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Different Types of Weighted Composition Operators on Banach Spaces of Analytic Functions

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Åbo, Finland 2023

ISBN 978-952-12-4289-2 (printe)
ISBN 978-952-12-4290-8 (digital)
Painosalama, Åbo, Finland 2023

Preface

This thesis is based on research carried out at the Faculty of Science and Engineering at Åbo Akademi University in the subject Mathematics between the years 2019 and 2023.

I would like to thank the Doctoral Network in Information Technologies and Mathematics at Åbo Akademi University and the Magnus Ehrnrooth foundation for the financial support, which allowed the creation of this thesis. I am also deeply grateful for the support I have received from my supervisor Professor Mikael Lindström and my assistant supervisor Dr Santeri Miihkinen. At the final phase, I would like to thank Professor Karl-Mikael Perfekt for agreeing to act as my opponent at the public defense of the thesis as well as Dr Boban Karapetrović and Professor Óscar Blasco for the pre-examination of the thesis.

Finally, I would like to thank colleagues, friends, my family and Siiri for their heart-warming support.

Åbo, April 2023

David Norrbo

Abstract

Many problems in real life are, at least approximatively, of linear nature and can be mathematically examined with the aid of linear spaces. It is natural to measure the size of objects occurring in the problems and such an operation is called a norm. If the space and the norm fit well together, they constitute a Banach space. A norm associates every vector with a nonnegative number or infinity, and the Banach space consists of those vectors whose associated number is finite. Different norms give rise to different Banach spaces.

In this dissertation, which contains three articles, different types of weighted composition operators on Banach spaces, consisting of analytic functions defined on the unit disc of the complex plane, are examined. Since the vectors are functions, there are two basic linear operations to consider. One way to modify the vector is by multiplying it with another function. Such an operator is said to be a multiplication operator if it is well defined. Another way to modify the vector is to first transform the input via a function and make the original vector act on the modified input. Such a transformation is done by a composition operator, since the resultant vector is a composition of the original vector and the function transforming the input. A combination of a multiplication operator and a composition operator is said to be a weighted composition operator.

In one of the articles, a certain class of integral operators on weighted Bergman spaces are examined. The exact value of the essential norm of such operators, which can be represented as a mean of weighted composition operators, is calculated. Another article deals with the connection between some operator-theoretic properties of a weighted composition operator on the Banach space BMOA and the behaviour of corresponding functions. Compactness, weak compactness and complete continuity are examined. In the so far not mentioned article, the spectrum and essential spectrum are determined for multiplication operators on some Banach spaces.

Svensk sammanfattning

Många naturligt förekommande problem i verkligheten kan lämpligen beskrivas matematiskt med hjälp av vektorrum. Det är vanligt att man vill kunna mäta storleken av en vektor; att förse ett vektorrum med en sådan operation (norm), ger oss ett så kallat Banachrum, givet att normen och rummet samarbetar väl. En given norm ger alla vektorer ett värde, större än eller lika med noll, eller oändligt. De med ändlig norm utgör det så kallade Banachrummet. Olika normer ger upphov till olika Banachrum.

I denna avhandling, som väsentligen består av tre artiklar, undersöks olika varianter av så kallade viktade kompositionsoperatorer på olika Banachrum bestående av funktioner, analytiska på den öppna enhetsdisken i det komplexa talplanet. Eftersom vektorerna är funktioner, existerar det två enkla typer av linjära operationer. Den enklaste är att förändra värdet av funktionen genom att multiplicerar den med en annan funktion. En sådan transformation kan utföras av en linjär operator, en så kallad multiplikationsoperator. Förutom värdet, kan indatat till funktionen förändras. Denna typ av transformation görs av en så kallad kompositionsoperator. Kombineras dessa två linjära operationer fås en viktad kompositionsoperator.

I en av artiklarna betraktas bland annat viktade Bergmanrum och den väsentliga normen av en klass integraloperatorer bestäms. Dessa operatorer kan uttryckas som ett kontinuerligt medeltal av viktade kompositionsoperatorer. Det är även intressant att veta hurudana funktioner som, vid bildandet av en viktad kompositionsoperator, ger upphov till vissa operatorteoretiska egenskaper hos operatorn. I en annan av de inkluderade artiklarna karakteriseras de funktioner som genererar en kompakt (compact), svagt kompakt (weakly compact) respektive fullständigt kontinuerlig (completely continuous) operator på Banachrummet BMOA. Vissa egenskaper av en linjär operator kan erhållas från dess så kallade spektrum, vilket berättar när en skalär förskjutning av operatorn är inverterbar. Det sista resultatet som behandlas i avhandlingen är spektrumet och väsentliga spektrumet av en multiplikationsoperator på vissa Banachrum.

Contents

Chapter 1

Introduction

Linear systems and differential equations are often the mathematical problems obtained, either in an exact manner from or as an approximation of, some problems encountered in life. A simple linear system can be represented as a matrix equation $Ax = b$, where the matrix A can be viewed as a linear operator and the unknown variables constitute a vector x in a vector space. The result of the operator acting on the vector of unknowns is a vector b, which is assumed to be known. A generalisation of these concepts give rise to (linear) functional analysis. It follows from the proof of [1, Theorem 3, p. 170] that all linear isometries $C(Q,\mathbb{R}) \to C(Q,\mathbb{R})$ are of the form $f \mapsto \psi f \circ \phi$, where $\psi(Q) \subset \{-1,1\}$, $\phi\colon Q \to Q$ is a homeomorphism and Q is a compact metric space. The map $f \mapsto \psi f \circ \phi$ is a weighted composition operator. In [8], Forelli proved that a certain subclass of weighted composition operators, mapping H^p, $p > 1, p \neq 2$ onto itself, is exactly the set of linear isometries on the given Hardy space. About 20 years later, Kolaski [12] proved a similar statement on the weighted Bergman spaces.

The Hilbert matrix operator on spaces of analytic functions on the unit disk is often represented as an integral operator $f \mapsto \int_0^1 f(x)(1-xz)^{-1}\,dx$ and it has arisen from the Hilbert matrix in connection with the double series theorem, stating that for $(a_n)_n \in \ell^p$ and $(b_n)_n \in \ell^q$, where $p^{-1} + q^{-1} = 1,\ p, q > 1$, the inequality

$$\left|\sum_n \sum_m a_n b_n (1+n+m)^{-1}\right| \leq \frac{\pi}{\sin(\pi/p)} \left(\sum_n a_n^p\right)^{\frac{1}{q}} \left(\sum_n b_n^q\right)^{\frac{1}{q}}$$

holds. At first sight, the Hilbert matrix operator has nothing to do with a weighted composition operator, but it turns out that in many cases an equivalent and more useful representation is given by $f \mapsto \int_0^1 w_t f \circ \phi_t dt$, where $w_t(z) = (1-(1-t)z)^{-1}$ and $\phi_t = tw_t$.

One may ask if the situation is different if the domain for the analytic functions changes. The Riemann mapping theorem gives a negative answer to that question in proper nice subsets of the complex plane; at least concerning many interesting properties, for example, the existence of non-tangential limits, which are invariant under the Riemann map. In higher dimensions $\mathbb{C}^n$, $n > 1$, things are different, but most of the focus in this thesis is on the one-dimensional case. When the spectrum of a generalised Hilbert matrix operator was determined on ℓ^2 by Rosenblum [23], he proved that there is an isometric map to an L^2-space converting the Hilbert matrix operator into a multiplication operator. Since the spectrum remains unchanged under such a transformation, if zero is neglected, it is sufficient to know the spectrum of the acquired multiplication operator. Therefore, knowledge of weighted composition operators can solve other, not immediately related, problems.

1.1 List of publications

This thesis is based on the following publications.

Paper I [16] M. Lindström, S. Miihkinen, and D. Norrbo, *Unified approach to spectral properties of multipliers*, Taiwanese J. Math. , 24(6) (2020), 1471-1495.
https://doi.org/10.11650/tjm/200205

Paper II [17] M. Lindström, S. Miihkinen, and D. Norrbo, *Exact essential norm of generalized Hilbert matrix operators on classical analytic function spaces*, Adv. Math. 408 (2022), Paper No. 108598, 34 pp. 47B38 (30H20)
https://doi.org/10.1016/j.aim.2022.108598

Paper III [18] J. Laitila, M. Lindström, and D. Norrbo, *Compactness and weak compactness of weighted composition operators on BMOA*, Proc. Amer. Math. Soc. 151 (2023), 1195-1207
https://doi.org/10.1090/proc/16203

The author of this thesis has made a significant contribution to all of the contained publications.

Chapter 2

Definitions and basic results

2.1 Preliminaries

For $n \in \mathbb{Z}_{\geq 1} := \{1, 2, \dots\}$ the open unit ball and its boundary in $\mathbb{C}^n$ are given by

$$\mathbb{B}_n = \left\{ z \in \mathbb{C}^n : |z| = \sqrt{\sum_{k=1}^{n} |z_k|^2} < 1 \right\} \quad \text{and} \quad \mathbb{S}_n = \left\{ z \in \mathbb{C}^n : |z| = \sqrt{\sum_{k=1}^{n} |z_k|^2} = 1 \right\}$$

respectively. For $n = 1$ the notations $\mathbb{D} = B_1$ and $\mathbb{T} = \mathbb{S}_1$ are used. The vector spaces of analytic functions are denoted $\mathrm{HOLO}(\mathbb{B}_n)$, $n \in \mathbb{Z}_{\geq 1}$, and to continue with spherical objects, the notations $B_n(a,r) = \{z \in \mathbb{C}^n : |z - a| < r\}$ and $B(a,r) = B_1(a,r)$ will be used, where $r \geq 0$ and $a \in \mathbb{C}^n$. For a normed space X and $f \in X$, the similar notation $B_X(f,r)$ denotes an open ball with center f and radius r, where the distance between f and some function $g \in X$ is measured by $\|f - g\|_X$.

A homogeneous polynomial $p \colon \mathbb{B}_n \to \mathbb{C}$ of degree $k \in \mathbb{Z}_{\geq 0}$ is a polynomial, not identically zero if $k \geq 1$, of the form

$$p(z_1, z_2, \dots, z_n) = \sum_{\substack{\sum_{v=1}^{n} j_v = k \\ j_v \in \mathbb{Z}_{\geq 0}}} a_{(j_1,\dots,j_n)} \prod_{u=1}^{n} z_u^{j_u},$$

where $a_{(j_1,\dots,j_n)} \in \mathbb{C}$, $(j_1, \dots, j_n) \in \mathbb{Z}_{\geq 0}^n$.

A function $f \in \mathrm{HOLO}(\mathbb{B}_n)$ can always be represented in its standard form, which is $f = \sum_{k=0}^{\infty} p_k$, where p_k, $k \in \mathbb{Z}_{\geq 0}$ are some homogeneous polynomials of degree k, uniquely determined by f.

The gradient of a function $f \in \mathrm{HOLO}(\mathbb{B}_n)$ is defined as

$$\nabla f \colon \mathbb{C}^n \to \mathbb{C}^n \colon z \mapsto \left(\frac{\partial f}{\partial z_1} \dots \frac{\partial f}{\partial z_n} \right),$$

which in one dimension is given by the standard derivative, Df.

The radial derivative of a function $f \in \mathrm{HOLO}(\mathbb{B}_n)$ is given by

$$(Rf)(z_1,\dots,z_n) = \sum_{k=1}^{n} z_k \frac{\partial}{\partial z_k} f(z_1,\dots,z_n) = \sum_{k=0}^{\infty} k\; p_k(z)$$

and the fractional radial derivative is given by

$$(R^\beta f)(z) = \sum_{k=0}^{\infty} k^\beta\; p_k(z), \quad \beta \in \mathbb{R}.$$

Operator-theoretic definitions

Let X and Y be Banach spaces. The Banach spaces encountered in this thesis are implicitly complex. A linear operator $T\colon X \to Y$ is said to be bounded if there exists a constant $C > 0$ such that

$$\|Tf\|_Y \le C\|f\|_X, \; f \in X,$$

and the smallest such C is called the norm of T, denoted $\|T\|_{X\to Y}$. The closed unit ball of a normed space X is given by $\{f \in X : \|f\|_X \le 1\}$ and is denoted B_X. The notation should not be confused with the notation for an open ball in a normed space, which always contains an explicitly stated center and radius. From linearity, it follows that $\|T\|_{X\to Y} = \sup_{f\in B_X}\|Tf\|_Y$. The dual of a Banach space X, denoted X^*, is the Banach space of all bounded linear functionals $l\colon X \to \mathbb{C}$ with the naturally induced norm

$$\|l\|_{X^*} = \sup_{x\in B_X} |l(x)|.$$

For a set $M \subset X$, the *annihilator* of M, denoted $M^\perp$, is a closed subspace of X^* given by

$$M^\perp := \{l \in X^* : l(x) = 0 \text{ for all } x \in M\}.$$

Weak and weak* sequential convergence are defined as

weak: Given $f_n, f \in X$, $n \in \mathbb{Z}_{\ge 1}$, letting $n \to \infty$ the convergence $f_n \xrightarrow{w} f$ means that $l(f_n - f) \to 0$ for every $l \in X^*$;

weak*: Given $l_n, l \in X^*$, $n \in \mathbb{Z}_{\ge 1}$, letting $n \to \infty$ the convergence $l_n \xrightarrow{w^*} l$ means that $(l_n - l)(f) \to 0$ for every $f \in X$.

Let τ be a topology on a set X and take a subset $B \subset X$. Then the relative topology of τ to B is denoted $\tau(B)$. The topological space $(B, \tau(B))$ will be denoted (B, τ). The topologies τ_π and τ_0 are the topologies induced by point-wise convergence on $\mathbb{D}$ and by uniform convergence on compact subsets of $\mathbb{D}$ respectively. These topologies are defined on $\mathrm{HOLO}(\mathbb{D})$. On Banach spaces, the weak topology $w(X)$ is defined as the coarsest topology yielding that all functionals in X^* are continuous. If (B, w) is metrizable, $w(B)$ coincides with the topology induced by weak sequential convergence on B. For X^*, the weak* topology $w^*(X^*)$ is the coarsest topology such that the map

$x \mapsto f_x$, $f_x(l) = l(x)$, $l \in X^*$ is continuous for every $x \in X$, that is, the topology is generated by the sets $\{f_x^{-1}(U),\ U \subset \mathbb{C} \text{ open},\ x \in X\}$. Again, if (B_*, w^*) is metrizable for some set $B_* \subset X^*$, the topology coincides the topology induced by sequential weak* convergence. A standard result, which can be found in for example [19, Theorem 2.6.23], states that the unit ball B_{X^*} is $w^*(B_{X^*})$-metrizable if and only if X is separable.

A bounded linear operator $T\colon X \to Y$ acting between normed spaces is said to be

1. *invertible* if it is bijective with a bounded inverse,
2. *compact* if it maps bounded sequences to sequences with a convergent subsequence,
3. *weakly compact* if it maps bounded sequences to sequences with a weakly convergent subsequence,
4. *completely continuous* if it maps weakly convergent sequences to norm convergent sequences,
5. an *isomorphism* if it is a bijection with a bounded inverse.
6. *Fredholm* if $\dim \operatorname{Ker} T < \infty$ and $\dim(Y/\operatorname{Ran} T) < \infty$; according to Atkinson's theorem an equivalent definition is: there exists a bounded linear operator S such that both $ST - I$ and $TS - I$ are compact.

In case (X, τ_X) and (Y, τ_Y) are topological spaces, a mapping $S\colon X \to Y$ is

7. a *homeomorphism* if it is a continuous bijection with a continuous inverse.

The algebra of bounded linear operators from a normed space X to a normed space Y is denoted by $\mathcal{L}(X,Y)$ and $\mathcal{L}(X) := \mathcal{L}(X,X)$. Similarly, the ideal to $\mathcal{L}(X,Y)$ of compact operators $X \to Y$ is denoted by $\mathcal{K}(X,Y)$ and $\mathcal{K}(X) := \mathcal{K}(X,X)$.

The essential norm of an operator $T \in \mathcal{L}(X,Y)$ is defined to be the real number

$$\inf_{L \in \mathcal{K}(X,Y)} \|T - L\|_{X \to Y}.$$

If X is a normed space, the *spectrum* of $T \in \mathcal{L}(X)$ is defined as the subset $\sigma(T)$ of $\mathbb{C}$, consisting of the numbers λ such that $T - \lambda I$ is not invertible. Similarly, the *essential spectrum* $\sigma_e(T) \subset \mathbb{C}$ is the set with the property that $\lambda \in \sigma_e(T)$ if and only if $T - \lambda I$ is not Fredholm. It is well known that both the spectrum and the essential spectrum are compact, non-empty sets.

If two maps $a, b\colon M \to \mathbb{R}$ satisfies $a(x) \gtrsim b(x)$ (or $a(x) \lesssim b(x)$), it means there is a constant $C > 0$ such that $a(x) \geq Cb(x)$ (or $a(x) \leq Cb(x)$) for all $x \in M$. The notion of a and b being equivalent means that both $a(x) \gtrsim b(x)$ and $a(x) \lesssim b(x)$ hold and is denoted by $a(x) \asymp b(x)$. If $\mathcal{I}$ is an isomorphism $X \to Y$, then $\|\mathcal{I} f\|_Y \asymp \|f\|_X$, $f \in X$ and $\|g\|_Y \asymp \left\|\mathcal{I}^{-1} g\right\|_X$, $g \in Y$. In this case X and Y are isomorphic, which is denoted by $X \simeq Y$. If the isomorphism is an isometry, that is $\|\mathcal{I} f\|_Y = \|f\|_X$ for all $f \in X$, the relation is denoted $X \cong Y$ and the spaces are said to be isometrically isomorphic.

The adjoint to an operator $T \in \mathcal{L}(X,Y)$ is given by $T^*\colon Y^* \to X^*$, $y^* \mapsto T^*y^*$, $y^* \in Y^*$, where $T^*y^* = y^*T\colon X \to \mathbb{C}$.

A Banach space X is *reflexive* if one of the following equivalent statements holds:

- The canonical embedding $\iota\colon X \to X^{**}, x \mapsto l_x$ is onto (an isomorphism), where $l_x(F) = F(x)$, $F \in X^*$.
- B_X is weakly compact.

General results

The first part of the following well-known result is also called the uniform boundedness principle and can be found in, for example, [19, p. 45].

Proposition 2.1.1 (Banach-Steinhaus theorem). *Let X be a Banach space and Y a normed space. If $\mathcal{F} \neq \emptyset$ is a family of bounded linear operators $X \to Y$ such that $\sup_{T\in\mathcal{F}}\|Tx\|_Y < \infty$ for each $x \in X$, then*

$$\sup_{T\in\mathcal{F}}\|T\|_{\mathcal{L}(X,Y)} < \infty.$$

Moreover, if there are $T, T_n \in \mathcal{L}(X,Y)$, $n \in \mathbb{Z}_{\geq 1}$ such that $\lim_{n\to\infty}\|T_nf - Tf\|_Y = 0$ for all $f \in X$, then $\lim_{n\to\infty}\sup_{f\in K}\|T_nf - Tf\|_Y = 0$ for every compact set $K \subset X$.

Proof of the second part: Let K be a compact subset of X and $\epsilon > 0$. Then there is a finite collection of open balls $B_X(f_j,\epsilon)$, $j \in \{1,\dots,J\}$ covering K. Let N be large enough to ensure

$$\sup_{n>N}\max_{j\in\{1,\dots,J\}}\left\|T_nf_j - Tf_j\right\|_Y < \epsilon.$$

For $n > N$, it holds for all $f \in K$ that

$$\begin{aligned}
\|(T_n - T)f\|_X &\leq \min_{j\in\{1,\dots,J\}}\left\|(T_n - T)(f - f_j + f_j)\right\|_X \\
&\leq \min_{j\in\{1,\dots,J\}}\left\|(T_n - T)(f - f_j)\right\|_X + \max_{j\in\{1,\dots,J\}}\left\|(T_n - T)f_j\right\|_X \\
&\leq \|(T_n - T)\|_{\mathcal{L}(X,Y)}\min_{j\in\{1,\dots,J\}}\left\|(f - f_j)\right\|_X + \epsilon \leq \|(T_n - T)\|_{\mathcal{L}(X,Y)}\epsilon + \epsilon.
\end{aligned}$$

From the first statement $\sup_n\|(T_n - T)\|_{\mathcal{L}(X,Y)} < \infty$ and the second statement follows. □

Another useful tool is the following corollary to Hahn-Banach extension theorem.

