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Applications of Max-Plus Algebra to Scheduling

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Preface

There are some people that I would like to thank for their contribution to this thesis and to the research that will be provided by it.

First, I want to express my gratitude and appreciation to my supervisors Prof. Göran Högnäs, who has introduced me to the area of max-plus algebra and discrete event systems and Dr. Jari Böling. I am most grateful for his infinite patience, which is more than I deserved. I want to thank them both especially for their contagious enthusiasm and creativity, for the pleasant and fruitful cooperation, for the support and the guidance they have given me during this project, and for the fact that they have encouraged me to continue my research on this topic.

I am also grateful to my colleagues for their support and assistance for scientific advice both at the theoretical and practical level.

Furthermore, I would like to express my gratitude to my family (wife and children), my mother, my brothers and the other members of my family that have supported and encouraged me during the realization of this thesis.

Finally, I would also like to thank my friends for their support and encouragement.

Abstract

Max-plus algebra provides mathematical theory and techniques for solving nonlinear problems that can be given the form of linear problems, when arithmetical addition is replaced by the operation of maximum and arithmetical multiplication is replaced by addition. Problems of this kind are sometimes of a managerial nature, arising in areas such as manufacturing, transportation, allocation of resources and information processing technology. Max-plus algebra also provides the linear-algebraic background to the rapidly developing field of tropical mathematics.

The aim of this thesis is to provide an introductory text to max-plus algebra and to present results on advanced topics and, in particular, how it is useful in applications. An overview of the basic notions of the max-plus algebra and max-plus linear discrete event systems (DES) is presented.

Train networks can be modelled as a directed graph, in which nodes correspond to arrivals and departures at stations, and arcs to traveling times. A particular difficulty is represented by meeting conditions in a single-track railway system. The stability and sensitivity of the timetable is analyzed, and different types of delays and delay behavior are discussed. Interpretation of the recovery matrix is also done. A simple train network with real-world background is used for illustration. Compared to earlier work, which typically includes numerical optimization, this study is fully done by using max-plus algebra.

In this thesis, the scheduling of production systems consisting of many stages and different units is considered, where some of the units can be used for various stages. If a production unit is used for various stages cleaning is needed in between, while no cleaning is needed between stages of the same type. Cleaning of units takes a significant amount of time, which is considered in the scheduling. The goal is to minimize the total production time, and such problems are often solved by using numerical optimization. In this thesis, the possibilities for using max-plus for the scheduling are investigated. Structural decisions, such as choosing one unit over another, proved to be difficult. Scheduling of a small production system consisting of 6 stages and 6 units is used as a case study.

Traffic systems, computer communication systems, production lines, and flows in networks are all based on discrete event systems and, thus, can be conveniently described and analyzed by means of max-plus algebra. Max-plus formalism can be used for modeling of train network and production systems.

Svensk sammanfattning

Max-plusalgebran tillhandhåller matematisk teori och teknik för lösning av icke-linjära problem som kan ges linjär form genom att vanlig aritmetisk addition ersätts av maximumoperationen medan aritmetisk multiplikation ersätts av addition. Problem av detta slag är ofta av organisatorisk natur. De uppträder på områden som tillverkningsindustri, transport, resurstilldelning och informationsbehandling. Maxalgebran utgör även den linjär-algebraiska bakgrunden till det snabbt växande området tropisk matematik.

Ändamålet med denna avhandling är att tillhandahålla en inledning till max-plusalgebran och presentera resultat av mer avancerad natur och i synnerhet visa hur den är användbar i tillämpningar. Grundbegreppen i max-plusalgebran och teorin för maxpluslinjära händelsedrivna system (Discrete Event Systems, DES) presenteras.

Tågnätverk kan modelleras som en orienterad graf där noderna representerar ankomster till och avgångar från stationer, medan kanterna svarar mot restider mellan stationerna. En speciell svårighet innebär modelleringen av enspåriga tågsystem där tåg gående i olika riktningar måste mötas. En tågtidtabells stabilitet och känslighet diskuteras, liksom olika typer av förseningar och strategier för att korrigera dessa. Återställningsmatrisen presenteras och tolkningen av den diskuteras. Teorin illustreras med hjälp av ett enkelt tågnätverk med verklighetsbakgrund.

En viktig tillämpning är tidsoptimeringen av produktionssystem bestående av många olika stadier och olika produktionsenheter (maskiner). Av enheterna kan en del användas för olika stadier i processen. I så fall måste de dock rengöras mellan de olika produktionsskedena. Däremot krävs ingen rengöring om enheten inte byter uppgift. Rengöringen tar en viss tid som måste beaktas i modelleringen. Målet är att minimera den totala produktionstiden. Detta har i litteraturen oftast gjorts med numerisk optimering. I denna avhandling har möjligheten att använda maxplusteknik undersökts. Strukturella beslut, såsom att besluta vid vilken tidpunktbyte av uppgift (och rengöring) ska göras, visade sig svåra att direkt modellera som ett maxplusproblem. För hela produktionsprocessen utvecklades därför ett hybridsystem med maxplusalgebraiska subproblem som central ingrediens. Tidtabellen för ett litet produktionssystem med 6 produktionsskeden och 6 produktionsenheter illustrerar tekniken.

Trafiksystem, datakommunikationssystem, produktionssystem och nätverksflöden baserar sig på DES och kan därför med fördel beskrivas och analyseras med hjälp av max-plusalgebra.

Outline of the Thesis

The thesis starts in Chapter 1 with an introduction to max-plus algebra. More specifically, we introduce the basic algebraic concepts and properties of max-plus algebra. The emphasis of the chapter is on modeling issues, that is, we will discuss what kind of discrete event systems can be modeled by max-plus algebra.

Chapter 2 deals with three different parts, part one deals with solvability of linear systems such as $A \otimes x = b$ and linear independence and dependence, part two with max-plus linear equations, finding the eigenvalues and eigenvectors by different methods (maximum cycle mean method and power method) and part three deals with max-plus linear discrete event systems and a real application, namely problems in railway networks and simple manufacturing systems. Analogue to characteristic equation and the Cayley–Hamilton theorem in max-plus algebra are introduced.

Chapter 3 discusses modeling and scheduling of a train network that can be modelled as a directed graph, in which vertices correspond to arrivals and departures at stations, and arcs to traveling times. A particular difficulty is represented by meeting conditions in a single-track railway system. Compared to earlier work which typically includes numerical optimization, max-plus formalism is used throughout this chapter. The stability and sensitivity of the timetable is analyzed, and different types of delays and delay behavior are discussed and simulated. Interpretation of the recovery matrix is also done. A simple train network with real-world background is used for illustration.

