

quantitatem Logarithmo  $\sin x$  ( $= a$ ) addendam, unde  
 $\log \sin(x+n) = a + \frac{b_n}{m}$ , quæ methodus in septem  
loca decimalia pro angulo  $> 6^\circ 3'$  nullum adfert  
errorem.