Proposition 2.1.2. *Let X be a normed space and $Y \subset X$ be a closed proper subspace. Then for each $x \in X \setminus Y$, there exists an $F \in X^*$ with $\|F\|_{X^*} = 1$, $F(x) = \inf_{y\in Y}\left\|x - y\right\|_X > 0$ and $F|_Y \equiv 0$.*

Choosing $Y = \{0\}$, one obtains a useful and well-known characterisation of the norm of a vector $x \in X$: $\|x\|_X = \sup_{l\in B_{X^*}}|l(x)|$. The section ends with some standard results, whose proofs are included for completeness.

Lemma 2.1.3. *Given two topologies τ_1, τ_2 and a set B, assume (B,τ_1) is a compact topological space and (B,τ_2) is a Hausdorff topological space. If $\tau_2 \subset \tau_1$, then $\tau_1 = \tau_2$, and as a consequence, the spaces (B,τ_1) and (B,τ_2) are identical.*

Proof. Since $\tau_2 \subset \tau_1$, the space (B,τ_1) is Hausdorff. Let $U \in \tau_1$. Then $C = B \setminus U$ is a closed subset of B, and hence compact. Indeed, for any open cover $\{U_j\}$ of C, the family $\{U_j \cup (B \setminus C)\}$ will be an open cover for B, and by compactness, there is a finite subcover $\{U_j \cup (B \setminus C)\}_{j\in J}$, which also covers $C \subset B$. Clearly, $\{U_j\}_{j\in J}$ is a finite subcover of C.

Furthermore, $\tau_2 \subset \tau_1$ yields that given a subset of B any cover consisting of elements in τ_2, will also be a cover with respect to τ_1, therefore, compactness is inherited to the space with a coarser topology. Since C is compact in (B,τ_2), it follows from the space being Hausdorff that C is closed in (B,τ_2), and hence $U \in \tau_2$. Indeed, given that C is compact in (B,τ_2), it can be separated from any $b \in B \setminus C$, by disjoint open sets $C \subset U_C \in \tau_2$ and $b \in U_b \in \tau_2$. Hence, $\bigcup_{b\in B\setminus C} U_b$ is a open subset disjoint from C, so that $B \setminus C = \bigcup_{b\in B\setminus C} U_b$ is open, which yields C is closed. □

Lemma 2.1.4. *In a normed space a set M is weakly bounded if and only if it is bounded in norm.*

Proof. Assume M is weakly bounded. Consider the family $\{T_m \colon X^* \to \mathbb{C},\ m \in M\}$, where $T_m(x^*) = x^*(m)$. For every $m \in M$, the boundedness of T_m is provided by an application of the Hahn-Banach theorem (Proposition 2.1.2). The assumption of M being weakly bounded yields that $\sup_{m\in M}|T_m(x^*)| < \infty$ for every $x^* \in X^*$ and by the uniform boundedness principle the following equality, obtained by Proposition 2.1.2, is finite:

$$\sup_{m\in M} \sup_{x^*\in B_{X^*}} |T_m(x^*)| = \sup_{m\in M} \|m\|_X.$$

The equality immediately gives the other direction of the statement, therefore, a set is bounded (in norm) if it is weakly bounded. □

Lemma 2.1.5. *Let X be a Banach space. Every operator $L \in \mathcal{K}(X)$ is completely continuous.*

Proof. Let $(x_n)_n$ be a sequence converging weakly to x_0. Lemma 2.1.4 yields that $(x_n)_n$ is bounded (in norm). Hence, the image $(L(x_n))_n$ is relatively compact and

$$K := \overline{\{L(x_n)\}_n \cup \{L(x_0)\}}^X$$

is compact. Since the weak topology w is the smallest topology granting $l \in X^*$ to be continuous $(X,w) \to (\mathbb{C},|\cdot|)$, it follows from Lemma 2.1.3 that the compact metric space $(K,\|\cdot\|_X)$ and the Hausdorff space (K,w) are identical from a topological standpoint. Since $L(x_n) \to L(x_0)$ in the weak topology, the convergence also holds with respect to the norm topology. □

The following result is well known (see for example [19, Propositions 1.11.8 and 1.12.9]).

Lemma 2.1.6. *An isomorphism $X \to Y$ preserves separability, reflexivity, dense sets and compact sets.*

2.2 Relevant Banach spaces

Let μ be a finite measure on a set $M \subset \mathbb{C}^n$, $p \in [1,\infty[$. The L^p-spaces are defined as

$$L^p(M,\mu) = \left\{ f : M \to \mathbb{C} \text{ is } \mu-\text{measurable} : \|f\|_{L^p(M,\mu)} := \left(\int_M |f|^p \, d\mu \right)^{\frac{1}{p}} < \infty \right\}.$$

The definition can naturally be extended to include $L^\infty(M,\mu)$ as the space

$$\begin{aligned} L^\infty(M,\mu) = \Bigg\{ f : M \to \mathbb{C} \text{ is } \mu-\text{measurable} : \|f\|_{L^\infty(M,\mu)} &:= \lim_{p\to\infty} \left(\int_M |f|^p \, d\mu \right)^{\frac{1}{p}} \\ &= \inf_{\substack{m\subset M \\ \mu(m)=\mu(M)}} \sup_{z\in m} |f(z)| < \infty \Bigg\}. \end{aligned}$$

The short form L^p will be used instead of $L^p(M,\mu)$ when there is no ambiguity.

Let $M = \mathbb{Z}_{\geq 1}$ and μ be the standard counting measure on M; the sequence space ℓ^p is defined as

$$\ell^p = \left\{ (t_n)_n \in \mathbb{C}^M : \|(t_n)_n\|_{\ell^p} := \left(\sum_{n=1}^{\infty} |t_n|^p \right)^{\frac{1}{p}} < \infty \right\}, \ p > 1;$$

$$\ell^\infty = \left\{ (t_n)_n \in \mathbb{C}^M : \|(t_n)_n\|_{\ell^\infty} := \sup_n |t_n| < \infty \right\}.$$

Due to some geometric properties involving c_0, this special closed subspace of ℓ^∞ should not be left out:

$$c_0 = \left\{ (t_n)_n \in \ell^\infty : \lim_{n\to\infty} t_n = 0 \right\}.$$

The vectors $(e_n)_n$ form a Schauder basis for c_0, where e_n is the sequence in which all elements are zero except for the n:th element, which is 1.

Banach spaces of analytic functions

The normalised Lebesgue measure on a measurable set $M \subset \mathbb{C}^n$ is denoted by m. For $p \in [1,\infty]$, the Hardy spaces are defined as

$$H^p(\mathbb{B}_n) = \left\{ f \in \mathrm{HOLO}(\mathbb{B}_n) : \|f\|_{H^p(\mathbb{B}_n)} := \sup_{0<r<1} \|f_r\|_{L^p(\mathbb{S}_n,m)} < \infty \right\}.$$

If $\beta \in \mathbb{R}$ and $v \in L^1(\mathbb{B}_n, m)$, the Bergman-Sobolev spaces are defined as

$$A^p_{v,\beta}(\mathbb{B}_n) = \left\{ f \in \mathrm{HOLO}(\mathbb{B}_n) : \left\| (I+R)^\beta f \right\|_{L^p(\mathbb{B}_n,A_v)} < \infty \right\},$$

where $dA_v(z) = v(z)dA(z) = v(z)c_n d\mathfrak{Re}z_1 \dots d\mathfrak{Re}z_n d\mathfrak{Im}z_1 \dots d\mathfrak{Im}z_n$ and c_n is a constant such that $\int_{\mathbb{B}_n} dA(z) = 1$. It is assumed that $v\colon \mathbb{B}_n \to]0,\infty[$ is radial, that is, $v(z) = v(|z|)$, $z \in$

$\mathbb{B}_n$; continuous. Without loss of generality, it is assumed that $\int_{\mathbb{B}_n} v(z)dA(z) = 1$. The standard weighted Bergman spaces, which is a closed subspace of $L_v^p := L^p(\mathbb{B}_n, dA_v)$ are obtained when $\beta = 0$. In the case of $n = 1$, the polar representation is useful, that is, $dA_v(re^{it}) = v(r)dA(re^{it}) = v(r)\frac{rdrd\theta}{\pi}$.

Some other natural types of spaces are the growth spaces

$$H_v^\infty(\mathbb{B}_n) = \left\{ f \in \mathrm{HOLO}(\mathbb{B}_n) : \|f\|_{H_v^\infty(\mathbb{B}_n)} := \sup_{z\in\mathbb{B}_n} v(z)|f(z)| \right\}$$

and Bloch-type spaces

$$B_v(\mathbb{B}_n) = \left\{ f \in \mathrm{HOLO}(\mathbb{B}_n) : \|f\|_{B_v} := \|\nabla f\|_{H_v^\infty(\mathbb{B}_n)} = |f(z)| + \sup_{z\in\mathbb{B}_n} v(z)|(\nabla f)(z)| \right\},$$

where the weight function v satisfies all of the properties mentioned above, except $\int_{\mathbb{B}_n} v(z)dA(z) = 1$, which is replaced by $\sup_{z\in\mathbb{B}_n}|v(z)| = 1$. In [17] and [20] they are referred to as weighted Banach spaces of analytic functions. This name has a less precise literal meaning, and hence, the name growth spaces is used in this thesis. A radial weight function implies that the space is rotationally symmetric, that is, if $f \in X$, then $wf \in X$ for all $w \in \mathbb{S}_n$, and hence, the norm of the evaluation functionals $\delta_z \colon f \mapsto f(z)$ satisfies $\|\delta_z\|_{X\to\mathbb{C}} = \left\|\delta_{|z|}\right\|_{X\to\mathbb{C}}$. The evaluation maps are, henceforth, assumed to be bounded.

The subscript α will denote the weight function $v(z) = v_\alpha(z) = M_{X,\alpha}(1-|z|^2)^\alpha$, where $M_{X,\alpha}$ is a normalisation constant dependent on both the type of space X, but also the parameter α.

Henceforth, for proper subspaces of $\mathrm{HOLO}(\mathbb{D})$, the notation $(\mathbb{B}_1)$ or $(\mathbb{D})$ will be dropped completely.

A necessary tool to achieve some of the results obtained in the articles is the small growth spaces. They, together with the small Bloch-type spaces are defined as

$$H_v^0 = \left\{ f \in H_v^\infty : \lim_{|z|\to 1} v(z)|f(z)| = 0 \right\} \text{ and } B_{0,v} = \left\{ f \in B_v : \lim_{|z|\to 1} v(z)|Df(z)| = 0 \right\},$$

and equipped with the norms from H_v^∞ and B_v respectively. The spaces H_v^0 and $B_{0,v}$ are closed subspaces of H_v^∞ and B_v respectively. Another useful subspace of $(H_v^\infty)^*$ is

$$^*H_v^\infty = \left\{ F \in (H_v^\infty)^* : F|_{B_{H_v^\infty}} \text{ is } \tau_0\text{-continuous} \right\}.$$

If $B_{H_v^0}$ is dense in $B_{H_v^\infty}$ with respect to τ_0, it has been proved in [2, Theorem 1.1] that $^*H_v^\infty$ is isometrically isomorphic to $(H_v^0)^*$ with the isomorphism given by the restriction of the restriction map $R \colon (H_v^\infty)^* \to (H_v^0)^*$. This is, hereafter, assumed to hold for the considered growth spaces. In the same article, it is mentioned that as a consequence of a result by Ng, [22, Theorem 1], the map $\iota_H \colon f \mapsto \delta_f,\ f \in H_v^\infty,\ \delta_f(F) = F(f),\ F \in {}^*H_v^\infty$ is an isometric isomorphism from H_v^∞ onto $({}^*H_v^\infty)^*$. For this, it is necessary that $(B_{H_v^\infty}, \tau_0)$ is compact, which is proved by elementary means below (see Lemma 2.2.1). As mentioned in Lemma 3.1.13 and [2, Corollary 1.2], one can conclude that $(H_v^0)^{**} \cong H_v^\infty$. Similarly, for $\alpha > 0$, the space $(B_{0,\alpha})^*$ is isomorphic to A^1 and $(A^1)^*$ is isomorphic to B_α (see [26, p. 1150]).

This thesis will also include some results on the space $BMOA$ (the space of analytic functions of bounded mean oscillation), defined as

$$BMOA = \left\{ f \in \mathrm{HOLO}(\mathbb{D}) : \|f\|_{BMOA} := |f(0)| + \sup_{a\in\mathbb{D}} \|f \circ \sigma_a - f(a)\|_{H^2} < \infty \right\},$$

where σ_a, $a \in \mathbb{D}$ is the automorphism of the disc, $\mathbb{D} \to \mathbb{D}$: $z \mapsto \frac{a-z}{1-\bar{a}z}$. The evaluation functionals are bounded and the following estimate holds for $f \in BMOA$ (see [9, p. 95]):

$$|f(z)| \leq |f(0)| + \frac{1}{2}\|f\|_{BMOA} \log \frac{1+|z|}{1-|z|}.$$

The counterpart to the small spaces for $BMOA$ is the space $VMOA$ (the space of analytic functions of vanishing mean oscillation), defined as

$$VMOA = \left\{ f \in \mathrm{HOLO}(\mathbb{D}) : \lim_{|a|\to 1} \|f \circ \sigma_a - f(a)\|_{H^2} = 0 \right\},$$

equipped with $\|\cdot\|_{BMOA}$. It also holds, for the closed subspace $VMOA$, that its dual is isomorphic to H^1, and $(H^1)^*$ is isomorphic to $BMOA$ (see [9, Theorem 7.3 and Theorem 7.1]). In particular, $VMOA^{**} \simeq BMOA$.

As a consequence of John-Nirenberg's lemma, $f \mapsto |f(0)| + \|f \circ \sigma_a - f(a)\|_{H^p}$ is for $p \in [1, \infty[$ an equivalent norm to $\|\cdot\|_{BMOA}$ on $BMOA$.

Next, some well-known results, which are useful considering Banach spaces of analytic functions, are presented:

Lemma 2.2.1. *The topological space $(B_{H_v^\infty}, \tau_0)$ is compact.*

Proof. By Montel's theorem, the bounded set $B_{H_v^\infty}$ is relatively compact in $\mathrm{HOLO}(\mathbb{D})$ with respect to τ_0. Let $f_n \in B_{H_v^\infty}$, $n \in \mathbb{Z}_{\geq 1}$ and $f \in \mathrm{HOLO}(\mathbb{D})$ be such that $f_n \to f$ with respect to τ_0. For every $0 < R < 1$, it holds that

$$\sup_{z\in R\mathbb{D}} v(z)|f(z)| \leq \sup_{z\in R\mathbb{D}} v(z)|f(z) - f_n(z)| + 1, \; n \in \mathbb{Z}_{\geq 1}.$$

Let $n \to \infty$ to conclude that $\sup_{z\in R\mathbb{D}} v(z)|f(z)| \leq 1$ for all $0 < R < 1$, which yields the statement.

□

Lemma 2.2.2. *Let $X \subset \mathrm{HOLO}(\mathbb{D})$ be a Banach space. The following statements are equivalent:*

(1) $\|\cdot\|_X$ is finer than the compact open topology, τ_0.

(2) $\|\cdot\|_X$ is finer than the topology of point-wise convergence, τ_π.

(3) $\delta_z \in X^*$, $z \in \mathbb{D}$ (the point evaluations are bounded).

(4) For every compact set $K \subset \mathbb{D}$ the evaluation maps $\{\delta_z : z \in K\}$ are uniformly bounded.

Proof. First, a topology τ_1 is finer than a topology τ_2 on X if and only if $\mathrm{Id} : (X, \tau_1) \to (X, \tau_2)$ is continuous.

(1) $\Rightarrow$ (2) is true, because the open sets in τ_π are included in τ_0.

(2) $\Rightarrow$ (3): Recall that δ_z is bounded if and only if its kernel is closed. Take $f_n \in \operatorname{Ker} \delta_z$ with $f_n \to f \in X$, with respect to the norm, as $n \to \infty$. It follows from the assumption that $\delta_z(f) = f(z) = \lim_n f_n(z) = 0$, hence, $\operatorname{Ker} \delta_z$ is closed.

(3) $\Rightarrow$ (4): For a compact set $K \subset \mathbb{D}$, it holds that $\sup_{z \in K} |\delta_z(f)| = \sup_{z \in K} |f(z)| < \infty$ for all $f \in X$. Since $\delta_z \in X^*$, $z \in K$, it follows from the Banach-Steinhaus theorem (uniform boundedness principle) that $\sup_{z \in K} \|\delta_z\|_{X^*} < \infty$.

(4) $\Rightarrow$ (1): Take an arbitrary compact set $K \subset \mathbb{D}$ and a sequence $f_n \in X$ such that $\|f_n\|_X \to 0$. It follows from (4) that

$$\sup_{z \in K} |f_n(z)| = \sup_{z \in K} |\delta_z(f_n)| \leq \sup_{z \in K} \|\delta_z\|_{X^*} \|f_n\|_X.$$

Therefore, $\lim_{n \to \infty} \sup_{z \in K} |f_n(z)| = 0$, proving the statement, since τ_0 is metrizable. □

Lemma 2.2.3. *Let $X \subset \mathrm{HOLO}(\mathbb{D})$ be a Banach space with $\delta_z \in X^*$, $z \in \mathbb{D}$. Then the topologies $\tau_\pi(B_X)$ and $\tau_0(B_X)$ coincide and are metrizable. If X is also reflexive, then $w(B_X)$ coincide with $\tau_0(B_X)$ and is metrizable. Furthermore, the space X is separable.*

Proof. According to Lemma 2.2.2, it holds that $\sup_{f \in B_X} \sup_{z \in K} |f(z)| < \infty$ for every compact set $K \subset \mathbb{D}$, so by Montel's theorem $B := \overline{B_X}^{\tau_0} \subset \mathrm{HOLO}(\mathbb{D})$ is compact. Since $\tau_\pi(B) \subset \tau_0(B)$, it follows from Lemma 2.1.3 that $\tau_\pi(B) = \tau_0(B)$ and hence

$$\tau_\pi(B_X) = \{U \cap B_X : U \in \tau_\pi(B)\} = \{U \cap B_X : U \in \tau_0(B)\} = \tau_0(B_X).$$

Since $\tau_0(B_X)$ is metrizable, so is $\tau_\pi(B_X)$.

If X is reflexive, $(B_X, w(B_X)$ is compact and Lemma 2.1.3 yields that $w(B_X) = \tau_\pi(B_X)$ and most of the statements follow from the first part. Since $(B_X, w(B_X))$ is metrizable, so is $(\iota B_X, w^*(\iota(B_X))) = (B_{X^{**}}, w^*(B_{X^{**}}))$. By [19, Theorem 2.6.23] X^* is separable and hence X is separable. □

2.3 Weighted composition operators

Weighted composition operators appear in some form in all articles included in this thesis. Given two Banach spaces of analytic functions $X, Y \subset \mathrm{HOLO}(\mathbb{B}_n)$, $n \in \mathbb{Z}_{\geq 1}$ containing the constant functions (written $\mathbb{C} \subset X, Y$), a weighted composition operator W is an operator that transforms $f \in X$ into an analytic function $\psi f \circ \varphi \in Y$. Some natural demands are that φ is an analytic selfmap of $\mathbb{B}_n$ and since $1 \in X$, it follows that $\psi \in Y$ for W to be well-defined. It is also sensible to write $W_{\psi,\varphi}$ instead of W.

If φ is the identity, the map $W_{\psi,\varphi} \colon f \mapsto \psi f$ is a multiplication operator, also denoted M_ψ, and hence, one can also write $W_{\psi,\varphi} = M_\psi C_\varphi$ as a composition of a multiplication operator $M_\psi \colon f \mapsto \psi f$ and a composition operator $C_\varphi \colon f \mapsto f \circ \varphi$. In general, for

$W_{\psi,\varphi}\colon X \to Y$ to be bounded, it is sufficient but not necessary that both $M_\psi\colon X \to Y$ and $C_\psi\colon X \to X$ are bounded. Indeed, $\|W_{\psi,\varphi}\|_{\mathcal{L}(X,Y)} \le \|M_\psi\|_{\mathcal{L}(X,Y)} \|C_\varphi\|_{\mathcal{L}(X)}$ proves sufficiency and if $X = Y = L^2(\mathbb{D}, dA) \cap \mathrm{HOLO}(\mathbb{D})$, the standard non-weighted Bergman Hilbert space, the operator $W_{\psi,\varphi}$ is bounded for the choice $\psi\colon z \mapsto (1-z)^{-\frac{2}{3}}$ and $\varphi\colon z \mapsto \frac{1}{2}z$, although $M_\psi f \notin Y$ for all $f \in X$.