In Chapter 4, the scheduling of production systems consists of many stages and different units are considered, where some of the units can be used for multiple stages. If a production unit is used for different stages, cleaning is needed in between, while no cleaning is needed between stages of the same type. Cleaning of units takes a significant amount of time, which is considered in the scheduling. The goal is to minimize the total production time, and such problems are often solved by using numerical optimization. In this chapter, max-plus formalism is used for modeling of such production systems, and the possibilities for using max-plus for the scheduling are also investigated. Structural decisions such as choosing one unit over another proved to be difficult. Scheduling of a small production system consisting of 6 stages and 6 units is used as a case study.

Chapter 5 reviews various stochastic extensions and the ergodic theory for stochastic max-plus linear systems. The common approaches are discussed, and the chapter may serve as a reference to max-plus ergodic theory.

Chapter 6 discusses the general conclusion.

List of Publications

This thesis is written as a monograph, but two chapters of it are related to the following manuscripts

Al Bermanei H, Böling JM, and Högnäs G (2016), Modeling and simulation of train networks using max-plus algebra. *The 9th EUROSIM Congress on Modelling and Simulation EUROSIM 2016*, Oulu, Finland, DOI: 10.3384/ecp17142: pp. 612-618.

Al Bermanei H, Böling JM, and Högnäs G (2017), Modeling and scheduling of production system by using max-plus algebra. *International Conference on Innovative Technologies (In-Tech 2017)*, Ljubljana, Slovenia, Vol. 4, SI-1291: pp.37-40. Awarded for science technology transfer at the conference (awarded one of the three best articles at the conference).

List of Symbols

ϕ	the empty set
N	set of natural numbers: $N = \{0, 1, 2, \dots \dots\}$
Z	set of integers
Q	set of rational numbers
$\mathbb{R}$	set of real numbers
C	set of complex numbers
P_n	set of permutations of the set $\{1, 2, \dots \dots., n\}$
P_n^e	set of even permutations of the set $\{1, 2, \dots \dots., n\}$
P_n^o	set of odd permutations of the set $\{1, 2, \dots \dots., n\}$

Matrices and Vectors

$\mathbb{R}^{m \times n}$	set of the *m* by *n* matrices with real entries
$\mathbb{R}^n$	set of real column vectors with *n* components
A^T	transpose of the matrix *A*
I_n	*n* by *n* identity matrix
a_i	i^{th} component of the vector a
a_{ij} , $(A)_{ij}$	entry of the matrix *A* on the i^{th} row and the j^{th} column

Max-Plus Algebra

$\oplus$	max-algebraic addition
$\otimes$	max-algebraic multiplication
ε	zero element in a semiring; in the max-plus semiring $\varepsilon = -\infty$
e	unit element in a semiring; in the max-plus semiring $e = 0$
$x^{\otimes n}$	max-algebraic power of *x*
I_n	*n* by *n* max-algebraic Identity matrix
$\varepsilon_{m \otimes n}$	*m* by *n* max-algebraic zero matrix
$A^{\otimes n}$	n^{th} max-algebraic power of the matrix *A*

$\mathbb{R}_{\max}$	max-plus algebra: $\mathbb{R}_{max} = \mathbb{R} \cup \{-\infty\}$
$G(A)$	precedence graph of the matrix A
$\underline{n}$	the set $\{1, \dots \dots, n\}$ for $n \in N \backslash \{0\}$
λ	the eigenvalue and in the stochastic case the Lyapunov exponent of $\{A(k): k \epsilon N\}$
λ^{top}	the top Lyapunov exponent of $\{A(k): k \epsilon N\}$
λ^{bot}	the bottom Lyapunov exponent of $\{A(k): k \epsilon N\}$
$\|\|A\|\|_{\max}$	the maximal finite element of matrix A
$\|\|A\|\|_{\min}$	the minimal finite element of matrix A

Contents

Chapter 1

Introduction

Max-plus algebra has been an active area of study since the 1970's. Much of the first interest and motivation in this area of mathematics can be seen in connection to the modeling and simulating of discrete event systems typically arising in areas involving allocation of resources.

The main motivation for this dissertation is to introduce the fascinating mathematical theory and then show how max-plus algebra fits naturally into the description and analysis of, e.g., graph theoretical problems and, in particular, can be used with great efficiency in scheduling applications. To make the presentation self-contained much introductory material on max-plus algebra is included in the first two chapters. This part is almost of textbook or lecture notes character. The exposition follows well-known works such as [1], [6] and [7]. Further references are given throughout the text. Chapter 5 is a brief introduction to stochastic max-plus systems with examples drawn from the previous material. Chapter 3 and 4 contain our main contribution. A deeper analysis of two scheduling applications is made: a train schedule and a production schedule in a manufacturing process, respectively. Even if the max-plus formalism is an efficient tool, great care must be given to system modeling. In Chapter 4, the final procedure turns out to be an interplay between many different methods with max-plus algebra in a key role. The proposed procedure is much faster and more versatile than the previously used optimization methods of [33].

1.1 Brief History of Max-plus

In max-plus algebra, we work with the max-plus semi-ring, which is the $\mathbb{R}_{\max} = \mathbb{R}\cup\{-\infty\}$ and the two binary operations addition $\oplus$ and multiplication $\otimes$, which are defined by:

$$a\oplus b = \max(a, b)\,,\ \ a\otimes b = a + b, \text{for all } a, b \in \mathbb{R}_{\max} \text{ and } \ (-\infty) + a = -\infty.$$

Furthermore, let $\varepsilon = -\infty$ and $e = 0$, the additive and multiplicative identities respectively. The operations $\oplus$ and $\otimes$ are associative, commutative and distributive as in conventional algebra.

Example 1.1

$$5 \oplus 3 = \max(5, 3) = 5, \qquad 5 \otimes 3 = 5 + 3 = 8$$

$$5 \oplus \varepsilon = \max(5, -\infty) = 5, \qquad 5 \otimes \varepsilon = 5 + (-\infty) = 5 - \infty = -\infty = \varepsilon$$

$$5 \otimes e = 5 + 0 = 5, \qquad 5 \oplus e = \max(5,0) = 5$$

$$e \oplus 3 = \max(0, 3) = 3 \quad \text{and} \quad e \oplus (-3) = \max(0, -3) = 0 = e$$

Max-plus algebra is one of many idempotent semi-rings, which have been considered in different fields of mathematics. Another one is min-plus algebra, the $\oplus$ means minimum and the additive identity is ∞. We shall here consider max-plus algebra only. It first appeared in 1956 in Kleene's paper and this paper has found applications in many areas such as mathematical physics, algebraic geometry, and optimization. It is also used in control theory, machine scheduling, discrete event processes, queuing systems, manufacturing systems, telecommunication networks, parallel processing systems and traffic theory. Many equations that are used to describe the behavior of these applications are nonlinear in conventional algebra but become linear in max-plus algebra. This is the main reason for its usefulness in various fields. Many of the theorems and techniques used in conventional linear algebra have counterparts in the max-plus semi-ring. Cuninghame-Green [9], Gaubert [4, 5], Gondran and Minoux [39] are among the researchers who have devoted a considerable amount of time to create a great deal of the max-plus linear algebra theory we have today. Many of Cuninghame-Green's results are found in [9]. They have studied concepts such as solving systems of linear equations, the eigenvalue problem, and linear independence in the max-plus sense.