Chapter 3

Summary of the results

3.1 The essential norm of some integral operators acting in a bounded manner on weighted Bergman spaces and growth spaces

Integral operators on weighted Bergman spaces

Let $f \in \mathrm{HOLO}(\mathbb{D})\colon z \mapsto \sum_{k=0}^{\infty} a_k z^k$. The Hilbert matrix operator $\mathcal{H}$ is defined on a subset of $\mathrm{HOLO}(\mathbb{D})$ as follows:

$$\mathcal{H}(f)(z) = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \frac{a_k}{k+n+1} z^n, \quad z \in \mathbb{D}.$$

For example, the function $z \mapsto \sum_{n=0}^{\infty} z^n$, $z \in \mathbb{D}$ does not have a Hilbert matrix operator transform. Following the proof of Diamantopoulos [5], the Hilbert matrix operator transform can be written as $\mathcal{H}(f)(z) = \int_0^1 \frac{f(x)}{1-xz}\,dx$ on a Banach space on which the polynomials are dense and $\int_0^1 \|\delta_t\| dt < \infty$. Given such a space, this representation can indeed be used on any function belonging to the space and it is, hence, valid on all linear subspaces.

The Hilbert matrix operator is a prime example of an integral operator of the form $I_K\colon f \mapsto \int_0^1 K(\cdot,x) f(x)\,dx$, where

- $K\colon \mathbb{D}\times\mathbb{D} \to \mathbb{C}$ is analytic in both arguments;
- $\lim_{z\to 1} K(z,t) \in [0,\infty[$ for all $t \in]0,1[$ and
- $\sup_{w\in\mathbb{D}} \sup_{z\in\mathbb{D}\setminus B(1,\epsilon)} |K(z,w)| < \infty$ for all $\epsilon > 0$.

These will be referred to as the three kernel conditions.

The following result is a special case of [17, Cor 5.3].

Theorem 3.1.1. *If $I_K \in \mathcal{L}(A^p_\alpha)$, $p-2>\alpha\geq 0$ has a kernel that satisfies the conditions above, and the two technical conditions (3.1.1) and (3.1.2), then*

$$\|I_K\|_{e,A^p_\alpha\to A^p_\alpha} = \int_0^1 \lim_{z\to 1} x_z'(t)K(z,x_z(t))\frac{t^{\frac{2+\alpha}{p}}}{(1-t)^{\frac{2+\alpha}{p}}}\,dt, \quad p-2>\alpha\geq 0,$$

where $x_z(t)=\frac{t}{1-(1-t)z}$.

A crucial part of the proof of the result is that I_K can be represented as

$$I_K(f)(z) = \int_0^1 x_z'(t)K(z,x_z(t))f(\phi_t(z))\,dt,$$

where $\phi_t(z)=x_z(t)$, $t\in]0,1[$, $z\in\mathbb{D}$.

The operator I_K can in other words be seen as a mean of weighted composition operators, namely $f\mapsto\int_0^1 T_tC_{\phi_t}(f)\,dt$, where $T_t\colon z\mapsto x_z'(t)K(z,x_z(t))$ is analytic in $\mathbb{D}$ for all $t\in]0,1[$. This representation for the Hilbert matrix operator was introduced by Diamantopoulos and Siskakis in [6], in which case $T_t(z)=\frac{\phi_t(z)}{t}$. This yields the following corollary to Theorem 3.1.1:

Corollary 3.1.2. *For $p-2>\alpha\geq 0$ the essential norm of the Hilbert matrix operator acting on A^p_α is given by*

$$\|\mathcal{H}\|_{e,A^p_\alpha\to A^p_\alpha} = \int_0^1 \frac{t^{\frac{2+\alpha}{p}-1}}{(1-t)^{\frac{2+\alpha}{p}}}\,dt = \frac{\pi}{\sin\left(\frac{2+\alpha}{p}\pi\right)}.$$

For Theorem 3.1.1 to hold true, the possible pole for $K(\cdot,\cdot)$ at $(1,1)$ cannot be of a high order. On the weighted Bergman spaces, a sufficient demand, in addition to the three kernel conditions, are the following technical assumptions: there exists $0<\epsilon<\frac{1}{p}$ such that

$$\int_0^1 \sup_{c\in]\frac{2}{p}-\epsilon,\frac{2}{p}[}\ \sup_{z\in B(1,\epsilon)\cap\mathbb{D}}\left|\left(\frac{1}{1-t}-z\right)^{c+\frac{\alpha}{p}} T_t(z)\right| < \infty \tag{3.1.1}$$

and for $0<r<1$,

$$\sup_{\theta\in]0,2\pi[}\left|T_t(re^{i\theta})\right| \lesssim |T_t(r)|. \tag{3.1.2}$$

Recall condition (CUBA) from [17]:

$$\int_0^1 \sup_{z\in\phi_t^{-1}(D_{>R_0},t)} \frac{|T_t(z)|v(z)^{\frac{1}{p}}}{v(\phi_t(z))^{\frac{1}{p}}}\frac{t^{\frac{2}{p}}\,dt}{(1-t)^{\frac{2}{p}}} < \infty. \tag{CUBA}$$

The following lemma could be generalised to other weights than v_α and the statement is a better version of the nonoptimal one in [17, Remark 5.5]:

Lemma 3.1.3. *An integral operator $I_K\colon A^p_\alpha\to A^p_\alpha$, $p-2>\alpha\geq 0$, which satisfies the three kernel conditions, (3.1.1) and (3.1.2) also satisfies condition (CUBA).*

Proof. First, although for a fixed $0 < R_0 < 1$, it holds that

$$\bigcup_{t\in]\delta,1[} \phi_t^{-1}(D_{>R_0,t}) = \mathbb{D} \quad \text{for all } \delta \in]0,1[,$$

it is also true that for every $\epsilon > 0$, using $U_\epsilon := \mathbb{D} \setminus B(1,\epsilon)$, it holds that

$$\begin{aligned}
\sup_{z\in U_\epsilon} \frac{|T_t(z)| v_\alpha(z)^{\frac{1}{p}}}{v_\alpha(\phi_t(z))^{\frac{1}{p}}} &\leq \sup_{z\in U_\epsilon} \left|K(z,\phi_t(z))\right| \frac{1}{(1-(1-t)(1-\epsilon))^2} \frac{2^{\frac{\alpha}{p}}(1-|z|)^{\frac{\alpha}{p}}}{(1-\phi_t(|z|))^{\frac{\alpha}{p}}} \\
&\leq 2^{\frac{\alpha}{p}} \frac{\sup_{w\in\mathbb{D}} \sup_{z\in U_\epsilon} |K(z,w)|}{\epsilon^2} \left|\frac{2-t}{1-t}\right|^{\frac{\alpha}{p}},
\end{aligned}$$

where $(1-|z|)^\alpha \leq v_\alpha(z) \leq 2^\alpha(1-|z|)^\alpha$ and $\phi_t(z) \leq \phi_t(|z|)$, $z \in \mathbb{D}$, $t \in]0,1[$ have been used. Since $t \mapsto (1-t)^{-\frac{2+\alpha}{p}}$ is integrable on $]0,1[$, there exists, for every $\epsilon > 0$, a constant $C_\epsilon > 0$ such that

$$\int_0^1 \sup_{z\in\phi_t^{-1}(D_{>R_0,t})} \frac{|T_t(z)| v_\alpha(z)^{\frac{1}{p}}}{v_\alpha(\phi_t(z))^{\frac{1}{p}}} \frac{t^{\frac{2}{p}}\,dt}{(1-t)^{\frac{2}{p}}} \leq C_\epsilon + \int_0^1 \sup_{z\in B(1,\epsilon)\cap\mathbb{D}} \frac{|T_t(z)| v_\alpha(z)^{\frac{1}{p}}}{v_\alpha(\phi_t(z))^{\frac{1}{p}}} \frac{t^{\frac{2}{p}}\,dt}{(1-t)^{\frac{2}{p}}}.$$

By applying $v_\alpha(z) \asymp (1-|z|)^\alpha$ and (3.1.2), it follows that

$$\begin{aligned}
\int_0^1 \sup_{z\in B(1,\epsilon)\cap\mathbb{D}} \frac{|T_t(z)| v_\alpha(z)^{\frac{1}{p}}}{v_\alpha(\phi_t(z))^{\frac{1}{p}}} \frac{t^{\frac{2}{p}}\,dt}{(1-t)^{\frac{2}{p}}} &\lesssim \int_0^1 \sup_{z\in B(1,\epsilon)\cap\mathbb{D}} \frac{|T_t(|z|)|(1-|z|)^{\frac{\alpha}{p}}}{(1-\phi_t(|z|))^{\frac{\alpha}{p}}} \left|\frac{1}{1-t} - |z|\right|^{\frac{2}{p}} dt \\
&\leq \int_0^1 \sup_{z\in B(1,\epsilon)\cap\mathbb{D}} |T_t(z)| \left|\frac{1}{1-t} - z\right|^{\frac{2+\alpha}{p}} dt,
\end{aligned}$$

which proves the statement.

□

The upper bound

The upper bound is obtained from the estimate

$$\begin{aligned}
\|I_K\|_{e,A_\alpha^p\to A_\alpha^p} \leq{}& \sup_{f\in B_{A_\alpha^p}} \sup_{|z|\leq R} |(I-L)(f)(z)| \int_0^1 \left\| T_t \chi_{\phi_t^{-1}(D_{\leq R,t})} \right\|_{L_\alpha^p} dt \\
&+ R^{-\frac{4}{p}} \|I-L\|_{A_\alpha^p\to A_\alpha^p} \int_0^1 \frac{t^{\frac{2}{p}}}{(1-t)^{\frac{2}{p}}} \sup_{z\in\phi_t^{-1}(D_{>R,t})} \frac{|T_t(z)| v_\alpha(z)^{\frac{1}{p}}}{v_\alpha(\phi_t(z))^{\frac{1}{p}}}\, dt,
\end{aligned}$$

where $L \in \mathcal{L}(A_\alpha^p)$ is a compact operator, $0 < R < 1$, $D_{\leq R,t} = R\mathbb{D} \cap \phi_t(\mathbb{D})$, and $D_{>R,t} = \mathbb{D} \setminus D_{\leq R,t}$. Notice that for $t \in]0,1[$ the function $\phi_t \colon \mathbb{D} \to \mathbb{D}$ is a bounded Möbius map, and hence, it is univalent and maps circles to circles. It follows that $\phi_t^{-1}(D_{>R,t}) = \mathbb{D} \setminus \phi_t^{-1}(D_{\leq R,t})$ and $\phi_t^{-1}(D_{\leq R,t}) = \phi_t^{-1}(R\mathbb{D}) \cap \mathbb{D}$ (see Figure 3.1 for a visualisation). For a fixed t, it holds that

$$\lim_{R\to 1} \bigcup_{t'\in]0,t[} \overline{\phi_{t'}^{-1}(D_{>R,t'})} = \{1\},$$

which has a clear connection with the expression for the essential norm given in Theorem 3.1.1. However, it is also worth pointing out that for a fixed $0 < R < 1$, it holds that

$$\lim_{t\to 1} \bigcup_{t'\in]0,t[} \phi_{t'}^{-1}(D_{>R,t'}) = \mathbb{D}.$$

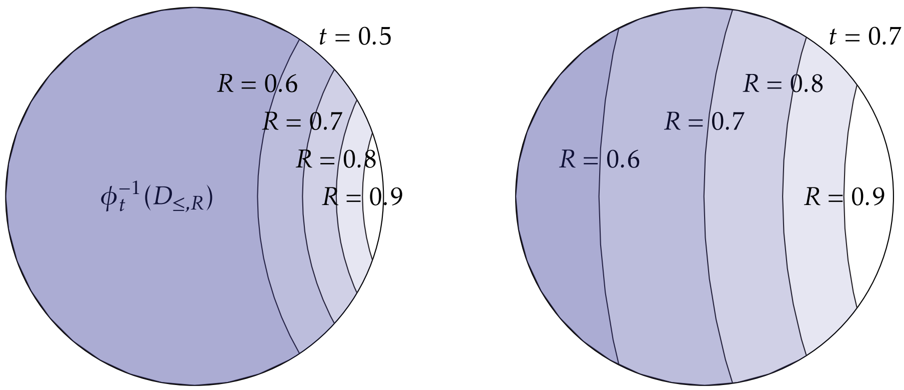


Figure 3.1: Two pictures of the region $\phi_t^{-1}(D_{\leq,R})$.

To continue, for a complex (or real) Banach space X, the concept of M-ideals, the metric compact approximation property, property (m_p) and the notion of a X containing a copy of another space Z are introduced.

A projection $P : X \to X$ is called an *L-projection* if $\|x\|_X = \|Px\|_X + \|x - Px\|_X$, for all $x \in X$.

A closed subspace $Y \subset X$ is called an *M-ideal* if $Y^\perp$ is the range of an L-projection.

The Banach space X is said to contain a copy of a space Z if there is a subspace $Y \subset X$ such that Y is isomorphic to Z.

The Banach space X has the *metric compact approximation property* if for every compact set $K \subset X$ and every $\epsilon > 0$, there is a compact operator $L \in \mathcal{L}(X)$ with $\|L\|_{\mathcal{L}(X)} \leq 1$ such that $\|(I - L)x\|_X \leq \epsilon$ for all $x \in K$.

A separable Banach space X satisfies property (m_p) if for every weakly null sequence $(x_n)_n \subset X$ and $x \in X$ it holds that

$$\limsup_{n\to\infty} \|x_n + x\|_X = \limsup_{n\to\infty} (\|x_n\|_X^p + \|x\|_X^p)^{\frac{1}{p}}.$$

The following lemma is inspired by [3].

Lemma 3.1.4. *Let $p \geq 1$. For every $\epsilon > 0$ there is a $C_\epsilon > 0$ such that*

$$\Big||a+b|^p - |a|^p\Big| \leq \epsilon |a|^p + C_\epsilon |b|^p \quad \textit{for all } a, b \in \mathbb{C}. \tag{3.1.3}$$

Proof. Observe that if $b = 0$ the statement is always true. From the case $a = 0$ it follows that $C_\epsilon \geq 1$. Excluding the case $a = 0$ or $b = 0$, and dividing (3.1.3) by $|a|^p$, the lemma can be restated as

$$\forall \epsilon > 0\ \exists C_\epsilon > 0 : \Big| |1+z|^p - 1 \Big| \le \epsilon + C_\epsilon |z|^p \quad \forall z \in \mathbb{C} \setminus \{0\}. \tag{3.1.4}$$

Let $\epsilon > 0$. A sufficient lower bound on C_ϵ is given by $2^p + 1$ when $|z| \ge 1$, because

$$\Big| |1+z|^p - 1 \Big| \le (1+|z|)^p + 1 \le |z|^p \left(\left(\frac{1}{|z|} + 1 \right)^p + \left| \frac{1}{z} \right|^p \right) \le (2^p+1)|z|^p.$$

Henceforth, assume that $C_\epsilon \ge (2^p + 1)$. If $|z| \le 1$, it follows that

$$\begin{aligned} \Big| |1+z|^p - 1 \Big| &= \max\{|1+z|^p - 1, 1 - |1+z|^p\} \le \max\{(1+|z|)^p - 1, 1 - (1-|z|)^p\} \\ &\le \sum_{k=1}^{\infty} \binom{p}{k} |z|^k = |z| \sum_{k=1}^{\infty} \binom{p}{k} |z|^{k-1} \le |z| \sum_{k=0}^{\infty} \binom{p}{k} = 2^p |z|. \end{aligned}$$

If $2^p|z| \le \epsilon$, the inequality (3.1.4) holds for any C_ϵ. Else, both $2^p|z| \le 2^p$ and

$$C_\epsilon |z|^p \ge C_\epsilon \left(\frac{\epsilon}{2^p} \right)^p,$$

are true, and hence, (3.1.4) holds if $C_\epsilon \left(\frac{\epsilon}{2^p} \right)^p \ge 2^p$, equivalently,

$$C_\epsilon \ge \frac{2^{p(p+1)}}{\epsilon^p}.$$

This proves (3.1.3) with $C_\epsilon = \max\{2^p + 1, \frac{2^{p(p+1)}}{\epsilon^p}\}$. □

To give more details to Lemma 3.2 in [17], in which case a more general weight v is used, the following results will be stated using a radial, continuous weight function v, which belongs to $L^1(\mathbb{D}, dA)$ and satisfies $v(r_2) \le v(r_1)$, $0 < r_1 < r_2 < 1$ and $\lim_{r \to 1} v(r) = 0$. Moreover, the proof of [17, Lemma 3.2] is not complete, since it is not evident that $\{I - L_n : n \in \mathbb{Z}_{\ge 1}\}$ is τ_0-τ_0 equicontinuous. A complete proof will be presented culminating in Lemma 3.1.11, which can be compared to [17, Lemma 3.2] considering the weighted Bergman spaces. To verify the statement, the convex combination L_n used consists of *dilation operators*, which are defined as

$$\mathcal{D}_r \colon \mathrm{HOLO}(\mathbb{D}) \to \mathrm{HOLO}(\mathbb{D}) \colon f \mapsto f_r,\ r \in [0,1[, \text{ where } f_r(z) = f(rz), \quad z \in \mathbb{D}.$$

In this thesis the dilation operator is considered as an operator $A_v^p \to A_v^p$.

The following lemma follows from Lemma 2.2.3 and the fact that A_v^p, $p > 1$ is reflexive.

Lemma 3.1.5. *Let $p > 1$. The space A_v^p is separable and the topologies $\tau := w(B_{A_v^p}) = \tau_0(B_{A_v^p}) = \tau_\pi(B_{A_v^p})$ renders $(B_{A_v^p}, \tau)$ a compact metrizable topological space.*

The following result is well known.

Lemma 3.1.6. *For $f \in A_v^p$*

$$\lim_{r \to 1} \|(I - \mathcal{D}_r) f\|_{A_v^p} = 0.$$

Proof. Let $f \in A_v^p$. First, by the maximum modulus principle

$$\int_0^{2\pi} \left|\mathcal{D}_r(f)(\rho e^{it})\right|^p dt = \int_0^{2\pi} \left|f(r\rho e^{it})\right|^p dt \leq \int_0^{2\pi} \left|f(\rho e^{it})\right|^p dt$$

holds for all $R, r, \rho \in]0,1[$, so that

$$\int_R^1 \int_0^{2\pi} \left|\mathcal{D}_r(f)(\rho e^{it})\right|^p dt\ v(\rho)\, d\rho \leq \int_R^1 \int_0^{2\pi} \left|f(\rho e^{it})\right|^p dt\ v(\rho)\, d\rho.$$

Take $\epsilon > 0$ and choose R close enough to 1 in order to ensure that

$$\|(I - \mathcal{D}_r)f\|^p_{L_v^p(\mathbb{D}\setminus R\mathbb{D})} < \epsilon \quad \text{for all } r \in]0,1[.$$

Considering the Taylor expansion of f, it follows that $\sup_{z\in R\mathbb{D}}|f(z) - f(rz)|$ tends to zero as $r \to 1$, and hence,

$$\lim_{r\to 1}\|(I - \mathcal{D}_r)f\|_{A_v^p} = 0.$$

□

A large part of the proof of property (m_p) is based on the proof of [3, Theorem 2].