In the coming chapters, we shall notice the extent to which max-plus algebra is an analogue of traditional linear algebra and look at many max-plus counterparts of conventional results.

Example 1.2 Consider the railroad network between two cities [2]. This is an example of how max-plus algebra can be applied to a discrete event system. Assume we have two cities, S_1 being the station in the first city, and S_2 the station in the second city. This system contains 4 trains. The time it takes a train to go from S_1 to S_2 is 3 hours where the train travels along track 1. It takes 5 hours to go from S_2 to S_1 where the train travels along track 2. These tracks can be referred to as long-distance tracks. There are two more tracks in this network, one of which runs through city 1 and one of which runs through city 2. We can refer to these as the inner-city tracks. Call them tracks 3 and 4 respectively. We can picture track 3 as a loop beginning and ending at S_1. Similarly, track 4 starts and ends at S_2. The time it takes to traverse the loop on track 3 is 2 hours. The time it takes to travel from S_2 to S_2 on track 4 is 3 hours. Track 3 and track 4 each contains a train. Two trains circulate along the two long-distance tracks. In this network, we also have the following criteria:

1. The travel times along each track indicated above are fixed
2. The frequency of the trains must be the same on all four tracks
3. Two trains must leave a station simultaneously in order to wait for the changeover of passengers
4. The two $(k + 1)^{st}$ trains leaving S_i cannot leave until the k^{th} train that left the other station arrives at S_i.

$x_i(k-1)$ will denote the k^{th} departure time for the two trains from station i. Therefore, $x_1(k)$ denotes the departure time of the pair of $k+1$ trains from S_1 and $x_2(k)$ is the departure time of the $k+1$ trains from S_2. $x(0)$ is a vector denoting the departure times of the first trains from S_1 and S_2. Thus, $x_1(0)$ denotes the departure time of the first pair of trains from station 1 and likewise $x_2(0)$ denotes the departure time of the first pair of trains from station 2 [2]. See Figure 1.1.

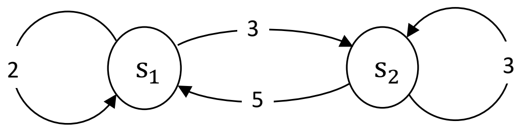

Figure 1.1: Train problem

If we want to determine the departure time of the k^{th} trains from station 1, then we can see that

$x_1(k + 1) \geq x_1(k) + a_{11} + \delta$ and

$x_1(k + 1) \geq x_2(k) + a_{12} + \delta,$

where a_{ij} denotes the travel time from station j to station i and δ is the time allowed for the passengers to get on and off the train. Thus, in our situation we have:

$$a_{11} = 2, a_{22} = 3, a_{12} = 5 \text{ and } a_{21} = 3.$$

We will assume $\delta = 0$ in this example. Thus, it follows that

$$x_1(k + 1) = \max\{x_1(k) + a_{11}, x_2(k) + a_{12}\}.$$

Similarly, we can see that

$$x_2(k + 1) = \max\{x_1(k) + a_{21}, x_2(k) + a_{22}\}.$$

In conventional algebra we would determine the successive departure times by iterating the nonlinear system,

$$x_i(k+1) = \max_{j=1,2,\ldots n}\{a_{ij} + x_j(k)\}$$

In max-plus we can write this as:

$$x_i(k+1) = \oplus_{j=1}^{n}\{a_{ij}\otimes x_j(k)\}, j = 1,2, \dots \dots, n,$$

where

$$\oplus_{j=1}^{n}\{a_{ij}\otimes x_j(k)\} = (a_{i1}\otimes x_1)\oplus(a_{i2}\otimes x_2)\oplus \dots \dots \dots \oplus(a_{in}\otimes x_n) \text{ for }, i = 1,2, \dots \dots, n.$$

In the example we have, $x_1(1) = 0 \oplus 5 = 5$ and $x_2(1) = 1 \oplus 3 = 3$, provided we are given $x_1(0) = -2$ and $x_2(0) = 0$.

Thus, $A = \begin{pmatrix} 2 & 5 \\ 3 & 3 \end{pmatrix}$ and $x(0) = \begin{pmatrix} -2 \\ 0 \end{pmatrix}$.

We can express this system using matrices and vectors such that $x(k) = A\otimes x(k-1)$. So, $x(1) = A\otimes x(0), x(2) = A\otimes x(1) = A\otimes A\otimes x(0) = A^{\otimes 2}\otimes x(0)$. Thus, in general

$$x(k) = A^{\otimes k}\otimes x(0).$$

This gives us a simple example of how a system of equations, which is not linear in the conventional algebra, is linear in max-plus algebra.

Example 1.3 Consider two flights from airports A and B arriving at a major airport C from which two other connecting flights depart [20].

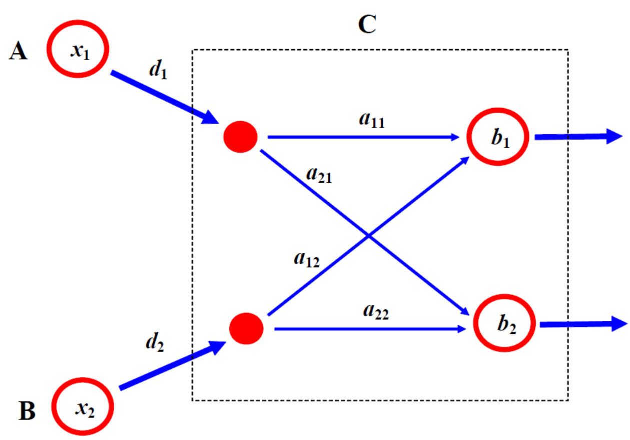

Figure 1.2: Transfer between connecting flights

The airport has many gates and the transfer time between them is nontrivial. Departure times from *C* are given and cannot be changed; for the above-mentioned flights, they are b_1 and b_2. The transfer times between the two arrival and two departure gates are given in the matrix

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}.$$

Durations of the flights from A to C and B to C are d_1 and d_2, respectively.

The task is to determine the departure times x_1 and x_2 from A and B, respectively, so that all passengers arrive at the departure gates on time, but as close as possible to the closing times (Figure 1.2). We can express the gate closing times in terms of departure times from airports A and B,

$b_1 = \max(x_1 + d_1 + a_{11}, x_2 + d_2 + a_{12}),$

$b_2 = \max(x_1 + d_1 + a_{21}, x_2 + d_2 + a_{22}).$

In max-plus algebraic notation, this system gets a more concise succinct form of a system of linear equations: $b = B \otimes x$, and the matrix $B = A \otimes d$ where

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \ d = \begin{pmatrix} d_1 & \varepsilon \\ \varepsilon & d_2 \end{pmatrix} \text{ and } b = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}.$$

We will see in Section 2.1 how to solve such systems. For those that have no solution, Section 2.1 provides a simple max-plus algebraic technique for finding the solution to the inequality $B \otimes x \le b$.

1.2 Definitions and Basic Properties

In this section we introduce max-plus algebra, give the essential definitions and study the concepts that play a key role in max-plus.

Definition 1.1 We denote $\mathbb{R}_{\max} = \mathbb{R} \cup \{-\infty\}$ and the two binary operations addition $\oplus$ and multiplication $\otimes$, which are defined by:

$$a \oplus b = \max(a, b), a \otimes b = a + b, \text{for all } a, b \in \mathbb{R}_{\max} \text{ and } (-\infty) + a = -\infty.$$

Define $\varepsilon = -\infty$ and $e = 0$. The additive and multiplicative identities are, thus, ε and e respectively and the operations are associative, commutative and distributive as in conventional algebra. The possibility of working in a formally linear way is because the following statements hold for $a, b, c \in \mathbb{R}_{\max}$.

<u>For the addition $\oplus$</u>

- **Commutativity**

  For all $a, b \in \mathbb{R}_{\max}$, $a \oplus b = \max(a, b) = \max(b, a) = b \oplus a$

- **Associativity**

  For all $a, b, c \in \mathbb{R}_{\max}$, $a \oplus (b \oplus c) = \max(a, (b \oplus c))$

  $= \max(a(\max(b, c)) = \max(a, b, c)$

  $= \max(\max(\mathrm{a}, \mathrm{b}), c) = \max(a, b) \oplus c = (a \oplus b) \oplus c$

- **Zero Element**

  $a \oplus \varepsilon = \max(a, \varepsilon) = a = \max(\varepsilon, a) = \varepsilon \oplus a$

- **Idempotency of Addition**

  $a \oplus a = a$

  $a \oplus b = \max(a, b) = a$ or b

  $a \oplus b = \max(a, b) \geq a$

  $a \oplus b = a \iff a \geq b$

<u>For the multiplication ⊗</u>

- **Commutativity**

  For all $a, b \in \mathbb{R}_{\max}, a \otimes b = a + b = b + a = b \otimes a$

- **Associativity**

  For all $a, b, c \in \mathbb{R}_{\max}, a \otimes (b \otimes c) = (a \otimes b) \otimes c$

- **Unit Element**

  $$a \otimes e = a = e \otimes a$$

- **Zero Element**

  $$a \otimes \varepsilon = \varepsilon = \varepsilon \otimes a$$

- **Multiplicative Inverse**

  $a \otimes a^{-1} = e = a^{-1} \otimes a$ for all $a \in \mathbb{R}$

  $a \otimes a^{-1} = a + (-a)$ because $a^{-1} = -a$ for all $a \in \mathbb{R}$

  So $a \otimes a^{-1} = e = a^{-1} \otimes a$

  Of course, ε has no multiplicative inverse

- $(\mathrm{a} \oplus \mathrm{b}) \otimes c = (a \otimes c) \oplus (b \otimes c)$ Distributive law

Proof: $(\mathrm{a} \oplus \mathrm{b}) \otimes c = \max(a, b) \otimes c = \max(a, b) + c = \max(a + c, b + c)$

$= \max(a \otimes c, b \otimes c) = (a \otimes c) \oplus (b \otimes c)$

- If $a \geq b \Longrightarrow a \oplus c \geq b \oplus c$ and

  if $a \geq b \Longrightarrow a \otimes c \geq b \otimes c$

- If $a \otimes c \geq b \otimes c, c \in \mathbb{R} \Longrightarrow a \geq b$ cancellative law

Lemma 1.1: For any $a \in \mathbb{R}_{\max} \setminus \{\varepsilon\}$, a does not have an additive inverse.

Proof: Let $a \in \mathbb{R}_{max}$ and $a \neq \varepsilon$ such that a has an inverse with respect to $\oplus$.

Let b be the inverse of $\Longrightarrow a \oplus b = \varepsilon$, adding a to both sides gives,

$a \oplus (a \oplus b) = a \oplus \varepsilon = a$, using the associative property of $\oplus$ gives

$a = (a \oplus a) \oplus b = a \oplus b$, a is an idempotent, i.e. $a \oplus a = a$.

Hence $a = a \oplus b = \varepsilon$ which is contradiction because $a \neq \varepsilon$.

Thus, a does not have an additive inverse.

1.3 Matrices and Vectors in Max-plus Algebra

In this section, we are mainly concerned with systems of linear equations. There are two kinds of linear systems in $\mathbb{R}_{\max}$ for which we can compute solutions of $y = A \otimes x \oplus b$. We also study the spectral theory of matrices. There exist good notions of the eigenvalue and the eigenvector but there is often only one eigenvalue; this occurs when the precedence graph associated with the matrix is strongly connected.

1.3.1 Matrices

The set of $n \times m$ matrices where $n, m \in N$ over $\mathbb{R}_{\max}$, denote by $\mathbb{R}_{\max}^{m \times n}$, where *n* is the numbers of rows and *m* is the number of columns. We write the matrix $A \in \mathbb{R}_{\max}^{m \times n}$ as:

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$, the entry in the i^{th} row and j^{th} column of A is denoted a_{ij}

Definition 1.2:

- For all $A, B \in \mathbb{R}_{\max}^{n \times n}$, define their sum by:

  $(A \oplus B)_{ij} = a_{ij} \oplus b_{ij} = \max\,(a_{ij}, b_{ij})$

- For all $A \in \mathbb{R}_{\max}^{m \times k}, B \in \mathbb{R}_{\max}^{k \times n}$ define their product by:

  $(A \otimes B)_{ij} = \oplus_{l=1}^{k} \left(a_{il} \oplus b_{lj}\right) = \max_{l=\{1,2,\dots,k\}} \left(a_{il} + b_{lj}\right)$

The transpose of the matrix $A \in \mathbb{R}_{\max}^{m \times n}$ is, denoted by $A^T \in \mathbb{R}_{\max}^{n \times m}$ and is define as:

$$[A^T]_{ij} = [A]_{ji}$$

- The $n \times n$ identity matrix $I_n \in \mathbb{R}_{max}^{n \times n}$ in max-plus is defined as:

$$(I_n)_{ij} = \begin{cases} e & \text{if} \quad i = j \\ \varepsilon & \text{if} \quad i \neq j \end{cases}$$