Lemma 3.1.7. *Let $p > 1$. The weighted Bergman space A_v^p has the metric compact approximation property, satisfies (m_p) and does not contain a copy of ℓ^1.*

Proof of the metric compact approximation property:
The dilation operators $\mathcal{D}_r \colon A_v^p \to A_v^p$ are compact and by subharmonicity their norm is less than or equal to 1. In the weighted Bergman spaces they tend strongly to the identity as $r \to 1$, and by the Banach-Steinhaus theorem the convergence is uniform in any compact set $K \subset A_v^p$. Some details of the statements are given in Lemma 3.1.6. □

Proof of (m_p):
Fix $\epsilon > 0$ and let $f, f_n \in A_v^p$, $n \in \mathbb{Z}_{\geq 1}$ with $f_n \to 0$ weakly as $n \to \infty$. Define

$$W_{\epsilon,n}(z) := \max\{\left||f(z) + f_n(z)|^p - |f_n(z)|^p - |f(z)|^p\right| - \epsilon|f_n(z)|^p, 0\}, \quad z \in \mathbb{D}.$$

Since $\delta_z \in (A_v^p)^*$, it follows that $\lim_{n\to\infty} W_{\epsilon,n}(z) = 0$ for all $z \in \mathbb{D}$. Furthermore, by Lemma 3.1.4

$$\left||f + f_n|^p - |f_n|^p - |f|^p\right| \leq \left||f + f_n|^p - |f_n|^p\right| + |f|^p \leq \epsilon|f_n|^p + C_\epsilon|f|^p + |f|^p$$

on $\mathbb{D}$. Therefore, $W_{\epsilon,n} \leq C_\epsilon|f|^p + |f|^p$ and by the dominated convergence theorem,

$$\lim_{n\to\infty}\int_{\mathbb{D}} W_{\epsilon,n}\, dA_v = 0.$$

Moreover, $\left||f + f_n|^p - |f_n|^p - |f|^p\right| \leq W_{\epsilon,n} + \epsilon|f_n|^p$, and since $f_n \overset{n\to\infty}{\to} 0$ weakly, $\sup_n\|f_n\|_{A_v^p}$ is a finite constant, according to Lemma 2.1.4, denoted by C. This yields

$$\int_{\mathbb{D}}\left||f + f_n|^p - |f_n|^p - |f|^p\right| dA_v \leq \int_{\mathbb{D}} W_{\epsilon,n}\, dA_v + \epsilon\|f_n\|^p_{A_v^p} \leq \int_{\mathbb{D}} W_{\epsilon,n}\, dA_v + \epsilon C,$$

so that

$$\limsup_{n\to\infty}\int_{\mathbb{D}}\Big||f+f_n|^p-|f_n|^p-|f|^p\Big|\,dA_v\le C\epsilon.$$

Let $\epsilon\to 0$ to obtain

$$\lim_{n\to\infty}\int_{\mathbb{D}}\Big||f+f_n|^p-|f_n|^p-|f|^p\Big|\,dA_v=0.$$

It follows that

$$\limsup_{n\to\infty}\int_{\mathbb{D}}|f+f_n|^p\,dA_v=\limsup_{n\to\infty}\int_{\mathbb{D}}|f_n|^p+|f|^p\,dA_v.$$

□

Proof of A_v^p not containing a copy of ℓ^1:
Since reflexivity is passed on to subspaces and images of isomorphisms (Lemma 2.1.6), the weighted Bergman spaces ($p\in]1,\infty[$) cannot contain a copy of the non-reflexive space ℓ^1. □

Since A_v^p, $p\in]1,\infty[$ is separable according to Lemma 3.1.5, an application of [11, Corollary 3.6] together with Lemma 3.1.7 yields that the subspace $\mathcal{K}(A_v^p)$ is an M-ideal in $\mathcal{L}(A_v^p)$.

The following two results are stated in, for example, [7, Theorem 4.10.1] and [25, Theorem III.2.4] respectively:

Proposition 3.1.8 (Riesz representation theorem)**.** *Let T be a compact topological space and $L\in C(T,\mathbb{C})^*$ be linear. Then there exists a uniquely determined complex Radon measure μ on T such that*

$$L(x)=\int_T x(t)\,d\mu(t),\quad x\in C(T,\mathbb{C}).$$

Proposition 3.1.9 (Hahn-Banach separation theorem)**.** *If X is a normed space and V_1,V_2 are disjoint convex sets of which V_1 is open. Then there is a linear functional $\phi\in X^*$ such that*

$$\mathfrak{Re}\,\phi(v_1)<\mathfrak{Re}\,\phi(v_2)\quad\textit{for all } v_1\in V_1,\ v_2\in V_2.$$

In the spirit of Kalton ([10, p. 151-152]), the following two lemmas are proved:

Lemma 3.1.10. *When $p>1$, every operator $\kappa\in\mathcal{K}(A_v^p)^*$ can be extended to $\nu_*\in\mathcal{L}(A_v^p)^*$ in a norm-preserving way such that $\nu_*(S)=\lim_{r\to 1}\nu_*(S\mathcal{D}_r)$, $S\in\mathcal{L}(A_v^p)$.*

Proof. The topological spaces $(B_{(A_v^p)^*},w^*)$ and $(B_{A_v^p},w)$ are compact according to Alaoglu's theorem and reflexivity, and (by Tychonoff's theorem) the product space $Q:=B_{(A_v^p)^*}\times B_{A_v^p}$ equipped with the product topology is also compact. Let $\kappa\in\mathcal{K}(A_v^p)^*$ and $V\colon\mathcal{K}(A_v^p)\to C(Q,\mathbb{C})$, where $V(S)\colon(u^*,g)\mapsto u^*(Sy)$, $S\in\mathcal{K}(A_v^p)$. By Proposition 2.1.2,

$$\|V(S)\|_{C(Q,\mathbb{C})}=\sup_{(u^*,g)\in Q}|V(S)(u^*,g)|=\sup_{(u^*,g)\in Q}|u^*(Sg)|=\|S\|_{A_v^p\to A_v^p},$$

that is, V is a linear isometry onto $V(\mathcal{K}(A_v^p)) \subset C(Q,\mathbb{C})$, which is therefore closed. This yields that $V(\mathcal{K}(A_v^p))$ is a Banach space. As a consequence, the bounded inverse theorem ensures that V has a bounded inverse. It follows that $\hat{\kappa} := \kappa \circ V^{-1} \in V(\mathcal{K}(A_v^p))^*$. Since Q is compact, the Riesz representation theorem (Proposition 3.1.8) can be applied to obtain

$$\hat{\kappa}(\hat{S}) = \int_Q \hat{S}(u^*,g)\, d\mu(u^*,g), \quad \hat{S} \in V(\mathcal{K}(A_v^p)) \subset C(Q,\mathbb{C}),$$

where μ is a uniquely determined complex Radon measure on Q. This means that

$$\kappa(S) = \int_Q V(S)(u^*,g)\, d\mu(u^*,g) = \int_Q u^*(Sg)\, d\mu(u^*,g), \quad S \in \mathcal{K}(A_v^p).$$

Let us define $\nu_* \colon \mathcal{L}(A_v^p) \to \mathbb{C}$ by

$$\nu_*(S) = \int_Q u^*(Sg)\, d\mu(u^*,g).$$

It follows from Proposition 2.1.2 that

$$\int_Q |u^*(Sg)| \left|d\mu(u^*,g)\right| \le \|S\|_{\mathcal{L}(A_v^p)} \left|\mu\right|(Q) < \infty, \quad S \in \mathcal{L}(A_v^p),$$

which yields $\nu_* \in \mathcal{L}(A_v^p)^*$. Since $\lim_{r\to 1} \|(I - \mathcal{D}_r)f\|_{A_v^p} = 0$ for all $f \in A_v^p$ (Lemma 3.1.6), it follows that for all $S \in \mathcal{L}(A_v^p)$,

$$\begin{aligned} |\nu_*(S) - \nu_*(S\mathcal{D}_r)| &= \left| \int_Q u^*(Sg)\, d\mu(u^*,g) - \int_Q u^*(S\mathcal{D}_r g)\, d\mu(u^*,g) \right| \\ &\le \int_Q |u^*(S(I-\mathcal{D}_r)g)| \left|d\mu(u^*,g)\right| \\ &\le \|S\| \int_Q \|(I-\mathcal{D}_r)g\|_{A_v^p} \left|d\mu(u^*,g)\right|. \end{aligned}$$

By the dominated convergence theorem, the right-hand side tends to zero as $r \to 1$. The norm is preserved, because for all $S \in \mathcal{L}(A_v^p)$ and $r \in]0,1[$

$$\|\kappa\|_{\mathcal{K}(A_v^p)^*} \ge \frac{|\kappa(S\mathcal{D}_r)|}{\|S\mathcal{D}_r\|_{A_v^p \to A_v^p}} \ge \frac{|\nu_*(S\mathcal{D}_r)|}{\|S\|_{A_v^p \to A_v^p} \|\mathcal{D}_r\|_{A_v^p \to A_v^p}} \ge \frac{|\nu_*(S\mathcal{D}_r)|}{\|S\|_{A_v^p \to A_v^p}}.$$

Let $r \to 1$ to conclude the statement.

□

The *convex hull* of a set A is defined as

$$\operatorname{co} A := \bigcup_{n \in \mathbb{Z}_{\ge 1}} \left\{ \sum_{j=1}^n a_j t_j : a_j \in A;\ t_j \ge 0;\ j = 1,2,\dots;\ \sum_{j=1}^n t_j = 1 \right\},$$

and it is the smallest convex set containing A.

Lemma 3.1.11. *There are increasing sequences* $(r_n)_n \subset]0,1[$ *and* $(N(n))_n$ *with* $N(n) \in \mathbb{Z}_{\geq n}$ *for each positive integer* n *such that* $\lim_{n\to\infty} r_n = 1$ *(and* $\lim_{n\to\infty} N(n) = \infty$*) for which*

$$\lim_{n\to\infty} \sup_{f\in B_{A_v^p}} \sup_{|z|\leq R} v(z)|(I-L_n)f(z)| = 0 \tag{3.1.5}$$

for all $R \in]0,1[$ *and*

$$\lim_{n\to\infty} \|I-L_n\|_{A_v^p\to A_v^p} = 1, \tag{3.1.6}$$

where $L_n = \sum_{k=n}^{N(n)} c_{k,n}\mathcal{D}_{r_k}$ *for some constants* $c_{k,n} \geq 0$ *with* $\sum_{k=n}^{N(n)} c_{k,n} = 1$ *for all* $n \in \mathbb{Z}_{\geq 1}$.

Proof. First, it will be proved that for all $f \in A_v^p$,

$$\lim_{n\to\infty} \|(I-L_n)f\|_{A_v^p} = 0. \tag{3.1.7}$$

Let $\epsilon > 0$, and consider any two increasing sequences $(r_n)_n \subset]0,1[$ and $(N(n))_n$ with $N(n) \in \mathbb{Z}_{\geq n}$ for each integer $n \in \mathbb{Z}_{\geq 1}$ such that $\lim_{n\to\infty} r_n = 1$. Furthermore, let $(c_{k,n})_{k=n}^{N(n)}$, $n \in \mathbb{Z}_{\geq 1}$ be any convex combination. By the Minkowski inequality and Lemma 3.1.6

$$\|(I-L_n)f\|_{A_v^p} \leq \sum_{k=n}^{N(n)} c_{k,n}\left\|(I-\mathcal{D}_{r_k})f\right\|_{A_v^p} < \sum_{k=n}^{N(n)} c_{k,n}\epsilon = \epsilon$$

whenever n is large enough, which proves (3.1.7).

Turning to (3.1.5), let r_n, $N(n)$, $c_{k,n}$ be as above and take $R \in]0,1[$. Since the norm topology is finer than the compact open topology $\tau_0(A_v^p)$, equation (3.1.7) implies

$$\lim_{n\to\infty} \sup_{|z|\leq R} |(I-L_n)(f)(z)| = 0 \quad \text{for every } f \in A_v^p.$$

Next, it is proved that

$$\lim_{\delta\to 0} \sup_{n\in\mathbb{Z}_{\geq 1}} \sup_{f\in B_{\tau_0}(0,\delta)\cap A_v^p} \sup_{|z|\leq R} |L_n f(z)| = 0$$

from which it follows that

$$\lim_{\delta\to 0} \sup_{n\in\mathbb{Z}_{\geq 1}} \sup_{f\in B_{\tau_0}(0,\delta)\cap A_v^p} \sup_{|z|\leq R} |(I-L_n)f(z)| = 0, \tag{3.1.8}$$

where

$$B_{\tau_0}(f,\delta) = \{g \in \mathrm{HOLO}(\mathbb{D}) : \sup_{|z|\leq R} |f(z)-g(z)| < \delta\} \in \tau_0.$$

Take $\epsilon > 0$ and observe that the maximum modulus principle yields

$$\sup_{|z|\leq R} |L_n f(z)| \leq \sum_{k=n}^{N(n)} c_{k,n} \sup_{|z|\leq R} \left|\mathcal{D}_{r_k} f(z)\right| \leq \sup_{|z|\leq R} |f(z)|.$$

It follows that for $0 < \delta < \epsilon$

$$\sup_{n\in\mathbb{Z}_{\geq 1}} \sup_{f\in B_{\tau_0}(0,\delta)\cap A^p_v} \sup_{|z|\leq R} |L_n f(z)| < \epsilon.$$

Next, notice that $(B_{A^p_v}, \tau_0)$ is compact according to Lemma 3.1.5, therefore, there is a finite subcover $\left\{B_{\tau_0}(f_j,\delta)\right\}_{j=1}^J \subset \left\{B_{\tau_0}(f,\delta)\right\}_{f\in B_{A^p_v}}$ of $B_{A^p_v}$. It follows that for $f \in B_{A^p_v}$

$$\frac{\sup_{|z|\leq R} v(z)|(I-L_n)(f)(z)|}{\sup_{|z|\leq R} v(z)} \leq \min_{j\in[1,J]} \sup_{|z|\leq R}\left|(I-L_n)(f-f_j)(z)\right| + \max_{j\in[1,J]} \sup_{|z|\leq R}\left|(I-L_n)(f_j)(z)\right|$$

$$\leq \sup_{f\in B_{\tau_0}(0,\delta)\cap 2B_{A^p_v}} \sup_{|z|\leq R} |(I-L_n)(f)(z)| + \max_{j\in[1,J]} \sup_{|z|\leq R} \|\delta_z\| \left\|(I-L_n)(f_j)\right\|_{A^p_v}.$$

In accordance with (3.1.8), the first term is arbitrarily small by the choice of $\delta > 0$ and by (3.1.7), the right-hand side tends to zero as $n \to \infty$. This proves (3.1.5).

Finally, it will be proved that for each $n \in \mathbb{Z}_{\geq 1}$ there is an $L_n \in \mathrm{co}\{\mathcal{D}_{r_n}, \mathcal{D}_{r_{n+1}}, \ldots\}$ such that

$$\|I - L_n\|_{\mathcal{L}(A^p_v)} < 1 + \frac{1}{n}.$$

Assume that this is not true, that is, there exists a $n_0 \in \mathbb{Z}_{\geq 1}$ such that for all $L \in C := \mathrm{co}\{\mathcal{D}_{r_{n_0}}, \mathcal{D}_{r_{n_0+1}}, \ldots\}$, it holds that $\|I-L\|_{\mathcal{L}(A^p_v)} \geq 1 + \frac{1}{n_0}$. The open convex set $B(I, 1+\frac{1}{n_0})$ and the convex set C are therefore disjoint. Let ϕ be the separating functional, obtained by Proposition 3.1.9, for which $\mathfrak{Re}\,\phi(T) < \mathfrak{Re}\,\phi(L)$ for all $T \in B(I, \frac{1}{n_0})$ and $L \in C$. It can be assumed that $\left\|\phi\right\|_{\mathcal{L}(A^p_v)^*} = 1$. Notice that for any $S \in \mathcal{L}(A^p_v)$,

$$\left|\phi(S)\right| = \min_{t\in\mathbb{T}}\left|t\phi(S)\right| = \min_{t\in\mathbb{T}}\left|\mathfrak{Re}(t\phi(S)) + i\,\mathfrak{Im}(t\phi(S))\right| \leq \left|\mathfrak{Re}\,\phi(t_0 S)\right| \leq \left|\phi(t_0 S)\right|,$$

where t_0 is chosen so that $\mathfrak{Im}(t_0\phi(S)) = 0$. It follows that $\left\|\mathfrak{Re}\,\phi\right\|_{A^p_v\to\mathbb{C}} = \left\|\phi\right\|_{\mathcal{L}(A^p_v)^*} = 1$. Moreover, using $T = I + rS \in B(I, 1+\frac{1}{n_0})$, where $\|S\|_{A^p_v\to A^p_v} = 1$ and $0 \leq r < 1 + \frac{1}{n_0}$, one can conclude that

$$\mathfrak{Re}\,\phi(T-I) = \mathfrak{Re}\,\phi(rS) = r\,\mathfrak{Re}\,\phi(S).$$

Hence, for every $0 \leq r < 1+\frac{1}{n_0}$ there exists a $T \in B(I, 1+\frac{1}{n_0})$ with $\mathfrak{Re}\,\phi(T-I) = r$, which yields

$$0 < \mathfrak{Re}\,\phi(L-I) - \mathfrak{Re}\,\phi(T-I) = \mathfrak{Re}\,\phi(L-I) - r,$$

that is, $r < -\mathfrak{Re}\,\phi(I-L)$ for all $L \in C$. Since $0 \leq r < 1 + \frac{1}{n_0}$ is arbitrary, it holds that

$$1 + \frac{1}{n_0} \leq -\mathfrak{Re}\,\phi(I-L) \tag{3.1.9}$$

for all $L \in C$. As a small remark, Proposition 2.1.2, yields that, for each $L \in C$, there exists ϕ_L such that

$$1 + \frac{1}{n_0} \leq \inf_{L'\in C}\left\|I - L'\right\|_{\mathcal{L}(A^p_v)} \leq \|I-L\|_{\mathcal{L}(A^p_v)} = -\mathfrak{Re}\,\phi_L(I-L),$$

The convexity of C allowed a functional that satisfies (3.1.9), independent of L.

Since $\mathcal{K}(A_v^p)$ is an M-ideal, it follows from the definition that $\mathcal{L}(A_v^p)^* = \mathcal{K}(A_v^p)^* \oplus_1 \mathcal{K}(A_v^p)^\perp$ and $\phi = (I^* - P_L)\phi + P_L\phi = \nu_* + \nu_\perp$, where ν_* is given in Lemma 3.1.10 and P_L is the L-projection with $P_L(\mathcal{L}(A_v^p)^*) = \mathcal{K}(A_v^p)^\perp$. The definition of ν_* does indeed grant $P_L\nu_* \equiv 0$. Together with $\inf_{L\in C}|\nu_*(I-L)| = 0$ (Lemma 3.1.10), this yields

$$\begin{aligned}1 + \frac{1}{n_0} &\leq \inf_{L\in C}(-\mathfrak{Re}\,\phi(I-L)) = \inf_{L\in C}(-\mathfrak{Re}\,\nu_*(I-L)) - \mathfrak{Re}\,\nu_\perp(I) = -\mathfrak{Re}\,\nu_\perp(I) \leq \|\mathfrak{Re}\,\nu_\perp\|_{\mathcal{L}(A_v^p)\to\mathbb{C}} \\ &= \|\nu_\perp\|_{\mathcal{L}(A_v^p)^*} \leq \left\|\phi\right\|_{\mathcal{L}(A_v^p)^*} = 1,\end{aligned}$$

where the last inequality holds true due to $\left\|\phi\right\|_{\mathcal{L}(A_v^p)^*} = \|\nu_*\|_{\mathcal{L}(A_v^p)^*} + \|\nu_\perp\|_{\mathcal{L}(A_v^p)^*}$. This is a contradiction with the conclusion that for all $n \in \mathbb{Z}_{\geq 1}$ there is an $L_n \in \mathrm{co}\{\mathcal{D}_{r_n}, \mathcal{D}_{r_{n+1}}, \ldots\}$ such that $\|I - L_n\|_{\mathcal{L}(A_v^p)} < 1 + \frac{1}{n}$. $\square$

Lower bound

For the lower bound a suitable approximate identity will be used. To reduce ambiguity of the term approximate identity, it is in this thesis defined to be a sequence or net $(f_c)_c$ in a Banach space $X \subset \mathrm{HOLO}(\mathbb{D})$, which tends to zero everywhere except at one point, $\xi \in \overline{\mathbb{D}}$, as $c \to \infty$. The norm of f_c is, however, 1 for all c. In many spaces $X \subset \mathrm{HOLO}(\mathbb{D})$ the sequence approximates an evaluation map, $\|f_c g\|_X \to g(\xi) = \delta_\xi(g)$ as $c \to \infty$ for suitable functions, g. It is often sufficient if $g \in X$ can be continuously extended to $\mathbb{D} \cup (B(\xi,\epsilon) \cap \overline{\mathbb{D}})$ for some $\epsilon > 0$. A final observation is that, as a consequence of the maximum modulus principle, it is impossible to create an approximate identity of analytic functions gathering mass to $\xi \in \mathbb{D}$. The mass must be moved to the boundary! Chapter 4 contains a small survey on the approximate identities used in [16] and [17].

The approximate identity is created by multiplying the body of the approximate identity on A^p, $z \mapsto (1-z)^{-c_n}$, $c_n \in]0, \frac{2}{p}[$, where c_n increases to $\frac{2}{p}$ as $n \to \infty$, with a function that neutralises the weight v in a radial fashion as $z \to 1$. If $v = v_\alpha$, $\alpha > 0$ the function is given by $g\colon z \mapsto (2(1-z))^{-\frac{\alpha}{p}}$. Notice that the body of the approximate identity is directing the mass towards 1. Finally, normalisation of the function yields an approximate identity, for which it is easy to see point-wise convergence to zero on $\mathbb{D}$ as $n \to \infty$. Lemma 3.1.5 grants that point-wise convergence implies weak convergence, and hence, the effect of the compact operators in the definition of the essential norm will be nullified. Indeed, for a Banach space X, every $L \in \mathcal{K}(X)$ is completely continuous by Lemma 2.1.5. If $L \in \mathcal{K}(X)$, $T \in \mathcal{L}(X)$ and $(f_n)_n \in B_X$ is a weak null sequence, then

$$\|T(f_n)\|_X \leq \|(T-L)(f_n)\|_X + \|L(f_n)\|_X,$$

which yields $\lim_{n\to\infty}\|T(f_n)\|_X \leq \|T-L\|_{X\to X}$ and

$$\lim_{n\to\infty}\|T(f_n)\|_X \leq \|T\|_{e,X\to X}.$$

There are other interesting relations between the integral operator I_K and the integrand in the form of a weighted composition operator. In fact, in [17] it is proved that for $p-2>\alpha\geq 0$,

$$\left\|T_t C_{\phi_t}\right\|_{e,A^p_\alpha\to A^p_\alpha}=\lim_{z\to 1}T_t(z)\frac{t^s}{(1-t)^s},\quad s=\frac{2+\alpha}{p},$$

which together with Theorem 3.1.1 yields the interesting identity ([17, Section 9]):

$$\left\|\int_0^1 T_t C_{\phi_t}\,dt\right\|_{e,A^p_\alpha\to A^p_\alpha}=\int_0^1\left\|T_t C_{\phi_t}\right\|_{e,A^p_\alpha\to A^p_\alpha}dt. \tag{3.1.10}$$

Again, it is the weighted composition operator representation, which apparently allows the essential norm to pass by the mean value operator in this case.

Growth spaces

A similar identity to (3.1.10) is obtained, in [17], for integral operators on the standard growth spaces:

$$\left\|\int_0^1 T_t C_{\phi_t}\,dt\right\|_{e,H^\infty_\alpha\to H^\infty_\alpha}=\int_0^1\left\|T_t C_{\phi_t}\right\|_{e,H^\infty_\alpha\to H^\infty_\alpha}dt,\quad \alpha\in]0,1[.$$

Concerning the essential norm of the Hilbert matrix operator the following result can be found in [17, Example 7.5]:

Theorem 3.1.12. *For $0<\alpha<1$ the essential norm of the Hilbert matrix operator acting on H^∞_α is given by*

$$\|\mathcal{H}\|_{e,H^\infty_\alpha\to H^\infty_\alpha}=\frac{\pi}{\sin(\alpha\pi)}.$$

The upper bound of the essential norm of such an integral operator can be obtained similarly to the corresponding result on weighted Bergman spaces. The result obtained in Lemma 3.1.11 is in [20, Proposition 2.1] proved to hold, without invoking the theory of M-ideals, using the standard sliding hump technique. The compact operators L_n, $n\in\mathbb{Z}_{\geq 1}$ are again convex combinations of the dilation operators induced by an increasing sequence $(r_n)_n$ tending to 1 as $n\to\infty$. The uniform weights $(c_{k,n})_n=(n^{-1})_n$ are sufficient. The following lemma is necessary to obtain the lower bound of the essential norm in the way it is done in [17].

Lemma 3.1.13. *It holds that $(H^0_v)^{**}\cong({}^*H^\infty_v)^*\cong H^\infty_v$, where the Banach spaces are equipped with their natural norms. Furthermore, the relative topology $w^*(({}^*H^\infty_v)^*)$ to $B_{({}^*H^\infty_v)^*}$ is metrizable and $(\iota_H^{-1}\circ R^*)|_{B_{(H^0_v)^{**}}}:(B_{(H^0_v)^{**}},w^*((H^0_v)^{**}))\to(B_{H^\infty_v},\tau_\pi)$ is a homeomorphism.*

Proof. It was mentioned in Chapter 2 that the restriction $R\colon {}^*H^\infty_v\to(H^0_v)^*$ is an isometric isomorphism. Thus, the adjoint operator $R^*\colon (H^0_v)^{**}\to({}^*H^\infty_v)^*$ is an isometric isomorphism. Together with the preliminaries this proves the first statement.

To prove that $w^*(B_{({}^*H_v^\infty)^*})$ is metrizable, recall that it is sufficient for ${}^*H_v^\infty$ to be separable (see for example [19, Theorem 2.6.23]). The separability follows from the following:
Claim: For a sequence $(z_n)_n \subset \mathbb{D}$ with an accumulation point in $\mathbb{D}$, it holds that

$$ {}^*H_v^\infty = \overline{\operatorname{span}\{\delta_z,\ z \in \mathbb{D}\}}^{(H_v^\infty)^*} = \overline{\operatorname{span}\{\delta_{z_n},\ n \in \mathbb{Z}_{\geq 1}\}}^{(H_v^\infty)^*}. $$

Proof of Claim: Since $(B_{H_v^\infty}, \tau_0)$ is metrizable, consider a sequence $(f_n)_n \subset B_{H_v^\infty}$ with $f_n \to f \in B_{H_v^\infty}$ on compact subsets of $\mathbb{D}$ as $n \to \infty$. Then clearly $\delta_z(f_n) = f_n(z) \to f(z) = \delta_z(f)$ for all $z \in \mathbb{D}$ as $n \to \infty$, which yields that $\delta_z \in {}^*H_v^\infty$ for all $z \in \mathbb{D}$. Let $(z_n)_n \subset \mathbb{D}$ be a sequence with an accumulation point in $\mathbb{D}$. The space $\overline{\operatorname{span}\{\delta_z,\ z \in \mathbb{D}\}}^{(H_v^\infty)^*}$ is the smallest closed subspace of $(H_v^\infty)^*$ containing $\{\delta_z,\ z \in \mathbb{D}\}$, hence,

$$ \overline{\operatorname{span}\{\delta_{z_n},\ n \in \mathbb{Z}_{\geq 1}\}}^{(H_v^\infty)^*} \subset \overline{\operatorname{span}\{\delta_z,\ z \in \mathbb{D}\}}^{(H_v^\infty)^*} \subset {}^*H_v^\infty. $$

Assume that at least one of the inclusions above is proper. Let

$$ F \in {}^*H_v^\infty \setminus \overline{\operatorname{span}\{\delta_{z_n},\ n \in \mathbb{Z}_{\geq 1}\}}^{(H_v^\infty)^*}. $$

By Proposition 2.1.2, there is a functional $F_* \in ({}^*H_v^\infty)^*$ with $F_*(F) > 0$ and $F_*(G) = 0$ for all $G \in \overline{\operatorname{span}\{\delta_{z_n},\ n \in \mathbb{Z}_{\geq 1}\}}^{(H_v^\infty)^*}$. Let $\iota_H \colon H_v^\infty \to ({}^*H_v^\infty)^*$ be the isometric isomorphism, $f \mapsto \delta_f$. It follows that there is an $f \in H_v^\infty$, not identically zero, which is mapped to F_*, and hence, $f(z_n) = \delta_{z_n}(f) = F_*(\delta_{z_n}) = 0$ for all $n \in \mathbb{Z}_{\geq 1}$. From the identity theorem for analytic functions, it follows that $f \equiv 0$, which is a contradiction. Therefore,

$$ \overline{\operatorname{span}\{\delta_{z_n},\ n \in \mathbb{Z}_{\geq 1}\}}^{(H_v^\infty)^*} = \overline{\operatorname{span}\{\delta_z,\ z \in \mathbb{D}\}}^{(H_v^\infty)^*} = {}^*H_v^\infty. $$

$\square$

Continuing the proof of Lemma 3.1.13, it is now clear that if a sequence $(F_{*,n})_n \subset B_{({}^*H_v^\infty)^*} = \iota_H(B_{H_v^\infty})$ converges in $w^*(B_{({}^*H_v^\infty)^*})$ to some $F_* \in B_{({}^*H_v^\infty)^*}$, then the sequence consisting of $F_{*,n}(\delta_z) = \delta_z(\iota_H^{-1}(F_{*,n})) = \iota_H^{-1}(F_{*,n})(z)$, $n \in \mathbb{Z}_{\geq 1}$ converges to $\iota_H^{-1}(F_*)(z)$ for all $z \in \mathbb{D}$ as $n \to \infty$. This means that $(f_n)_n := (\iota_H^{-1}(F_{*,n}))_n \subset B_{H_v^\infty}$ converges in τ_π. According to Alaoglu's theorem $(B_{({}^*H_v^\infty)^*}, w^*(({}^*H_v^\infty)^*))$ is compact, which yields

$$ \iota_H^{-1}|_{\iota_H(B_{H_v^\infty})} \colon (B_{({}^*H_v^\infty)^*}, w^*(({}^*H_v^\infty)^*)) \to (B_{H_v^\infty}, \tau_\pi) $$

is a homeomorphism.

To conclude the proof it follows from R being an isometric isomorphism from $({}^*H_v^\infty)$ to $(H_v^0)^*$ that $R^*|_{B_{(H_v^0)^{**}}} : (B_{(H_v^0)^{**}}, w^*(B_{(H_v^0)^{**}})) \to (B_{({}^*H_v^\infty)^*}, w^*(B_{({}^*H_v^\infty)^*}))$ is a homeomorphism. Indeed, it is clearly a bijection since R^* is an isometric isomorphism $(H_v^0)^{**} \to ({}^*H_v^\infty)^*$. Let $(y_\alpha^{**})_\alpha \subset B_{(H_v^0)^{**}}$ be a net converging to y^{**} with respect to $w^*((H_v^0)^{**})$. Take $x^* \in {}^*H_v^\infty$. Now $Rx^* \in (H_v^0)^*$ and

$$ R^* y_\alpha^{**}(x^*) = y_\alpha^{**}(Rx^*) \to y^{**}(Rx^*) = (R^* y^{**})(x^*), $$

proving that the net $(R^*y^{**}_\alpha) \in ({}^*H^\infty_v)^*$ converges to $R^*y^{**} \in ({}^*H^\infty_v)^*$, again by Alaoglu's theorem $(B_{(H^0_v)^{**}}, w^*((H^0_v)^{**}))$ is compact, and therefore, R^* is a homeomorphism.

Finally, the homeomorphism $(B_{(H^0_v)^{**}}, w^*(B_{(H^0_v)^{**}}) \to (B_{H^\infty_v}, \tau_\pi(B_{H^\infty_v}))$ is given by

$$(\iota_H^{-1}|_{B_{({}^*H^\infty_v)^*}} \circ R^*|_{B_{(H^0_v)^{**}}}) = (\iota_H^{-1} \circ R^*)|_{B_{(H^0_v)^{**}}}.$$

□

When dealing with growth spaces, H^∞_v, in comparison to the weighted Bergman spaces, it is not as easy to find a sufficient condition for point-wise convergence to imply weak convergence, because the lack of reflexivity. However, Lemma 3.1.13 is a stable bridge, leading past the obstacle. Weak convergence on H^0_v can be compared to weak* convergence of the image of H^∞_v under the injection, $\iota_H|_{H^0_v}$, into the bidual, $(H^0_v)^{**}$, which in turn has a nice relationship with H^∞_v. From the different structure of the space the map $z \mapsto (1-z)^{-c_n}$ will be replace by $z \mapsto z^n$ and if $v = v_\alpha$, then the weight-neutraliser is given by $g\colon z \mapsto (2(1-z))^{-\alpha}$, $\alpha > 0$, which will guide the mass to 1. To ensure that our test-functions belong to the smaller space, H^0_v, the candidate test-function are multiplied with $z \mapsto H_n(z)$, $H_n(z) = (1-z)^{c_n}$, where $c_n > 0$ and $\lim_{n\to\infty} c_n = 0$ suitably fast. As in the A^p_v-case the approximate identity is obtained by normalisation. The fact that $(B_{H^\infty_v}, \tau_\pi)$ is homeomorphic to $(B_{(H^0_v)^{**}}, w^*(B_{(H^0_v)^{**}}))$ (see Lemma 3.1.13) is crucial. It follows that if $f_n \to 0$ in τ_π as $n \to \infty$, where $(f_n)_n \in B_{H^0_v} \subset B_{H^\infty_v}$, it also holds that $((R^*)^{-1} \circ \iota_H)(f_n) \to 0$ with respect to $w^*((H^0_v)^{**})$. For $y^* \in (H^0_v)^*$ it holds that

$$((R^*)^{-1} \circ \iota_H)(f_n)(y^*) = (\iota_H(f_n) \circ R^{-1})(y^*) = \iota_H(f_n)(R^{-1}y^*) = (R^{-1}y^*)f_n = y^*(f_n),$$

from which it can be concluded that $f_n \to 0$ in $w(H^0_v)$.

To be able to conclude that $f_n \to 0$ in $w(H^\infty_v)$, one only has to realise that $\hat{\iota}\colon H^0_v \to H^\infty_v$ is continuous, so for every functional $l \in (H^\infty_v)^*$ the map $l \circ \hat{\iota} \in (H^0_v)^*$, and hence, $f_n \to 0$ in $w(H^0_v)$ implies $f_n \to 0$ in $w(H^\infty_v)$. Notice that the adjoint operator of the inclusion map is a restriction map (compare to [20, p. 878]).

A final remark of interest is the different subsets of $\mathbb{D}$ that are used to obtain the lower bound. The crucial region is the intersection of $\mathbb{D}$ and a small disc centered at 1 in the case of A^p_v, and for H^∞_v the crucial region is a small part of the real line inside $\mathbb{D}$ touching 1. These regions are reasonable, because integration of an area, which is done in the A^p_v-norm needs a set with positive area and due to the approximate identity, it is clear that the set must touch 1. The H^∞_v-norm is constructed with a supremum, and hence, a line is sufficient.

A special case of an operator $X \to X$ defined as

$$f \mapsto \int_0^1 T_t C_{\phi_t}(f)\,dt,$$

where $T_t C_{\phi_t}(f)(z) = T_t(z) f(\phi_t(z))$, $z \in \mathbb{D}$, $T_t \in X$ and $\phi\colon \mathbb{D} \to \mathbb{D}$ analytic, is obtained if the symbols T_t and ϕ_t are constant with respect to t. Such an operator is, in fact, a weighted composition operator.

3.2 A characterisation for (weak) compactness and complete continuity of a weighted composition operator on BMOA

The compactness of weighted composition operators $W_{\psi,\varphi} = \psi C_\varphi\colon BMOA \to BMOA$ was characterised in [13] by Laitila using three conditions. As a part of [18] it is showed that one of the conditions is redundant. Before proceeding, for $z, a \in \mathbb{D}$, $\psi \in BMOA$ and φ an analytic selfmap of $\mathbb{D}$, the following notations will be used: $L(a) := \log \frac{2}{1-|a|^2}$, $\sigma_a(z) := \frac{a-z}{1-\bar{a}z}$, $\varphi_a(z) := \sigma_{\varphi(a)} \circ \varphi \circ \sigma_a$, $\alpha(\psi, \varphi, a) := \left|\psi(a)\right| \|\varphi_a\|_{H^2}$ and $\beta(\psi, \varphi, a) := L(\varphi(a)) \left\|\psi \circ \sigma_a - \psi(a)\right\|_{H^2}$.

In [18] the following is proved:

Theorem 3.2.1. *If $W_{\psi,\varphi} \in \mathcal{L}(BMOA)$, then the following statements are equivalent:*

(i) $\lim_{|\varphi(a)|\to 1} \left|\psi(a)\right| \|\varphi_a\|_{H^2} = 0$ *and* $\lim_{|\varphi(a)|\to 1} \log \frac{2}{1-|\varphi(a)|^2} \left\|\psi \circ \sigma_a - \psi(a)\right\|_{H^2} = 0;$

(ii) $W_{\psi,\varphi} : BMOA \to BMOA$ *is compact;*

(iii) $W_{\psi,\varphi} : BMOA \to BMOA$ *is weakly compact;*

(iv) $W_{\psi,\varphi} : BMOA \to BMOA$ *is completely continuous.*

Concerning (i)⇒(ii), what is proven is in fact that the two conditions given in (i), which are two of the three conditions Laitila used in [13], are in fact sufficient to prove Laitila's third condition, hence, the result follows from Laitila's result in [13]. The redundant condition is

$$\lim_{t\to 1} \sup_{|\varphi(b)|\le R} \int_{\tilde{E}(\varphi,b,t)} \left|\psi \circ \sigma_b(\xi)\right|^2 dm(\xi) = 0 \text{ for all } R \in (0,1), \tag{3.2.1}$$

where $\tilde{E}(\varphi, b, t) := \{\xi \in \mathbb{T} \colon |(\varphi_b)(\xi)| > t\}$, $t \in [0, 1[$.

Notice that the limit in (3.2.1) exists if the expression is bounded, and in [13] Laitila proved that the expression is bounded if $W_{\psi,\varphi} \in \mathcal{L}(BMOA)$.

The proof in [18] is done by contradiction, and the first part is to show that: if the limit in (3.2.1) is finite and strictly greater than zero, then a similar integral, which is associated with a larger real value, is also bounded from above and away from zero ([18, Claim 4.1]). Using this estimate it is in [18, Claim 4.2] proved, with the aid of [18, Lemma 3.4], that for every n large enough, there is an $\eta \in E_n$ such that

$$\inf_{r\in]0,1[} \frac{1}{I(r\eta)} \int_{E_n \cap I(r\eta)} \left|\psi \circ \sigma_{b_n}\right|^2 dm$$

is bounded away from zero, where $I(re^{i\theta}) := \{e^{it} : |\theta - t| \leq pi(1-r)\}$, $r \in [0,1[$, $\theta \in [0,2\pi[$,

$$E_n := \left\{\xi \in \mathbb{T} \colon \left|(\varphi \circ \sigma_{b_n})(\xi)\right| > \sqrt{1 - \frac{1+R_0}{1-R_0}(1-t_n^2)}\right\},$$

and $(t_n)_n \subset]0,1[$ and $(b_n)_n \subset \mathbb{D}$ with $|\varphi(b_n)| \leq R$ yield a sequential version of the redundant condition (3.2.1).

Claim 4.3 in [18] provides an upper bound, which can be made arbitrarily small in accordance with (i) and [18, Lemma 3.3]) resulting in a contradiction. Concerning [18, Lemma 3.4], which has the longest proof in [18], the first part is similar to [14], but the weight ψ creates some asymmetry, which causes problems concerning the approximation of arbitrary intervals with dyadic intervals with respect to a specific asymmetric measure given in **C**laim 4.2. The second part of the proof takes care of the problems related to the measure being asymmetric. On the one hand, the approximation is far from exact, but on the other hand, the result it yields, Lemma 3.4, is quite versatile since the demand of symmetry is dropped. The reverse, (i)$\Leftarrow$(ii) follows directly from [13].

Any compact operator $T \colon X \to Y$ between Banach spaces is clearly weakly compact and also completely continuous by Lemma 2.1.5. A sufficient condition to grant that an operator $T \in \mathcal{L}(X,Y)$ between Banach spaces is neither weakly compact nor completely continuous is given by the following lemma, which demands some more concepts.

A series $\sum_{n=1}^{\infty} x_n \subset X$, denoted $\sum_n x_n$, in a Banach space is said to be *unconditionally convergent* (UC) if $\sum_n x_{\sigma(n)}$ converges (in norm) for every permutation σ of $\mathbb{Z}_{\geq 1}$, that is, a bijective map on $\mathbb{Z}_{\geq 1}$.

A weaker property that a series can enjoy is being *weakly unconditionally Cauchy* (wuC). This means that the partial sums of $x^*(x_{\sigma(n)})$ form a Cauchy sequence in $\mathbb{C}$ for every permutation σ and $x^* \in X^*$, and hence, $\sum_n x^*(x_n)$ is unconditionally convergent, which in $\mathbb{C}$ is equivalent to $\sum_n |x^*(x_n)| < \infty$.

A sequence $(x_n)_n$ in a Banach space X is a basic sequence if it constitutes a basis for its closed linear span, $\overline{\operatorname{span}\{x_n,\ n \in \mathbb{Z}_{\geq 1}\}}^X$, which is a closed subspace of X. A standard result (see for example [19, Corollary 4.1.25]) is that $(x_n)_n$, where $x_n \neq 0$ for all $n \in \mathbb{Z}_{\geq 1}$, is a basic sequence in X if and only if there exists a constant C such that

$$\left\|\sum_{n=1}^{k_1} a_n x_n\right\|_X \leq C \left\|\sum_{n=1}^{k_2} a_n x_n\right\|_X \tag{3.2.2}$$

for all $(a_n)_n \subset \mathbb{C}^{\mathbb{Z}_{\geq 1}}$ and $k_2 > k_1 \geq 1$.