- For $A \in \mathbb{R}_{max}^{n\times n}, I_n \otimes A = A \otimes I_n = A$
- The $n \times n$ zero matrix $\varepsilon_n \in \mathbb{R}_{max}^{n\times n}$ in max-plus is defined as:

$$(\varepsilon_n)_{ij} = \varepsilon \text{ for all } i, j$$

- For a square matrix A and positive integer n the n^{th} power of A is written as $A^{\otimes n}$ and it is defined by: $A^{\otimes n} = \underbrace{A \otimes A \otimes \ldots\ldots\ldots \otimes A}_{n \text{ times}}$
- For any matrix A and any scalar $\propto \in \mathbb{R}_{\max}$

$$[\propto \otimes A]_{ij} = \propto \otimes [A]_{ij}$$

Example 1.4: Let A, B be two 2×2 matrices in $\mathbb{R}_{\max}^{n\times n}$ where

$$A = \begin{pmatrix} 2 & e \\ 1 & 3 \end{pmatrix} \text{ and } B = \begin{pmatrix} 3 & -1 \\ 2 & 4 \end{pmatrix}, \text{ so}$$

$$\boldsymbol{A} \oplus \boldsymbol{B} = \begin{pmatrix} \max(2,3) & \max(e,-1) \\ \max(1,2) & \max(4,3) \end{pmatrix} = \begin{pmatrix} 3 & e \\ 2 & 4 \end{pmatrix} = \boldsymbol{B} \oplus \boldsymbol{A}.$$

And

$$\boldsymbol{A} \otimes \boldsymbol{B} = \begin{pmatrix} (2 \otimes 3) \oplus (e \otimes 2) & (2 \otimes -1) \oplus (e \otimes 4) \\ (1 \otimes 3) \oplus (3 \otimes 2) & (1 \otimes -1) \oplus (3 \otimes 4) \end{pmatrix}$$

$$= \begin{pmatrix} \max(2+3, 0+2) & \max(2-1, 0+4) \\ \max(1+3, 3+2) & \max(1-1, 3+4) \end{pmatrix} = \begin{pmatrix} \max(5,2) & \max(1,4) \\ \max(4,5) & \max(0,7) \end{pmatrix}$$

$$= \begin{pmatrix} 5 & 4 \\ 5 & 7 \end{pmatrix}$$

But $\boldsymbol{B} \otimes \boldsymbol{A} = \begin{pmatrix} (3 \otimes 2) \oplus (-1 \otimes 1) & (3 \otimes e) \oplus (-1 \otimes 3) \\ (2 \otimes 2) \oplus (4 \otimes 1) & (2 \otimes e) \oplus (4 \otimes 3) \end{pmatrix}$

$$= \begin{pmatrix} \max(3+2, -1+1) & \max(3+0, -1+3) \\ \max(2+2, 4+1) & \max(2+0, 4+3) \end{pmatrix} = \begin{pmatrix} \max(5,0) & \max(3,2) \\ \max(4,5) & \max(2,7) \end{pmatrix}$$

$$= \begin{pmatrix} 5 & 3 \\ 5 & 7 \end{pmatrix} \neq \boldsymbol{A} \otimes \boldsymbol{B}$$

In general, we can say that for some $A, B \in \mathbb{R}_{\max}^{n\times n}$, $A \otimes B \neq B \otimes A$ i.e. $\otimes$ is not commutative in $\mathbb{R}_{\max}^{n\times n}$

1.3.2 Properties of Matrix Operations

For the matrices, A, B, C in $\mathbb{R}_{\max}^{n\times n}$ we have:

<u>For the addition $\oplus$</u>

- Commutativity

  For all $A, B \in \mathbb{R}_{\max}^{n\times n}$, $A\oplus B = B\oplus A$

- Associativity

  For all $A, B, C \in \mathbb{R}_{\max}^{n\times n}$, $(A\oplus B)\oplus C = A\oplus(B\oplus C)$

- $A\oplus\varepsilon = A = \varepsilon \oplus A$

- We define $\geq$ by:

  $B \geq A$ if and only if $b_{ij} \geq a_{ij}$ for all i, j

  So, $A\oplus B \geq A$

- $A\oplus B = A \iff A \geq B$

<u>Proof</u>: Let $A = (a_{ij}), B = (b_{ij})$ for all i, j

1) If $A\oplus B = A \Rightarrow a_{ij}\oplus b_{ij} = a_{ij}$ for all i, j

   $\Rightarrow \max(a_{ij}, b_{ij}) = a_{ij} \Rightarrow a_{ij} \geq b_{ij} \Rightarrow A \geq B$

2) If $A \geq B = A \Rightarrow a_{ij} \geq b_{ij}$ for all i, j

   $\Rightarrow \max(a_{ij}, b_{ij}) = a_{ij} \Rightarrow a_{ij}\oplus b_{ij} = a_{ij} \Rightarrow A \oplus B = A$

<u>For the multiplication $\otimes$:</u>

- $A\otimes B \neq B\otimes A$ in general

- **Associativity**

  For all $A, B, C \in \mathbb{R}_{\max}^{n\times n}$, $A\otimes(B\otimes C) = (A\otimes B)\otimes C$

Proof: $A\otimes(B\otimes C) = \bigoplus_{k=1}^{n} \left[a_{ik}\otimes(B\otimes C)_{kj}\right] = \max\limits_{1\leq k\leq n}\left[a_{ik}\otimes(B\otimes C)_{kj}\right]$

$$= \max_{1\leq k\leq n}\left[(a_{ik} + \max_{1\leq k\leq n}(b_{kl} + c_{lj})\right]$$

$$= \max_{1\leq l\leq n}\left[\max_{1\leq k\leq n}(a_{ik} + b_{kl} + c_{lj})\right] = \max_{1\leq l\leq n}\left[\max_{1\leq k\leq n}(a_{ik} + b_{kl}) + c_{lj}\right]$$

$$= \max_{1\le l\le n}\left[(A \otimes B)_{il} + c_{lj}\right] = (A \otimes B) \otimes C$$

- **Unit Matrix**

  For all $A, I_n \in \mathbb{R}_{\max}^{n\times n}$, $A{\otimes}I_n = A = I_n{\otimes}A$

- **Zero Matrix**

  For all $A, \varepsilon_n \in \mathbb{R}_{\max}^{n\times n}$, $A{\otimes}\varepsilon_n = \varepsilon_n = \varepsilon_n \otimes A$
  Note that ε_n is the $n \times n$ zero matrix.

- **Distributivity**

  For all $A, B, C \in \mathbb{R}_{\max}^{n\times n}$, $(A{\oplus}B){\otimes}C) = (A{\otimes}C){\oplus}(B{\otimes}C$ and

$$A{\otimes}(B{\oplus}C) = (A{\otimes}B){\oplus}(A{\otimes}C)$$

Proof: $(A{\oplus}B){\otimes}C)_{ij} = \oplus_{j=1}^{n} a_{ij}{\otimes}(\max(b_{ij}, c_{ij}))$

$$= \max_j[\max\ (a_{ij} + b_{ij}),(a_{ij} + c_{ij})] = \max(\max_j (a_{ij} + b_{ij}) + \max_j(a_{ij} + c_{ij}))$$

$$= \max[(A{\otimes}B)_{ij}, (A{\otimes}C)_{ij}] = [(A{\otimes}B){\oplus}(A{\otimes}C)]_{ij} = (A{\otimes}B){\oplus}(A{\otimes}C)$$

And for all $\boldsymbol{a} \in \mathbb{R}$

i. $a \otimes (B \oplus C) = a \otimes B \oplus a \otimes C$ and

ii. $a \otimes (B \otimes C) = B \otimes a \otimes C$

proof: i. $a \otimes (B \oplus C) = a \otimes \max(b_{ij}, c_{ij}) = a + \max(b_{ij}, c_{ij}) = \max(a + b_{ij}, a + c_{ij})$

$$= \max(a \otimes B, a \otimes C) = (a \otimes B) \oplus (a \otimes C)$$

ii. $a \otimes (B \otimes C) = a \otimes (\oplus_{k=1}^{n} (b_{ik} + c_{kj}) = a \otimes (\max_{1\le k\le n} (b_{ik} + c_{kj}) = \max_{1\le k\le n} (a + b_{ik} + c_{kj})$

$$= \max(b_{ik} + a + c_{kj}) = B \otimes \left[\max(a + c_{kj})\right] = B \otimes a \otimes C$$

Definition 1.3 The set is $\mathbb{R}_{\max}$ with the two operations $\oplus$ and $\otimes$ is called a max-plus algebra and is denoted by $\mathbb{R}_{\max} = (\mathbb{R}, \oplus, \otimes, \varepsilon, e)$.

Definition 1.4 A semiring is a nonempty set $\mathbb{R}$ with two operations $\oplus$,$\otimes$, and two elements ε and e such that:

- $\oplus$ is associative and commutative with zero element ε;

- $\otimes$ is associative, distributes over $\oplus$, and has identity element e,
- ε is absorbing for $\otimes$ i.e. $a \otimes \varepsilon = \varepsilon \otimes a = \varepsilon$, for all $a \in \mathbb{R}$.

Such a semiring is denoted by $(\mathbb{R}, \oplus, \otimes, \varepsilon, e)$.

In addition if $\otimes$ is commutative then $\mathbb{R}$ is called a commutative semiring, and if $\oplus$ is such that $a \oplus a = \max(a, a) = a$, for all $a \epsilon\ \mathbb{R}$ then it is called idempotent.

Definition 1.5 (Subsemiring) let $(\mathbb{R}, \oplus, \otimes, \varepsilon, e)$ be an (idempotent) semiring. A subset $S \subseteq \mathbb{R}$ is a subsemiring of $(\mathbb{R}, \oplus, \otimes, \varepsilon, e)$ if

(i) $\varepsilon \in S$ and $e \in S$

(ii) S is closed under addition and multiplication, i.e. for all.$a, b \in S, a \oplus b \in S$ and $a \otimes b \in S$

A subsemiring S of a semiring $(\mathbb{R}, \oplus, \otimes, \varepsilon, e)$gain the addition and multiplication of the last and $(S, \oplus, \otimes, \varepsilon, e)$ is itself a semiring. If $(\mathbb{R}, \oplus, \otimes, \varepsilon, e)$ is idempotent (or commutative) then so are its subsemirings.

Theorem 1.1 The set is $\mathbb{R}_{\max} = (\mathbb{R}, \oplus, \otimes, \varepsilon, e)$ has the algebraic structure of a commutative and idempotent semiring.

Proof: The proof follows immediately using the definitions given by

$$a \oplus b = \max(a, b) \text{ and } a \otimes b = a + b, \text{for all } a, b \in \mathbb{R}_{\max}$$

(In a similar way to the case for addition and multiplication over the real numbers) just being careful when one substitutes multiplication for the max-plus operation.

As, for example, in the distributive property for $a, b, c \in \mathbb{R}_{\max}$ it holds that:

$$\begin{aligned} a \otimes (b \oplus c) &= a + (b \oplus c) \quad \text{by def. of } \otimes \\ &= a + \max(b, c) \quad \text{by def. of } \oplus \\ &= \max(a + b, a + c) \\ &= (a \otimes b) \oplus (a \otimes c) \quad \text{by def. of } \oplus \text{ and } \otimes. \end{aligned}$$

Since $(\mathbb{R}_{\max}, \oplus, \otimes)$ is a commutative idempotent semiring, many of the tools known from linear algebra are available in max-plus algebra as well. The neutral elements are different ε is neutral for $\oplus$ and e for $\otimes$. In the case of matrices, the neutral elements are the matrix (of appropriate dimensions) with all entries ε for $\oplus$ and I for $\otimes$. On the other hand, in contrast to linear algebra, the operation $\oplus$ is not invertible. However, $\oplus$ is idempotent and this provides the possibility of constructing alternative tools, such as transitive closures of matrices or conjugation, for solving problems such as the eigenvalue-eigenvector problem and systems of linear equations or inequalities. One of the most frequently used elementary property is isotonicity of both $\oplus$ and $\otimes$. This can be formulated in the following lemma.

Lemma 1.2 If A, B, C are matrices over $\mathbb{R}_{\max}$ of compatible size and $c \in \mathbb{R}_{\max}$ and we say that $A \geq B$ if $a_{ij} \geq b_{ij}$ for all i, j then

1. $A \geq B \Longrightarrow A \oplus C \geq B \oplus C$
2. $A \geq B \Longrightarrow A \otimes C \geq B \otimes C$
3. $A \geq B \Longrightarrow C \otimes A \geq C \otimes B$
4. $A \geq B \Longrightarrow c \otimes A \geq c \otimes B$

Proof:

1. $A \geq B \Longrightarrow a_{ij} \geq b_{ij}$ for all i, j

   $\Longrightarrow a_{ij} \oplus c_{ij} \geq b_{ij} \oplus c_{ij} \Longrightarrow A \oplus C = B \oplus C$
2. $A \geq B \Longrightarrow A \oplus B = A \Longrightarrow (A \oplus B) \otimes C = A \otimes C$

   $\Longrightarrow (A \otimes C) \oplus (B \otimes C) = A \otimes C$

   Hence $(A \otimes C) \geq (B \otimes C)$
3. $A \geq B \Longrightarrow A \oplus B = A$

   $\Longrightarrow C \otimes (A \oplus B) = C \otimes A \Longrightarrow (C \otimes A) \oplus (C \otimes B) = C \otimes A$

   Hence $(C \otimes A) \geq (C \otimes B)$
4. $A \geq B \Longrightarrow a_{ij} \geq b_{ij}$ for all i, j

   $\Longrightarrow c \otimes a_{ij} \geq c \otimes b_{ij} \Longrightarrow c \otimes A \geq c \otimes B$

Definition 1.6: $A^0 = I, A^{\otimes 1} = A, A^{\otimes 2} = A \otimes A$ and $A^{\otimes k} = A^{\otimes (k-1)} \otimes A$, where $A^{\otimes k} = A^k$.