To be able to prove Theorem 3.2.9, which states that if $W_{\psi,\varphi} \in \mathcal{L}(BMOA)$ is not compact, then it can be neither completely continuous nor weakly compact, a handful of lemmas are necessary. The common factor in proving the two statements lies in the inability of a non-compact weighted composition operator on $BMOA$ to fix a copy of c_0. As with the essential norm and the power of M-ideal theory, a geometric touch connects compactness with both weak compactness and complete continuity concerning an operator $W_{\psi,\varphi} \in \mathcal{L}(BMOA)$. Therefore, some results concerning the space c_0 are appropriate. The following results (Lemmas 3.2.2–3.2.7) are classical.

Lemma 3.2.2. *Let $(x_n)_n$ be a bounded sequence in a Banach space X. The series $\sum_n t_n x_n$ converges for every $(t_n)_n \in c_0$ if and only if $\sum_n x_n$ is a wuC-series if and only if there exists a constant C such that for every integers $k_2 \geq k_1 \geq 1$ and $(t_n)_n \in c_0$ it holds that*

$$\left\| \sum_{n=k_1}^{k_2} t_n x_n \right\|_X \leq C \sup_{n \in \mathbb{Z}_{[k_1,k_2]}} |t_n|.$$

Proof. On the one hand, assume that $\sum_n t_n x_n$ converges for every $(t_n)_n \in c_0$. For $k \in \mathbb{Z}_{\geq 1}$ define $T_k \colon c_0 \to X$ such that $T_k((t_n)_n) = \sum_{n=1}^k t_n x_n$, $k \in \mathbb{Z}_{\geq 1}$. Clearly, T_k is a bounded operator and for a given $(t_n)_n \in c_0$ the quantity $\sup_k \|T_k((t_n)_n)\|_X$ is finite, because the series $\sum_n t_n x_n$ converges. The uniform boundedness principle yields that

$$C := \sup_{k \in \mathbb{Z}_{\geq 1}} \sup_{(t_n)_n \in B_{c_0}} \left\| \sum_{n=1}^{k} t_n x_n \right\| < \infty,$$

and hence, for every $x^* \in X^*$

$$\sup_{(t_n)_n \in B_{c_0}} \sup_{k \in \mathbb{Z}_{\geq 1}} \left| \sum_{n=1}^{k} t_n x^*(x_n) \right| = \sup_{(t_n)_n \in B_{c_0}} \sup_{k \in \mathbb{Z}_{\geq 1}} \left| x^* \left(\sum_{n=1}^{k} t_n x_n \right) \right| \leq C$$

as a consequence of Proposition 2.1.2. By a suitable choice of $(t_n)_n$, it is clear that

$$\sup_{(t_n)_n \in B_{c_0}} \sum_{n=1}^{\infty} |t_n| |x^*(x_n)| \leq C.$$

Since B_{c_0} is elementwise dense in B_{ℓ^∞}, it follows by dominated convergence that

$$\sum_{n=1}^{\infty} |x^*(x_n)| \leq C.$$

On the other hand, for a wuC-series $\sum_n x_n$, define $S_{k,(t_n)_n} \colon X^* \mapsto \mathbb{C}$ as

$$x^* \mapsto x^* \left(\sum_{n=1}^{k} t_n x_n \right), \quad (k, (t_n)_n) \in \mathbb{Z}_{\geq 1} \times B_{\ell^\infty}.$$

Clearly, $S_{k,(t_n)_n} \in X^{**}$ for all k and $(t_n)_n$, and from to the assumption of $\sum_n x_n$ being wuC, it follows that

$$\sup_{k \in \mathbb{Z}_{\geq 1}} \sup_{(t_n)_n \in B_{\ell^\infty}} \left| S_{k,(t_n)_n}(x^*) \right| \leq \sum_{n=1}^{\infty} |x^*(x_n)| < \infty$$

for all $x^* \in X^*$. The uniform boundedness principle yields that

$$\begin{aligned} \sup_{k \in \mathbb{Z}_{\geq 1}} \sup_{(t_n)_n \in B_{\ell^\infty}} \left\| \sum_{n=1}^{k} t_n x_n \right\|_X &= \sup_{k \in \mathbb{Z}_{\geq 1}} \sup_{(t_n)_n \in B_{\ell^\infty}} \sup_{x^* \in B_{X^*}} \left| x^* \sum_{n=1}^{k} t_n x_n \right| \\ &= \sup_{(k,(t_n)_n) \in \mathbb{Z}_{\geq 1} \times B_{\ell^\infty}} \left\| S_{k,(t_n)_n} \right\|_{X^{**}} \end{aligned}$$

is a finite constant, $C' > 0$. Furthermore, for a sequence $(t_n)_n \in c_0$ and $k_1 \leq k_2$, it holds that

$$\left\| \sum_{n=k_1}^{k_2} \chi_{t_n \neq 0}(n) \frac{t_n}{\sup\limits_{n \in \mathbb{Z}_{[k_1,k_2]}} |t_n|} x_n \right\|_X \leq C' \iff \left\| \sum_{n=k_1}^{k_2} t_n x_n \right\|_X \leq C' \sup_{n \in \mathbb{Z}_{[k_1,k_2]}} |t_n|,$$

where the leftmost sum is assumed to be 0 if $t_n = 0$ for all integers $n \in [k_1, k_2]$. It follows that $\sum_{n=k_1}^{k_2} t_n x_n$ is a Cauchy sequence in X, and by completeness, $\sum_n t_n x_n$ converges in X. □

Lemma 3.2.3. *A convex set M in a normed space X is weakly closed if and only if it is closed with respect to the norm topology.*

Proof. Since the weak topology is coarser than the norm topology, it is sufficient to prove that a closed convex set M is also weakly closed. Assume this is not the case and let M be a convex set, closed with respect to the norm, such that there is an $x_0 \in \overline{M}^w \setminus M$. As a consequence, there exists an open ball, $B(x_0, r)$ for some $r > 0$, that is disjoint from M. In a similar manner to Lemma 3.1.11, an application of Hahn-Banach separation theorem (Proposition 3.1.9) yields that there is a $\phi \in X^*$ with $\mathfrak{Re}\,\phi x_0 \leq c \leq d \leq \mathfrak{Re}\,\phi x$ for all $x \in M$, where $|d - c| = r > 0$. Since x_0 is a weak cluster point to M, every set U that is open in the weak topology and contains x_0 should have a nonempty intersection with M. However, $\{x \in X : \left|\phi(x - x_0)\right| < r\}$ is an open set in the weak topology with no common elements with M. Therefore, there are no $x_0 \in \overline{M}^w \setminus M$, so the sets are equal. □

The following assumption can be worded as: Assume that $T \in \mathcal{L}(X, Y)$ fixes a copy of c_0.

Lemma 3.2.4. *Assume $X_0 \subset X$ is isomorphic to c_0 and that $T \in \mathcal{L}(X, Y)$ such that $T|_{X_0}$ is an isomorphism onto some subspace Y_0. Then T is neither completely continuous nor weakly compact.*

Proof. First, the operator $T_0 := T|_{X_0} : X_0 \to Y_0$ inherits the property of being completely continuous or weakly compact. Using indirect proof, one can therefore neglect the operator T and only consider its restriction.

Let I_0 be an isomorphism $c_0 \to X_0$ and assume that T_0 is completely continuous. Since every norm-norm continuous operator is weak-weak continuous, it follows from the assumption that $T_0 \circ I_0 : c_0 \to Y_0$ is completely continuous, in which case, the identity operator on c_0, $I = I_0^{-1} T_0^{-1} \circ T_0 \circ I_0 : c_0 \to c_0$, is completely continuous. From Lemma 3.2.2, it follows that $\sum_n e_n$ is a wuC-series in c_0, and hence, $(e_n)_n$ is a weakly null sequence. This could also be seen from the explicit representation of an element in $(c_0)^* \cong \ell^1$. Since the identity is completely continuous, it should converge in norm to zero, which is impossible, because $\|e_n\|_{c_0} = 1$ for all $n \in \mathbb{Z}_{\geq 1}$.

Assume instead that T is weakly compact. In a similar fashion to the procedure used to disprove complete continuity, it follows that B_{c_0} is weakly relatively compact and in fact weakly compact since it is closed according to Lemma 3.2.3. According to the Eberlein-Šmulian theorem B_{c_0} is weakly sequentially compact. The partial sums, $y_k := \sum_{n=1}^{k} e_n$, $k \in \mathbb{Z}_{\geq 1}$, belong to B_{c_0} so there must exists a convergent subsequence of $(y_k)_k$. The candidates to limits are taken in c_0, but every subsequence $(y_{k_m})_m$ converges pointwise to $(1,1,\ldots) \notin c_0$ and a contradiction is achieved.

□

Lemma 3.2.5. *Given a basic sequence $(x_n)_n \subset X$ such that $\sum_n x_n$ is wuC and $\|x_n\|_X \geq c$ for all $n \in \mathbb{Z}_{\geq 1}$ and some $c > 0$, it holds that $(x_n)_n$ is equivalent to a basis of c_0. On the contrary, if $T\colon c_0 \to T(c_0) \subset X$ is an isomorphism, then $(T(e_n))_n$ is a basic sequence such that $\sum_n T(e_n)$ is wuC and $\|T(e_n)\|_X \geq c$ for all $n \in \mathbb{Z}_{\geq 1}$ and some $c > 0$.*

Proof. Consider the map $(t_n)_n \mapsto \sum_n t_n x_n$. It is clearly a linear bijection if it is well defined and surjective, since $(x_n)_n$ is a basic sequence. The following diagram illustrates how the assumptions come into play to prove that the map is well defined and surjective $c_0 \to \overline{\operatorname{span}\{x_n : n \in \mathbb{Z}_{\geq 1}\}}^X$:

$$(t_n) \in c_0 \quad \underset{\inf_n \|x_n\|_X > 0}{\overset{(x_n)\text{ wuC}}{\underset{\Longleftarrow}{\Longrightarrow}}} \quad \sum_n t_n x_n \text{ converge.}$$

Lemma 3.2.2 proves that the map is well defined and the surjectivity follows from $\sum_n t_n x_n$ being Cauchy and $\inf_n \|x_n\|_X > 0$:

$$\left\| \sum_{k=1}^{n} t_k x_k - \sum_{k=1}^{n-1} t_k x_k \right\|_X = |t_n| \|x_n\|_X.$$

Lemma 3.2.2 also provides that $(t_n)_n \mapsto \sum_n t_n x_n$ is bounded from above. By the bounded inverse theorem, $(t_n)_n \mapsto \sum_n t_n x_n$ is an isomorphism $c_0 \to \overline{\operatorname{span}\{x_n : n \in \mathbb{Z}_{\geq 1}\}}^X$ for which $e_n \mapsto x_n$.

On the contrary, T being an isomorphism yields immediately that $\inf_n \|T(e_n)\|_X = c$ for some $c > 0$, and for $\phi \in X^*$, the functional $\phi T \in (c_0)^*$, which yields that $\sum_n T(e_n)$ is wuC in X, because $\sum_n e_n$ is wuC in c_0 according to Lemma 3.2.2. To see that $T(e_n)$ is a basic sequence, one applies the characterisation (3.2.2), and because $(e_n)_n$ is a basic sequence, the statement follows from the linear operators T and T^{-1} being bounded.

□

Lemma 3.2.6. *Let X be a Banach space and $(x_n)_n \in X$ a sequence, equivalent to the standard basis of c_0. Then for any $(t_n)_n \in \ell^\infty$ such that $\inf_n |t_n| > 0$ the sequence $(t_n x_n)_n$ is equivalent to the standard basis of c_0.*

Proof. Due to Lemma 3.2.5 and the assumption, the sequence $(x_n)_n$ is a basic sequence with $\inf_n \|x_n\|_X > 0$ and $\sum_n x_n$ is wuC. None of these properties are affected by elementwise multiplication by $(t_n)_n$, where $0 < \inf_n |t_n| < \sup_n |t_n| < \infty$. Another application of Lemma 3.2.5 gives the result. □

Lemma 3.2.7. *If $T \in \mathcal{L}(X,Y)$, $(x_n)_n \subset X$ is equivalent to the standard basis of c_0 and $\|T(x_n)\|_Y$ is bounded away from zero. Then there is a subsequence $(n_k)_k \subset \mathbb{Z}_{\geq 1}$ such that both $(T(x_{n_k}))_k$ and $(x_{n_k})_k$ are equivalent to the standard basis of c_0. In other words, the assumption yields that T pfixes a copy of c_0.*

Proof. Under the given assumptions, $\sum_n T(x_n)$ is wuC, and so is the series of an arbitrary subsequence. It follows that $(T(x_n))_n$ is a weakly null sequence, bounded from above and away from zero, and by Bessaga-Pełczyński selection principle there is a subsequence $(n_k)_k \subset \mathbb{Z}_{\geq 1}$ rendering $(T(x_{n_k}))_k$ a basic sequence. By Lemma 3.2.5, it follows that $(T(x_{n_k}))_k$ is equivalent to the standard basis of c_0 and so is also $(x_{n_k})_k$. In fact, as a consequence of Lemma 3.2.5, a subsequence to a sequence equivalent to the standard basis of c_0, is also equivalent to the standard basis of c_0. □

The following lemma is found in [14, Proposition 6] and the proof uses the sliding hump technique.

Lemma 3.2.8. *Let $(f_n)_n$ be a sequence in $VMOA$ with unit norm such that $\lim_{n\to\infty}\|f_n\|_{H^2} = 0$. Then there exists a subsequence $(f_{n_k})_k$ of $(f_n)_n$ such that the map $c_0 \to \overline{\operatorname{span}}\{f_{n_k} : k \in \mathbb{Z}_{\geq 1}\}$ that maps $(t_n)_n \mapsto \sum_k t_k f_{n_k}$ is an isomorphism.*

Many of the $BMOA$ (or $VMOA$) specific ideas and calculations in the following theorem can be found in some form in [13] (see also [15]).

Theorem 3.2.9. *If $W_{\psi,\varphi} \in \mathcal{L}(BMOA)$ is not compact, then it can be neither completely continuous nor weakly compact.*

Proof. If $W_{\psi,\varphi} \in \mathcal{L}(BMOA)$ is not compact, then there is a number $\lambda > 0$ and a sequence $(a_n)_n \in \mathbb{D}$ with $\lim_{n\to\infty}|\varphi(a_n)| = 1$ such that at least one of the following holds:

1. $\alpha(\psi,\varphi,a_n) \geq \lambda$ for all n,
2. $\beta(\psi,\varphi,a_n) \geq \lambda$ for all n.

Laitila proved in [13] that $W_{\psi,\varphi}\colon BMOA \to BMOA$ is bounded if and only if

$$\sup_{a\in\mathbb{D}} \alpha(\psi,\varphi,a) < \infty \text{ and } \sup_{a\in\mathbb{D}} \beta(\psi,\varphi,a) < \infty.$$

Lemma 3.2.4 yields that it suffices to prove that $W_{\psi,\varphi} \in \mathcal{L}(BMOA)$ fixes a copy of c_0. According to Lemma 3.2.7 and Lemma 3.2.8, it suffices to find a sequence $(x_n)_n \in VMOA$ with unit norm such that $\|x_n\|_{H^2} \to 0$ as $n \to \infty$ and $\left\|W_{\psi,\varphi}(x_n)\right\|_{BMOA}$ is bounded away from zero. According to Lemma 3.2.6, the demand of unit norm can be relaxed to $0 < \inf_n\|x_n\|_{BMOA} \leq \sup_n\|x_n\|_{BMOA} < \infty$.

Assume (1) holds and let $f_n := \sigma_{\varphi(a_n)} - \varphi(a_n)$, $n \in \mathbb{Z}_{\geq 1}$. It follows that for $b \in \mathbb{D}$

$$f_n \circ \sigma_b - f_n(b) = \sigma_{\varphi(a_n)} \circ \sigma_b - \sigma_{\varphi(a_n)} \circ \sigma_b(0)$$

and with $\omega = \omega(a_n, b) = -\sigma_b(\varphi(a_n)) = \frac{\varphi(a_n)-b}{1-\overline{b}\varphi(a_n)}$

$$\sigma_{\varphi(a_n)} \circ \sigma_b(z) = \frac{1-\overline{b}\varphi(a_n)}{1-\overline{\varphi(a_n)}b}\frac{\omega-(-z)}{1-(-z)\overline{\omega}}$$

so that

$$f_n \circ \sigma_b(z) - f_n(b) = \frac{1-\overline{b}\varphi(a_n)}{1-\overline{\varphi(a_n)}b}\left(\frac{\omega-(-z)}{1-(-z)\overline{\omega}} - \omega\right)$$

and

$$|f_n \circ \sigma_b(z) - f_n(b)|^2 = \frac{|z|^2(1-|-\omega|^2)^2}{\left|1-\overline{(-\omega)}z\right|^2}.$$

This yields that

$$\|f_n \circ \sigma_b - f_n(b)\|_{H^2}^2 = (1-|-\omega|^2)\int_{\mathbb{T}} \frac{(1-|-\omega|^2)}{\left|1-\overline{(-\omega)}z\right|^2}\, dm(z) = (1-|\omega|^2). \tag{3.2.3}$$

The last equality is due to the integrand resembling the Poisson Kernel in an exact way. Since $\lim_{|b|\to 1}|\omega(\varphi(a_n), b)| \to 1$, it follows that $f_n \in VMOA$. From the symmetry of ω, it can be concluded that $\lim_{n\to\infty}|\omega(\varphi(a_n), b)| = \lim_{|\varphi(a_n)|\to 1}|\sigma_b(\varphi(a_n))| \to 1$ proving that

$$\lim_{n\to\infty}\|f_n \circ \sigma_b - f_n(b)\|_{H^2}^2 = 0 \tag{3.2.4}$$

for every fixed $b \in \mathbb{D}$. However, since $\omega(a_n, \cdot)$ is an automorphism of the disc, equation (3.2.3) yields

$$\sup_{b\in\mathbb{D}}\|f_n \circ \sigma_b - f_n(b)\|_{H^2} = \sup_{b\in\mathbb{D}}\sqrt{(1-|\omega(a_n, b)|^2)} = 1$$

and since $f_n(0) = 0$ it follows that $\|f_n\|_{BMOA} = 1$ for all n. To prove that $\left\|W_{\psi,\varphi}(f_n)\right\|$ is bounded away from zero, one can with some simple calculations obtain

$$\begin{aligned}\left\|W_{\psi,\varphi}(f_n)\right\|_{BMOA} &\geq \left\|\psi(a_n)\varphi_{a_n} + (\psi \circ \sigma_{a_n} - \psi(a_n))(\varphi_{a_n} - \varphi(a_n))\right\|_{H^2} \\ &\geq \alpha(\psi, \varphi, a_n) - 2\left\|\psi \circ \sigma_{a_n} - \psi(a_n)\right\|_{H^2} = \alpha(\psi, \varphi, a_n) - 2\frac{\beta(\psi, \varphi, a_n)}{L(\varphi(a_n))}.\end{aligned}$$

Since $\sup_{a\in\mathbb{D}}\beta(\psi, \varphi, a) < \infty$ and $|\varphi(a_n)| \to 1$ as $n \to \infty$, it follows that

$$\liminf_{n\to\infty}\left\|W_{\psi,\varphi}(f_n)\right\|_{BMOA} \geq \liminf_{n\to\infty}\alpha(\psi, \varphi, a_n) \geq \lambda > 0.$$

Since $f_n \not\equiv 0$ for every $n \in \mathbb{Z}_{\geq 1}$, it can be concluded that $\left\|W_{\psi,\varphi}(f_n)\right\|_{BMOA}$ is bounded away from zero, and hence, $W_{\psi,\varphi} \in \mathcal{L}(BMOA)$ is neither completely continuous nor weakly compact in the case (1) holds.