Note that in case of A being invertible; we can extend the definition of A^k to negative powers.

As an example, consider the matrix A where $A = \begin{pmatrix} \varepsilon & 1 \\ 2 & \varepsilon \end{pmatrix}$.

So $A \otimes A^{-1} = I = \begin{pmatrix} 0 & \varepsilon \\ \varepsilon & 0 \end{pmatrix}$

If $A^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then $\begin{pmatrix} \varepsilon & 1 \\ 2 & \varepsilon \end{pmatrix} \otimes \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 0 & \varepsilon \\ \varepsilon & 0 \end{pmatrix}$

$$\begin{pmatrix} \varepsilon & 1 \\ 2 & \varepsilon \end{pmatrix} \otimes \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \varepsilon \otimes a \oplus 1 \otimes c & \varepsilon \otimes b \oplus 1 \otimes d \\ 2 \otimes a \oplus \varepsilon \otimes c & 2 \otimes b \oplus \varepsilon \otimes d \end{pmatrix}$$

$$= \begin{pmatrix} 1+c & 1+d \\ 2+a & 2+b \end{pmatrix} = \begin{pmatrix} 0 & \varepsilon \\ \varepsilon & 0 \end{pmatrix}$$

So, $1 + c = 0 \Rightarrow c = -1, 1 + d = \varepsilon \Rightarrow d = \varepsilon, 2 + a = \varepsilon \Rightarrow a = \varepsilon, 2 + b = 0 \Rightarrow b = -2$

$$\therefore A^{-1} = \begin{pmatrix} \varepsilon & -2 \\ -1 & \varepsilon \end{pmatrix}$$

Lemma 1.3 For every $A \in \mathbb{R}_{\max}^{n \times n}$ and for any nonnegative integer k

$$(I \oplus A)^k = I \oplus A \oplus A^2 \oplus \dots\dots\dots\dots \oplus A^k$$

Proof: We need to prove this lemma by using mathematical induction theorem

1. When $k = 1 \Longrightarrow (I \oplus A)^1 = I \oplus A$, so it is true when k=1
2. Suppose the statement is true for k

i.e. $(I \oplus A)^k = I \oplus A \oplus A^2 \oplus \dots \oplus A^k$, we need to show that the statement is true for $(k+1)$

i.e. we need to show that, by using idempotent property of $\oplus$

$$(I \oplus A)^{k+1} = I \oplus A \oplus A^2 \oplus \dots\dots \oplus A^k \oplus A^{k+1}$$

L.H.S = $(I \oplus A)^{k+1} = (I \oplus A)^k \otimes (I \oplus A)$

$= (I \oplus A \oplus A^2 \oplus \dots\dots \oplus A^k) \otimes (I \oplus A) = I \oplus A \oplus \dots\dots \oplus A^k \oplus A \oplus A^2 \oplus \dots\dots \oplus A^{k+1}$

$= I \oplus A \oplus A \oplus A^2 \oplus A^2 \oplus \dots\dots\dots \oplus A^k \oplus A^k \oplus A^{k+1}$

$= I \oplus A \oplus A^2 \oplus \dots\dots\dots \oplus A^k \oplus A^{k+1}$ = R.H.S

So it is true for $(k+1)$

By mathematical induction theorem

$(I \oplus A)^k = I \oplus A \oplus A^2 \oplus \dots\dots\dots\dots \oplus A^k$ for all nonnegative integers k

1.3.3 Vectors

The elements $x \in \mathbb{R}_{\max}^{n \times 1}$ are called vectors (or max plus vectors). The j^{th} coordinate of a vector x is denoted by x_j or $[x_j]$. The j^{th} column of the identity matrix I_n is known as the j^{th} basis vector of the $\mathbb{R}_{\max}^{n}$. This vector is denoted by $e_j = (\varepsilon, \varepsilon, \dots\dots, \varepsilon, e, \varepsilon, \varepsilon, \dots\dots, \varepsilon)^T$, i.e. e is the j^{th} entry of the vector.

1.4 Matrices and Graphs

In this section, we consider matrices with entries belonging to $\mathbb{R}_{\max}$ in which some algebraic operations are defined in section 1.2. Some relationships between these matrices and 'weighted graphs' will be introduced.

Consider a graph $G = (V, E)$ where V is a non-empty set of vertices (or nodes) and $E \subseteq V \times V$ (the set of edges (or set of arcs), and associate an element $A_{ij} \in \mathbb{R}_{\max}$ with each arc $(j, i) \in E$, then G is called a **weighted graph**. The quantity A_{ij} is called the weight of arc (j, i). Note that the second subscript of A_{ij} refers to the initial (and not the final) node. The reason is that, in the algebraic context, we will work with column vectors (and not with row vectors).

Definition 1.8 (Bipartite graph)[2]: If the set of vertices V of a graph G can be partitioned into two disjoint subsets V_1 and V_2 such that every arc of G connects an element of V_1 with one of V_2 or the other way around, then G is called bipartite graph.

Definition 1.9 (Transition graph)[2]: If an $m \times n$ matrix $A = (A_{ij})$ with entries in $\mathbb{R}_{\max}$ is given, the transition graph of A is a weighted bipartite graph with $n + m$ vertices, labeled $1, \ldots, m, m + 1, \ldots, m + n$, such that each row of A corresponds to one of the vertices 1, . . . , m and each column of A corresponds to one of the vertices $m + 1, \ldots, m + n$. An arc from j to $n + i, 1 \leq i \leq m, 1 \leq j \leq n$, is introduced with weight A_{ij} if $A_{ij} \neq \varepsilon$. If there is no arc, the corresponding weight is defined as ε.

As an example, consider the matrix A [2]. Where

$$A = \begin{pmatrix} 3 & \varepsilon & \varepsilon & 7 & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & 2 & \varepsilon & \varepsilon & \varepsilon & 1 \\ \mathrm{e} & \varepsilon & \varepsilon & 2 & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & 5 & \varepsilon \\ \varepsilon & 4 & \varepsilon & \varepsilon & \varepsilon & 8 & 6 \\ 4 & \varepsilon & 1 & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & e & \varepsilon & \varepsilon \end{pmatrix}$$

Its transition graph is depicted in Figure 1.3.