The other possibility is that (2) holds, in which case it can be assumed that (2) does not hold. In these settings, the function

$$z \mapsto g_n(z) := \left(\log\frac{2}{1-\overline{\varphi(a_n)}z}\right)^2\left(\log\frac{2}{1-|\varphi(a_n)|^2}\right)^{-1} \quad (z \in \mathbb{D})$$

turned out to be useful. It is shown in [13] that $g_n \in VMOA$ with a uniform upper bound of the norm with respect to n. This was achieved by an application of the Littlewood-Paley identity and an estimation of the Nevanlinna function. The uniform lower bound is achieved by the standard estimate of the norm of the evaluation map (see [9, p. 95])

$$\|g_n\|_{BMOA} \gtrsim \frac{g_n(z)}{L(z)} = \frac{\left(\log \frac{2}{1-\overline{\varphi(a_n)}z}\right)^2}{\left(\log \frac{2}{1-|z|^2}\right)\left(\log \frac{2}{1-|\varphi(a_n)|^2}\right)}, \quad z \in \mathbb{D}.$$

Using $z = \varphi(a_n)$, it follows that $\|g_n\|_{BMOA} \geq 1$. It remains to prove that $\lim_{n\to\infty}\|g_n\|_{H^2} = 0$, which is a bit cumbersome. The following sketch shows that $\lim_{n\to\infty}\|g_n\|_{H^p} = 0$ for all $p \geq 1$. It suffices to prove that $\int_{\mathbb{T}}\left|\log \frac{1}{1-\overline{\varphi(a_n)}z}\right|^p dm(z) < \infty$, $p \geq 2$ and from rotational symmetry of the norm, it can be assumed that $r_n = \varphi(a_n)$ is real and positive. The convergence holds true if and only if there exists $\epsilon > 0$ such that

$$\int_0^{\epsilon} \left|\log(1 - r_n e^{it})\right|^p dt < \infty.$$

The argument of $(1 - r_n e^{it})$ is bounded, so the above holds true if and only if

$$\int_0^{\epsilon} \left|\log\left|1 - r_n e^{it}\right|\right|^p dt = \int_0^{\epsilon} \left|\frac{1}{2}\log(1 - 2r_n \cos t + r_n^2)\right|^p dt < \infty.$$

For $0 < \epsilon < 2$, it holds that $\cos t \leq 1 - \frac{t^2}{3}$, $t \in [0, \epsilon]$, which gives a new sufficient condition to prove

$$\int_0^{\epsilon} \left|\log\left((1 - r_n)^2 + \frac{2r_n}{3}t^2\right)\right|^p dt = \int_0^{\sqrt{\frac{2r_n}{3}}\epsilon} \left|\log((1 - r_n)^2 + t^2)\right|^p \frac{\sqrt{3}\,dt}{\sqrt{2r_n}} < \infty.$$

For $r_n \geq \frac{1}{2}$ and $\epsilon < \frac{1}{2}$, it can be concluded that

$$\int_0^{\epsilon} \left|\log((1 - r_n)^2 + t^2)\right|^p dt \leq \int_0^{\epsilon} \left|\log(t^2)\right|^p dt = 2^p \int_0^{\epsilon} |\log t|^p \, dt = 2^p \int_{-\ln\epsilon}^{\infty} t^p e^{-t} \, dt < \infty$$

is true, and hence, the statement $\lim_{n\to\infty}\|g_n\|_{H^2} = 0$ follows from the fact that the other part of g_n, namely, $\left(\log \frac{2}{1-|\varphi(a_n)|^2}\right)^{-1}$ is independent of z and tends to zero as $n \to \infty$. The existence of a positive lower bound of $\left\|W_{\psi,\varphi}(g_n)\right\|_{BMOA}$ is similarly granted by some estimates from [13].

□

Remark 3.2.10. An alternative way of proving that

$$\int_0^{\epsilon} \left|\log(1 - r_n e^{it})\right|^p dt < \infty.$$

holds for some $\epsilon > 0$ is to use the formula

$$\int_0^{\epsilon} \left|f(e^{it})\right|^p \frac{dt}{2\pi} = p\int_0^{\infty} m(\{t \in [0,\epsilon] : \left|f(e^{it})\right| \geq \lambda\})\, d\lambda^p,$$

which is obtained by integrating with respect to values instead of arguments in the definition of the norm. The approximations of $\left|\log(1 - r_n e^{it})\right|$ would still be the same as above, and the result would be that the measure of the set in the integrand has an upper bound, independent of n, which decays exponentially in accordance with the John-Nirenberg lemma.

If the composition-symbol $\varphi \in \mathrm{HOLO}(\mathbb{D})$ is the identity, that is, $\varphi(z) = z$, $z \in \mathbb{D}$, then $W_{\psi,\varphi} = M_\psi$ is a multiplication operator.

3.3 Spectrum and essential spectrum of multiplication operators

In [16] the spectrum and essential spectrum of a multiplication operator $M_u \in \mathcal{L}(X)$, induced by a suitable analytic function u, is determined for quite general Banach spaces $X \subset \mathrm{HOLO}(\mathbb{D})$. These functions, u, form an algebra,

$$M(X) := \{u \in \mathrm{HOLO}(\mathbb{D}) : uf \in X \text{ for all } f \in X\},$$

which is a subset of X given that the constant functions belong to X. Another algebra, which eases the use of approximate identities, is the *disc algebra*, defined as

$$A(\mathbb{D}) = \{f \in (\mathbb{D}) \colon f \text{ has a continuous extension to } \overline{\mathbb{D}}\}.$$

Since continuity is preserved under uniform convergence, the disc algebra is a closed subspace of H^∞. As a consequence of Mergelyan's theorem (see [24, p. 386]), the disc algebra is the uniform closure of analytic polynomials on the open unit disc $\mathbb{D}$.

Concerning the results, the methods used are largely based on [4], and for the spectrum the following holds ([16, Theorem 3.2]):

Theorem 3.3.1. *Assume that $\mathbb{C} \subset X \subset \mathrm{HOLO}(\mathbb{B}_n)$ is a Banach space and that the evaluation functionals are bounded. Furthermore, assume that there is another Banach space Y such that $\|f\|_X \asymp \left\|R^N f\right\|_Y$ for some $N \in \mathbb{Z}_{\geq 1}$ and all $f \in \mathrm{HOLO}(\mathbb{B}_n)$, Y satisfies the given properties for X and whose multiplier algebra $M(Y) \supset H^\infty(\mathbb{B}_n)$. If $M_u \in \mathcal{L}(X)$, then*

$$\sigma(M_u) = \overline{u(\mathbb{B}_n)}.$$

The next useful result can be found in, for example, [21]:

Lemma 3.3.2 (Hartogs' extension theorem)**.** *Let f be analytic in $r < \sum_{j=1}^n \left|z_j\right|^2 < R$, where $0 < r < R$. Then f can be continued analytically to $\sum_{j=1}^n \left|z_j\right|^2 < R$.*

For the essential spectrum, similarly to [4], an application of Hartogs' theorem yields that:

Theorem 3.3.3. *Under the additional assumption that $P_j\colon \mathbb{B}_n \to \mathbb{C}$, $P_j(z) = z_j$, $j = 1,\dots,n$ are multipliers to a Banach space X, that is, and $P_j \in M(X(\mathbb{B}_n))$ for every j, the assumptions given in Theorem 3.3.1 are sufficient to ensure that*

$$\sigma_e(M_u) = \bigcap_{0<r<1} \overline{u(\mathbb{B}_n \setminus r\mathbb{B}_n)} = \overline{u(\mathbb{B}_n)} = \sigma(M_u).$$

Notice that in [16, Theorem 4.1], the assumptions are insufficient for the proof given, because the proof makes use of [16, Lemma 3.1], whose proof heavily depend on the existence of a space Y with the given properties.

The method to obtain the essential spectrum when $n = 1$ is, however, quite space specific and the results are given in the theorem below, which is part of [16, Theorem 4.13]:

Theorem 3.3.4. *If X is one of the following spaces:*

(a) $\mathbb{B}_\alpha(\mathbb{D})$, $0 < \alpha < 1$, *with* $u \in M(\mathbb{B}_\alpha(\mathbb{D})) = \mathbb{B}_\alpha(\mathbb{D}) \subset A(\mathbb{D})$;

(b) $\mathbb{B}(\mathbb{D})$ *with* $u \in M(\mathbb{B}(\mathbb{D})) \cap A(\mathbb{D})$;

(c) $A^p_{\alpha,\beta}(\mathbb{D})$ *with* $u \in M(A^p_{\alpha,\beta}(\mathbb{D})) = A^p_{\alpha,\beta}(\mathbb{D}) \subset A(\mathbb{D})$, *where* $p > 1$, $\alpha > -1$ *and* $\beta > \frac{2+\alpha}{p}$;

then

$$\sigma_e(M_u) = \bigcap_{0<r<1} \overline{u(\mathbb{D} \setminus r\mathbb{D})}.$$

As a disclaimer, the $A^p_{\alpha,\beta}$-case when $p = 2$, follows immediately from [4], but other values of $p > 1$ demanded new precise estimations in the creation of [16]. For example, in [16, Lemma 4.11], it is proved that, for a given $p \geq 1$, $\alpha > -1$ and $\beta \geq 0$, if $k \in \mathbb{Z}_{\geq 1}$ is large,

$$\left\|f_{\xi,k}\right\|^p_{A^p_{\alpha,\beta}} \asymp (k+1)^{-\alpha+\beta p-\frac{3}{2}},$$

where $f_{\xi,k}$ is given by

$$f_{\xi,k}\colon z \mapsto \left(\frac{1+\overline{\xi}z}{2}\right)^k, \quad \xi \in \mathbb{T},\ k \in \mathbb{Z}_{\geq 1}.$$

The upper estimate

$$\sigma_e(M_u) \subset \bigcap_{0<r<1} \overline{u(\mathbb{D} \setminus r\mathbb{D})},$$

can be achieved by adding one more general condition to the space X, in addition to the assumptions made in Theorem 3.3.1. The result is the following:

Theorem 3.3.5. *Assume that*

(i) $\mathbb{C} \subset X, Y \subset \mathrm{HOLO}(\mathbb{D})$ *is a Banach space and that the evaluation functionals are bounded,*

(ii) $M(Y) \supset H^\infty$,

(iii) there is an integer $N \in \mathbb{Z}_{\geq 1}$ *such that* $\|f\|_X \asymp \left\|R^N f\right\|_Y$ *for all* $f \in \mathrm{HOLO}(\mathbb{D})$, *and*

(iv) if $f(z_0) = 0$ *for a function* $f \in X$ *and* $z_0 \in \mathbb{D}$, *then* $z \mapsto \frac{f(z)}{z - z_0} \in X$.

If $M_u \in \mathcal{L}(X)$, *then*

$$\sigma_e(M_u) \subset \bigcap_{0<r<1} \overline{u(\mathbb{D} \setminus r\mathbb{D})}.$$

For spaces X properly contained in H^∞, approximate identities are used to obtain

$$\bigcap_{0<r<1} \overline{u(\mathbb{D} \setminus r\mathbb{D})} \subset \sigma_e(M_u), \tag{3.3.1}$$

to be more specific, normalised versions of the functions $f_{\xi,k}$, $\xi \in \mathbb{T}$, $k \in \mathbb{Z}_{\geq 1}$ are used. These calculations are, however, very space specific, and the calculations demand that u can be continuously extended to the closed disc $\overline{\mathbb{D}}$, in which case $\bigcap_{0<r<1} \overline{u(\mathbb{D} \setminus r\mathbb{D})}$ is the image of the complex unit circle $\mathbb{T}$ under the extension of u.

If X is such that all $u \in H^\infty$ induce a bounded operator $M_u \in \mathcal{L}(X)$, in which case every condition involving Y can be neglected, one can also conclude (3.3.1) given that $M_u \in \mathcal{L}(X)$, and X satisfies conditions (i) and (iv), given in Theorem 3.3.5. The proof scratches the theory of interpolation sequences, and it is proved that: if $M_u \in \mathrm{HOLO}(\mathbb{D})$ and $\lambda \in \bigcap_{0<r<1} \overline{u(\mathbb{D} \setminus r\mathbb{D})}$, a certain sequence of perturbations of $M_u - I\lambda$ are not Fredholm, and that this sequence tends strongly to $M_u - I\lambda$. Since the set of non-Fredholm operators is closed, (3.3.1) follows.

The final section contains some information about the two approximate identities used in [16] and [17] respectively. The result concerning the approximate identity used in [16] is an improvement of the main part of [16, Lemma 4.11].

Chapter 4

Approximate identities

The definition of an approximate identity, in the sense the term is used in this thesis, can be found in subsection 3.1 where the lower bound of the essential norm is discussed.

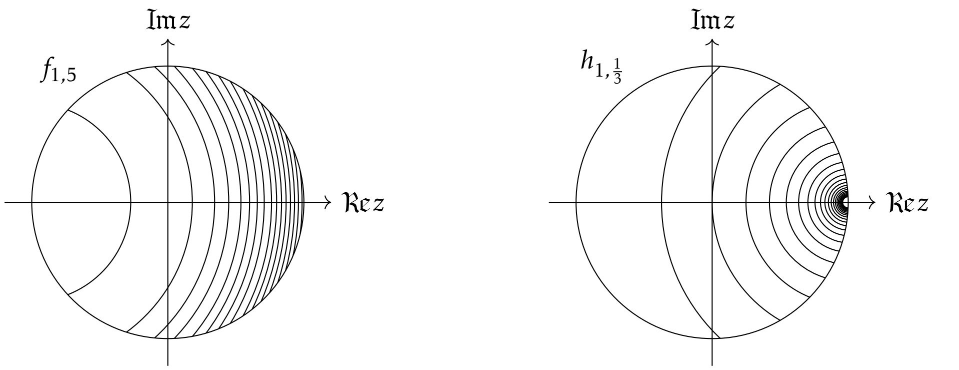
One can observe that the achievement of a lower bound for the essential spectrum of a multiplication operator on a Bergman-Sobolev space $A^p_{\alpha,\beta}(\mathbb{D})$, $\beta > \frac{2+\alpha}{p}$, $p > 1$, $\alpha > -1$ used the peak function:

$$(g_{\xi,k})_k := \left(\frac{f_{\xi,k}}{\left\| f_{\xi,k} \right\|_{A^p_{\alpha,\beta}}} \right)_k, \text{ where } f_{\xi,k}(z) := \left(\frac{1+\overline{\xi}z}{2} \right)^k, \quad \xi \in \mathbb{T}, z \in \mathbb{D}$$

as the approximate identity. However, the lower bound of the essential norm of the Hilbert matrix operator $\mathcal{H}\colon A^p_\alpha \to A^p_\alpha$, $p > 1$, $\alpha \geq 0$ was obtained using another approximate identity, namely,

$$(h_{\xi,c})_{0 \leq c < \frac{2}{p}} = \left(\frac{F_{\xi,c}}{\left\| F_{\xi,c} \right\|_{A^p_\alpha}} \right)_{0 \leq c < \frac{2}{p}}, \text{ where } F_{\xi,c}(z) = \frac{1}{2^{\frac{\alpha}{p}} (1-\overline{\xi}z)^{c+\frac{\alpha}{p}}}, \quad \xi \in \mathbb{T}, z \in \mathbb{D}.$$

The parameter ξ is the point on the boundary $\mathbb{T}$ where the mass is concentrated as $k \to \infty$ or $c \to \frac{2}{p}$. One can ask, what is the difference between these two approximate identities? The obvious part of the answer is that each of them consists of different mathematical functions and depending on context, one of them yields easier calculations. The function $f_{\xi,k}$ can be used on many Banach spaces in its current form in contrast to $F_{\xi,c}$, where at least the value of c must be adapted to the space the approximate identity is used on. The image of one of the functions from each approximate identity gives a hint about some other properties:



Each line is either an increase or a decrease in value by a fixed amount. Denser lines means, therefore, a steeper slope. Normalisation will not change the structure of the image and a different k or c will change the denseness of the contour lines and placement, but not the characteristics of the shape. It is also evident from construction that a contour line for $f_{1,k}$ is a circle with center -1, where the mass is larger the further away from -1 one observes. In comparison, the contour lines of the function $h_{1,c}$ are circles with center at it's mass concentration point 1. Normalising the functions, and considering the set

$$M_{>x}(f) = \{z \in \mathbb{D} : |f(z)| \geq x\},$$

it is evident that the shape of $M_{>x}(h_{1,c})$ is a circle, gathering mass equally from all directions. However, for k large enough (so that the following set is not empty), the set $M_{>x}(g_{1,k})$ is a narrow lune. It turns out that in the limit case, as $k \to \infty$ all mass is pushed tangentially to 1 from inside $\mathbb{D}$, which renders this approximate identity useless when considering inscribed polygons with a corner at the mass concentration point, or equivalently, suitable Stolz angles. To finish this section, some results of how well the approximate identities gathers mass from a smaller disc $B(1-R,R)$, $R \in]0,1]$ inscribed in $\mathbb{D}$, touching 1, are presented. This will prove the uselessness of $(g_{1,k})_k$ as an approximate identity on inscribed polygons. It will be proved that for $\alpha > -1$ and $R \in]0,1]$,

$$\int_{B(1-R,R)} |1+z|^q (1-|z|^2)^\alpha \, dA(z) \overset{q\to\infty}{\sim} \frac{2^{q+\frac{5}{2}+2\alpha}}{\pi} \frac{\Gamma(\alpha+\frac{3}{2})}{q^{\alpha+\frac{3}{2}}} \int_{\frac{2-2R}{2-R}}^{1} \frac{r^\alpha \, dr}{\sqrt{1-r}} \tag{4.0.1}$$

and

$$\int_{B(1-R,R)} |1-z|^{-c} (1-|z|^2)^\alpha dA(z) \overset{c\to 2+\alpha}{\sim} \frac{2^\alpha}{2+\alpha-c} \frac{1}{\pi} \beta\left(\frac{\alpha}{2}+\frac{1}{2}, \frac{1}{2}\right) = \frac{\left(\frac{\Gamma(\alpha+1)}{\Gamma(\frac{\alpha}{2}+1)^2}\right)}{2+\alpha-c},$$

where $A(q) \overset{q\to q_0}{\sim} B(q)$ means $\lim_{q\to q_0} \frac{A(q)}{B(q)}$ exists and lies in $]0,\infty[$. As R is smaller, some of the mass gathered tangentially in $\mathbb{D}$ to 1, will not make a contribution to the total mass gathered by the approximate identities on $B(1-R,R)$ as the limit is taken. The mass gathered non-tangentially is invariant. What can also be seen is, by putting $R=1$ in (4.0.1), an improvement of the constant in the main part of [16, Lemma 4.11] is obtained in the case p is not an integer, where the pair (q,α) here, can be compared to $((k-j)p,\gamma)$ in the article. Only considering $R=1$ would simplify the proof the asymptotic behaviour of the peak function $f_{1,k}$ (equation (4.0.1)), however, to obtain the above formula (4.0.1) for general $R\in]0,1]$, the following lemma is crucial:

Lemma 4.0.1. *If $\eta>\xi>-1$ and $f,g\colon [0,1[\to[0,\infty[$ are continuous an integrable with $f(0)\in]0,\infty[$ and g not indentically zero on $]0,b[$ for every $0<b<1$, then*

$$\int_0^1\int_0^1(1-tr)^q t^\eta r^\xi f(r)g(t)\,dt\,dr \overset{q\to\infty}{\sim} f(0)\int_0^1 r^{\eta-\xi-1}g(r)\,dr\frac{\Gamma(\xi+1)}{q^{\xi+1}}.$$

Proof. Take $\rho\in]0,\frac{f(0)}{2}[$ and choose $\epsilon\in]0,1[$ such that $0<f(0)-\rho<f(x)<f(0)+\rho$ when $0<x<\epsilon$. Let $q>0$.

On the one hand, there exists $M=M(\epsilon)\in]0,\infty[$ such that

$$\begin{aligned}\int_\epsilon^1\int_0^1(1-rt)^q t^\eta r^\xi f(r)\,dt\,dr &\le \int_\epsilon^1\int_0^1(1-\epsilon t)^q t^\eta \max\{1,\epsilon^\xi\}f(r)\,dt\,dr\\
&\le \max\{1,\epsilon^\xi\}\left(\int_0^1 f(r)\,dt\right)\int_0^1(1-\epsilon t)^q t^\eta\,dt\\
&\le \frac{\max\{1,\epsilon^\xi\}}{\epsilon^{\eta+1}}\left(\int_0^1 f(r)dt\right)\int_0^1(1-t)^q t^\eta\,dt\\
&= M(\epsilon)\left(\int_0^1 f(r)\,dt\right)\beta(q+1,\eta+1).\end{aligned}$$

Notice that

$$\begin{aligned}\frac{\int_\epsilon^1\int_{\frac12}^1(1-rt)^q t^\eta r^\xi f(r)g(t)\,dt\,dr}{\int_\epsilon^1\int_0^1(1-rt)^q t^\eta r^\xi f(r)g(t)\,dt\,dr} &\le \frac{(1-\frac{\epsilon}{2})^q\int_\epsilon^1\int_{\frac12}^1 t^\eta r^\xi f(r)g(t)\,dt\,dr}{(1-\frac{\epsilon}{4})^q\int_\epsilon^1\int_0^{\frac{\epsilon}{4}} t^\eta r^\xi f(r)g(t)\,dt\,dr}\\
&\le\left(\frac{1-\frac{\epsilon}{2}}{1-\frac{\epsilon}{4}}\right)^q\frac{\int_\epsilon^1\int_{\frac12}^1 t^\eta r^\xi f(r)g(t)\,dt\,dr}{\int_\epsilon^1\int_0^{\frac{\epsilon}{4}} t^\eta r^\xi f(r)g(t)\,dt\,dr}\\
&\overset{q\to\infty}{\to}0.\end{aligned}\tag{4.0.2}$$

Since $\beta(q+1,\eta+1)\sim\frac{\Gamma(\eta+1)}{q^{\eta+1}}$, it holds that

$$\begin{aligned}q^{\xi+1}\int_\epsilon^1\int_0^1(1-tr)^q t^\eta r^\xi f(r)g(t)\,dt\,dr &\underset{(4.0.2)}{\overset{q\to\infty}{\sim}} q^{\xi+1}\int_\epsilon^1\int_0^{\frac12}(1-tr)^q t^\eta r^\xi f(r)g(t)\,dt\,dr\\
&\le \left\|g\chi_{]0.\frac12[}\right\|_\infty M(\epsilon)\left(\int_0^1 f(r)\,dt\right)q^{\xi+1}\beta(q+1,\eta+1)\\
&\overset{q\to\infty}{\sim}\left\|g\chi_{]0.\frac12[}\right\|_\infty M(\epsilon)\left(\int_0^1 f(r)\,dt\right)\Gamma(\eta+1)q^{\xi-\eta}\\
&\overset{q\to\infty}{\to}0.\end{aligned}\tag{4.0.3}$$

On the other hand,

$$\int_0^\epsilon \int_0^1 (1-tr)^q t^\eta r^\xi f(r) g(t)\,dt\,dr \in B_{\mathbb{R}}(f(0),\rho) \int_0^\epsilon \int_0^1 (1-tr)^q t^\eta r^\xi g(t)\,dt\,dr$$

and

$$\begin{aligned}\int_0^\epsilon \int_0^1 (1-tr)^q t^\eta r^\xi g(t)\,dt\,dr &= \int_0^\epsilon \int_0^r (1-t)^q t^\eta r^{\xi-\eta-1} g(\frac{t}{r})\,dt\,dr \\ &= \int_0^\epsilon (1-t)^q t^\eta \left(\int_t^\epsilon r^{\xi-\eta-1} g(\frac{t}{r})\,dr \right) dt \\ &= \int_0^\epsilon (1-t)^q t^\xi \left(\int_{\frac{t}{\epsilon}}^1 r^{\eta-\xi-1} g(r)\,dr \right) dt.\end{aligned}$$

Now choose $\delta = \delta(\epsilon,\rho) \in]0,\epsilon[$ small enough so that

$$\int_0^{\frac{t}{\epsilon}} r^{\eta-\xi-1} g(r)\,dr < \rho \qquad \text{(recall that } \eta-\xi-1>-1 \text{ and } g \text{ bounded)},$$

and

$$\int_{\frac{t}{\epsilon}}^1 r^{\eta-\xi-1} g(r)\,dr > 0 \qquad \text{(recall that } g \not\equiv 0 \text{ and continuous)},$$

whenever $0 < t < \delta$ and define $G := \int_0^1 r^{\eta-\xi-1} g(r)\,dr$. Now

$$\int_0^\delta (1-t)^q t^\xi \left(\int_{\frac{t}{\epsilon}}^1 r^{\eta-\xi-1} g(r)\,dr \right) dt \in B_{\mathbb{R}}(G,\rho) \int_0^\delta (1-t)^q t^\xi\,dt,$$

which yields

$$\begin{aligned}&\frac{\int_0^\epsilon (1-t)^q t^\xi \left(\int_{\frac{t}{\epsilon}}^1 r^{\eta-\xi-1} g(r)\,dr \right) dt}{\int_0^\delta (1-t)^q t^\xi\,dt} \\ &= \frac{\int_0^\epsilon (1-t)^q t^\xi \left(\int_{\frac{t}{\epsilon}}^1 r^{\eta-\xi-1} g(r)\,dr \right) dt}{\int_0^\delta (1-t)^q t^\xi \left(\int_{\frac{t}{\epsilon}}^1 r^{\eta-\xi-1} g(r)\,dr \right) dt} \frac{\int_0^\delta (1-t)^q t^\xi \left(\int_{\frac{t}{\epsilon}}^1 r^{\eta-\xi-1} g(r)\,dr \right) dt}{\int_0^\delta (1-t)^q t^\xi\,dt} \\ &\in B_{\mathbb{R}}(G,\rho) \frac{\int_0^\epsilon (1-t)^q t^\xi \left(\int_{\frac{t}{\epsilon}}^1 r^{\eta-\xi-1} g(r)\,dr \right) dt}{\int_0^\delta (1-t)^q t^\xi \left(\int_{\frac{t}{\epsilon}}^1 r^{\eta-\xi-1} g(r)\,dr \right) dt},\end{aligned}$$

and hence, it follows that

$$
\begin{aligned}
&\frac{\int_0^\epsilon \int_0^1 (1-tr)^q t^\eta r^\xi f(r) g(t)\,dt\,dr}{\int_0^\delta (1-t)^q t^\xi\,dt} \\
&= \frac{\int_0^\epsilon \int_0^1 (1-tr)^q t^\eta r^\xi f(r) g(t)\,dt\,dr}{\int_0^\epsilon \int_0^1 (1-tr)^q t^\eta r^\xi g(t)\,dt\,dr} \frac{\int_0^\epsilon \int_0^1 (1-tr)^q t^\eta r^\xi g(t)\,dt\,dr}{\int_0^\delta (1-t)^q t^\xi\,dt} \\
&\in B_{\mathbb{R}}(f(0),\rho) B_{\mathbb{R}}(G,\rho) \frac{\int_0^\epsilon (1-t)^q t^\xi \left(\int_{\frac{t}{\epsilon}}^1 r^{\eta-\xi-1} g(r)\,dr \right) dt}{\int_0^\delta (1-t)^q t^\xi \left(\int_{\frac{t}{\epsilon}}^1 r^{\eta-\xi-1} g(r)\,dr \right) dt}.
\end{aligned}
\tag{4.0.4}
$$

The last quotient tends to 1 as $q \to \infty$ in accordance with the following formula, obtained similarly to (4.0.2):

$$
\begin{aligned}
\frac{\int_\delta^\epsilon (1-t)^q t^\xi \left(\int_{\frac{t}{\epsilon}}^1 r^{\eta-\xi-1} g(r)\,dr \right) dt}{\int_0^\epsilon (1-t)^q t^\xi \left(\int_{\frac{t}{\epsilon}}^1 r^{\eta-\xi-1} g(r)\,dr \right) dt} &\le \frac{\int_\delta^\epsilon (1-\delta)^q t^\xi \left(\int_{\frac{t}{\epsilon}}^1 r^{\eta-\xi-1} g(r)\,dr \right) dt}{\int_0^{\frac{\delta}{2}} (1-\frac{\delta}{2})^q t^\xi \left(\int_{\frac{\delta}{2\epsilon}}^1 r^{\eta-\xi-1} g(r)\,dr \right) dt} \\
&\le \left(\frac{1-\delta}{1-\frac{\delta}{2}} \right)^q \frac{G \int_\delta^\epsilon t^\xi\,dt}{\int_0^{\frac{\delta}{2}} t^\xi \left(\int_{\frac{\delta}{2\epsilon}}^1 r^{\eta-\xi-1} g(r)\,dr \right) dt} \\
&\overset{q\to\infty}{\longrightarrow} 0.
\end{aligned}
$$

Summarising, (notice that all parameters chosen are independent of q)

$$
\lim_{q\to\infty} \frac{\int_0^\epsilon \int_0^1 (1-tr)^q t^\eta r^\xi f(r) g(t)\,dt\,dr}{\int_0^\delta (1-t)^q t^\xi\,dt} \in B_{\mathbb{R}}(f(0),\rho) B_{\mathbb{R}}(G,\rho).
$$

Finally, a simple version of (4.0.2) ensures that

$$
\int_0^\delta (1-t)^q t^\xi\,dt \sim \int_0^1 (1-t)^q t^\xi\,dt = \beta(q+1,\xi+1) \sim \frac{\Gamma(\xi+1)}{q^{\xi+1}}
$$

as $q \to \infty$, which yields

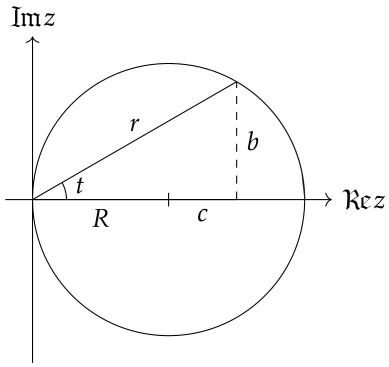
$$
\begin{aligned}
&\lim_{q\to\infty} \frac{q^{\xi+1}}{\Gamma(\xi+1)} \int_0^1 \int_0^1 (1-tr)^q t^\eta r^\xi f(r) g(t)\,dt\,dr \\
&\overset{(4.0.3)}{=} \lim_{q\to\infty} \frac{q^{\xi+1}}{\Gamma(\xi+1)} \int_0^\epsilon \int_0^1 (1-tr)^q t^\eta r^\xi f(r) g(t)\,dt\,dr \in B_{\mathbb{R}}(f(0),\rho) B_{\mathbb{R}}(G,\rho).
\end{aligned}
$$

Let $\rho \to 0$ to finish the proof. □

Theorem 4.0.2. *Let $\alpha > -1$. Then*

$$
\int_{B(1-R,R)} |1+z|^q (1-|z|^2)^\alpha\,dA(z) \overset{q\to\infty}{\sim} \frac{2^{q+\frac{5}{2}+2\alpha}}{\pi} \frac{\Gamma(\alpha+\frac{3}{2})}{q^{\alpha+\frac{3}{2}}} \int_{\frac{2-2R}{2-R}}^1 \frac{r^\alpha\,dr}{\sqrt{1-r}}.
$$

Proof. At the first equality below, the domain of integration is reflected through $z = \frac{1}{2}$, and the second inequality is the substitution to polar coordinates $z \mapsto re^{it}$ (see Figure 4.1 for information about the transformed path of integration):



$$\begin{gathered} r^2 = (R+c)^2 + b^2 \\ R^2 = c^2 + b^2 \\ \tfrac{R+c}{r} = \cos t \\ \Downarrow \\ r = 2R\cos t \end{gathered}$$

Figure 4.1: Change of variables

$$\begin{aligned}
&\int_{B(1-R,R)} |1+z|^q(1-|z|^2)^\alpha \, dA(z) = \int_{B(R,R)} |2-z|^q(1-|z-1|^2)^\alpha \, dA(z) \\
&= 2\int_0^{\frac{\pi}{2}} \int_0^{2R\cos t} \left(4 - 4r\cos t + r^2\right)^{\frac{q}{2}} (2r\cos t - r^2)^\alpha r \, dr \frac{dt}{\pi} \\
&= 2\int_0^{\frac{\pi}{2}} \int_0^{R} \left(4 - 8r\cos^2 t + 4r^2\cos^2 t\right)^{\frac{q}{2}} (4r\cos^2 t - 4r^2\cos^2 t)^\alpha 4\cos^2 t \; r \, dr \frac{dt}{\pi} \\
&= 2^{q+3+2\alpha} \int_0^{\frac{\pi}{2}} \int_0^{R} \left(1 - 2r\cos^2 t + r^2\cos^2 t\right)^{\frac{q}{2}} (\cos t)^{2\alpha+2} r^{1+\alpha}(1-r)^\alpha \, dr \frac{dt}{\pi} \\
&= 2^{q+3+2\alpha} \int_0^{\frac{\pi}{2}} \int_0^{R} \left(1 - \cos^2 t + (1-r)^2\cos^2 t\right)^{\frac{q}{2}} (\cos t)^{2\alpha+2} r^{1+\alpha}(1-r)^\alpha \, dr \frac{dt}{\pi} \\
&= 2^{q+3+2\alpha} \int_0^{\frac{\pi}{2}} \int_{1-R}^{1} \left(1 - \cos^2 t + r^2\cos^2 t\right)^{\frac{q}{2}} (\cos t)^{2\alpha+2} (1-r)^{1+\alpha} r^\alpha \, dr \frac{dt}{\pi} \\
&= 2^{q+2+2\alpha} \int_0^1 \int_0^1 \chi_{]0,1-(1-R)^2[}(r)(1-rt)^{\frac{q}{2}} t^{\alpha+1} r^{1+\alpha} \left(\frac{1-\sqrt{1-r}}{r}\right)^{1+\alpha} (\sqrt{1-r})^{\alpha-1} \frac{dr\,dt}{2\pi\sqrt{1-t}\sqrt{t}} \\
&= \frac{2^{q+1+2\alpha}}{\pi} \int_0^1 \int_0^1 (1-rt)^{\frac{q}{2}} t^{\alpha+\frac{1}{2}} r^{1+\alpha} \left(\chi_{]0,1-(1-R)^2[}(r) \left(\frac{1-\sqrt{1-r}}{r}\right)^{1+\alpha} (\sqrt{1-r})^{\alpha-1} \right) \frac{dr\,dt}{\sqrt{1-t}}.
\end{aligned}$$

By Lemma 4.0.1, it holds that

$$\begin{aligned}
&\int_{B(1-R,R)} |1+z|^q(1-|z|^2)^\alpha \, dA(z) \\
&\overset{q\to\infty}{\sim} \frac{2^{q+1+2\alpha}}{\pi} \int_0^1 r^{-\frac{1}{2}} \chi_{]0,1-(1-R)^2[}(r) \left(\frac{1-\sqrt{1-r}}{r}\right)^{1+\alpha} (\sqrt{1-r})^{\alpha-1} \, dr \frac{\Gamma(\alpha+\frac{3}{2})}{\left(\frac{q}{2}\right)^{\alpha+\frac{3}{2}}}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{2^{q+\frac{5}{2}+3\alpha}}{\pi}\frac{\Gamma(\alpha+\frac{3}{2})}{q^{\alpha+\frac{3}{2}}}\int_0^{1-(1-R)^2}\left(1-\sqrt{1-r}\right)^{1+\alpha}\frac{(\sqrt{1-r})^{\alpha-1}}{r^{\alpha+\frac{3}{2}}}\,dr\\
&= \frac{2^{q+\frac{9}{2}+3\alpha}}{\pi}\frac{\Gamma(\alpha+\frac{3}{2})}{q^{\alpha+\frac{3}{2}}}\int_{\frac{1}{2}}^{\frac{1}{2(1-R)}}\frac{dr}{\sqrt{2r-1}(1+2r)^{\alpha+\frac{3}{2}}}\\
&= \frac{2^{q+\frac{5}{2}+2\alpha}}{\pi}\frac{\Gamma(\alpha+\frac{3}{2})}{q^{\alpha+\frac{3}{2}}}\int_{\frac{1}{2}}^{\frac{1}{2(1-R)}}\frac{dr}{\sqrt{r-\frac{1}{2}}(\frac{1}{2}+r)^{\alpha+\frac{3}{2}}}\\
&= \frac{2^{q+\frac{5}{2}+2\alpha}}{\pi}\frac{\Gamma(\alpha+\frac{3}{2})}{q^{\alpha+\frac{3}{2}}}\int_{\frac{2-2R}{2-R}}^{1}\frac{r^\alpha\,dr}{\sqrt{1-r}}.
\end{aligned}
$$

□

Theorem 4.0.3. *Let $\alpha > -1$ and $0 < R \le 1$. Then*

$$
\int_{B(1-R,R)}|1-z|^{-c}(1-|z|^2)^\alpha\,dA(z) \overset{c\to 2+\alpha}{\sim} \frac{2^\alpha}{2+\alpha-c}\frac{1}{\pi}\beta\left(\frac{\alpha}{2}+\frac{1}{2},\frac{1}{2}\right) \quad \left(=\frac{\left(\frac{\Gamma(\alpha+1)}{\Gamma(\frac{\alpha}{2}+1)^2}\right)}{2+\alpha-c}\right).
$$

Proof. First, for every $0<\lambda<1$ and $c<2+\alpha$,

$$
\begin{aligned}
\frac{\int_\lambda^1 r^{1+\alpha-c}(1-r)^\alpha\,dr}{\int_0^1 r^{1+\alpha-c}(1-r)^\alpha\,dr} &\le \frac{\int_\lambda^1 \max\{1,\lambda^{1+\alpha-c}\}(1-r)^\alpha\,dr}{\int_0^1 r^{1+\alpha-c}(1-r)^\alpha\,dr}\\
&\le \frac{\max\{1,\frac{1}{\lambda}\}(1-\lambda)^{\alpha+1}}{(\alpha+1)\beta(2+\alpha-c,\alpha+1)} \overset{c\to 2+\alpha}{\to} 0.
\end{aligned} \tag{4.0.5}
$$

Take $0<\rho<1$ and choose $0<\delta<1$ such that $(1-r)^\alpha \in B_{\mathbb{R}}(1,\rho)$ whenever $0<r<\delta$. Similarly to the previous section, it holds that

$$
\begin{aligned}
&\int_{B(1-R,R)}|1-z|^{-c}(1-|z|^2)^\alpha\,dA(z) = \int_{B(R,R)}|z|^{-c}(1-|z-1|^2)^\alpha\,dA(z)\\
&= 2\int_0^{\frac{\pi}{2}}\int_0^{2R\cos t} r^{-c}(2r\cos t-r^2)^\alpha r\,dr\frac{dt}{\pi}\\
&= 2\int_0^{\frac{\pi}{2}}\int_0^{R}(2\cos t)^{-c}r^{-c}(4r\cos^2 t-4r^2\cos^2 t)^\alpha 4\cos^2 t\; r\,dr\frac{dt}{\pi}\\
&= \frac{2^{3+2\alpha-c}}{\pi}\left(\int_0^{\frac{\pi}{2}}(\cos t)^{2\alpha+2-c}\,dt\right)\left(\int_0^R r^{1+\alpha-c}(1-r)^\alpha\,dr\right)\\
&= \frac{2^{3+2\alpha-c}}{\pi}\left(\int_0^1 t^{\alpha+1-\frac{c}{2}}\frac{dt}{2\sqrt{1-t}\sqrt{t}}\right)\left(\int_0^R r^{1+\alpha-c}(1-r)^\alpha\,dr\right)\\
&= \frac{2^{2+2\alpha-c}}{\pi}\beta\left(\alpha+\frac{3}{2}-\frac{c}{2},\frac{1}{2}\right)\left(\int_0^R r^{1+\alpha-c}(1-r)^\alpha\,dr\right)
\end{aligned}
$$

$$\underset{(4.0.5)}{\overset{c\to 2+\alpha}{\sim}} \frac{2^\alpha}{\pi}\beta\left(\frac{\alpha}{2}+\frac{1}{2},\frac{1}{2}\right)\left(\int_0^\delta r^{1+\alpha-c}(1-r)^\alpha\,dr\right)$$
$$\in B_{\mathbb{R}}(1,\rho)\frac{2^\alpha}{\pi}\beta\left(\frac{\alpha}{2}+\frac{1}{2},\frac{1}{2}\right)\int_0^\delta r^{1+\alpha-c}\,dr$$
$$= B_{\mathbb{R}}(1,\rho)\frac{2^\alpha}{\pi}\beta\left(\frac{\alpha}{2}+\frac{1}{2},\frac{1}{2}\right)\frac{\delta^{2+\alpha-c}}{2+\alpha-c}.$$

The conclusion is that

$$\operatorname*{lim^*}_{c\to 2+\alpha}\frac{\int_{B(1-R,R)}|1-z|^{-c}(1-|z|^2)^\alpha\,dA(z)}{\frac{2^\alpha}{\pi}\beta(\frac{\alpha}{2}+\frac{1}{2},\frac{1}{2})\frac{1}{2+\alpha-c}} = \operatorname*{lim^*}_{c\to 2+\alpha}\frac{\int_{B(1-R,R)}\frac{(1-|z|^2)^\alpha}{|1-z|^c}\,dA(z)}{\frac{2^\alpha}{\pi}\beta(\frac{\alpha}{2}+\frac{1}{2},\frac{1}{2})\frac{\delta^{2+\alpha-c}}{2+\alpha-c}} \in B_{\mathbb{R}}(1,\rho),$$

where $\lim^*$ can be either $\liminf$ or $\limsup$. Let $\rho\to 0$ to obtain the statement. □

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ISBN 978-952-12-4290-8