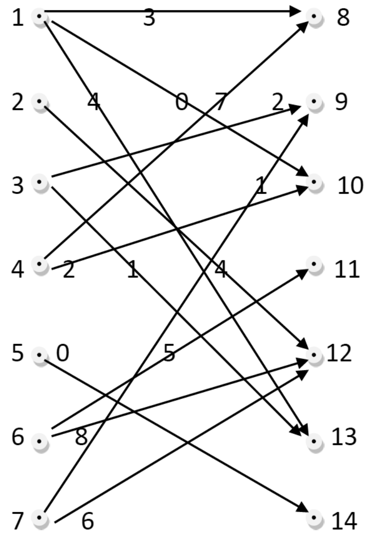

Figure 1.3: The transition graph of A

Definition 1.10 (Precedence graph)[2]: The precedence graph of a square $n \times n$ matrix A with entries in $\mathbb{R}_{\max}$ is a weighted directed graph with n vertices and an $\operatorname{arc}(j, i)$ if $A_{ij} \neq \varepsilon$, in which case the weight of this arc receives the numerical value of A_{ij}. The precedence graph is denoted $G\ (A)$.

As an example, consider the matrix

$$A = \begin{pmatrix} 2 & 6 & \varepsilon & -3 \\ 1 & \varepsilon & -1 & \varepsilon \\ \varepsilon & 5 & -2 & \varepsilon \\ \varepsilon & 4 & -2 & \varepsilon \end{pmatrix}$$

and the precedence graph of A is shown in Figure 1.4

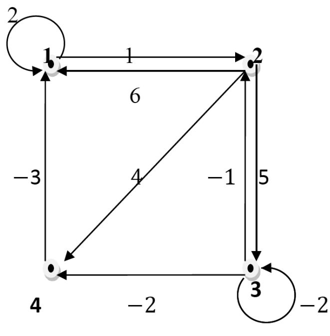

Figure 1.4: The precedence graph of A

Definition 1.11 (Adjacency matrix)[2]: The adjacency matrix $G = (G_{ij})$ of a graph $G = (V, E)$ is a matrix the numbers of rows and columns of which are equal to the number of vertices of the graph. The entry G_{ij} is equal to 1 if $(i, j) \in E$ and to 0 otherwise.

Note that if $G = G(A)$, then $G_{ij} = 1$ if and only if $A_{ji} \neq \varepsilon$, where $G(A)$ is the corresponding precedence graph

1.4.1 Matrices and Digraphs

We will sometimes use the language of directed graphs (digraphs) [20]. A digraph is a pair $D = (V, E)$ where V is a non-empty set of vertices (or nodes) and $E \subseteq V \times V$ the set of edges (or set of arcs).

A subgraph of D is any digraph $D' = (V', E')$ such that $V' \subseteq V$ and $E' \subseteq E$. If $e = (u, v) \in E$ for some $u, v \in V$ then we say that *e* is leaving *u* and entering *v*.

Any arc of the form (u, u) is called a loop.

Let $D = (V, E)$ be a given digraph. A sequence $\pi = (v_1, v_2, \dots \dots, v_p)$ of vertices in *D* is called a path (in *D*) if $p = 1$ or $p > 1$ and $(v_i, v_{i+1}) \in E$ for all $i = 1, \dots, p-1$.

The node v_1 is called the starting node and v_p the end node of π respectively.

The number $p - 1$ is called the length of π and will be denoted by $l(\pi)$. If u is the starting node and v is the end node of π then we say that π is a $u - v$ path. If there is a $u - v$ path in *D* then v is said to be reachable from u, notation $u \rightarrow u$. Thus, $u \rightarrow v$ for any $u \in V$. A path $(v_1, v_2, \dots \dots, v_p)$ is called a cycle if $v_1 = v_p$ and $p > 1$ and it is called an elementary cycle if, moreover, $v_i \neq v_j$ for $i, j = 1, \dots, p-1, i \neq j$ [20].

A digraph *D* is called strongly connected if $u \rightarrow v$ for all vertices u, v in D. A subdigraph D' of *D* is called a strongly connected component of D if it is a maximal strongly connected subdigraph of *D*, that is D' is a strongly connected subdigraph of *D* and if D' is a subdigraph of a strongly connected subdigraph D'', then $D' = D''$.

Note that a digraph consisting of one node and no arc is strongly connected and acyclic (a graph without cycles), however, if a strongly connected digraph has at least two vertices then it obviously cannot be acyclic. Because of this singularity we will have to assume in some statements that $|V| > 1$ [20].

If $= (a_{ij}) \in \mathbb{R}_{\max}^{n \times n}$, then the symbols F_A (Z_A) will denote the digraphs with the node set *N* and arc sets $E = \{(i, j);\ a_{ij} \neq \varepsilon\}$, ($E = \{(i, j);\ a_{ij} \neq e\}$).

F_A (Z_A) will be called the finiteness (zero) digraph of A. If F_A is strongly connected, then A is called irreducible and reducible otherwise.

Definition 1.12 We say that A is row (or column) R-astic if every row (or column) of A has a finite entry and A is doubly R-astic if every row and column of A has a finite entry [20].

Lemma 1.4 If $A = (a_{ij}) \in \mathbb{R}_{\max}^{n\times n}$ is irreducible and $n > 1$, then A is doubly R-astic.
Proof: It follows from irreducibility that an arc leaving and an arc entering a node exist for every node in F_A. Hence, every row and column of A has a finite entry [20].

Lemma 1.5 If $A = (a_{ij}) \in \mathbb{R}_{\max}^{n\times n}$ is row or column R-astic, then F_A contains a cycle [20].

Proof: Suppose that $A = (a_{ij}) \in \mathbb{R}_{\max}^{n\times n}$ is row R-astic and let $i_1 \in V$ be any node.
Then $a_{i_1 i_2} > \varepsilon$ for some $i_2 \in V$.
Similarly, $a_{i_2 i_3} > \varepsilon$ for some $i_3 \in V$ and so on. Hence, F_A has arcs $(i_1, i_2), (i_2, i_3), \dots \dots \dots$
By finiteness of V in the sequence $i_1, i_2, \dots \dots$ some i_r will eventually this proves the existence of a cycle in F_A.

A weighted digraph is $D = (V, E, w)$ where (V, E) is a digraph and $w: E \to R$. All definitions for digraphs are naturally extended to weighted digraphs. If $\pi = (v_1, v_2, \dots \dots, v_p)$ is a path in (V, E, w) then the weight of π is:

$$w(\pi) = w(v_1, v_2) + w(v_2, v_3) + \dots + w(v_{p-1}, v_p) \text{ if } p > 1 \text{ and } \varepsilon \text{ if } p = 1.$$

A path π is called positive if $w(\pi) > 0$. In contrast, $\sigma = (u_1, u_2, \dots \dots, u_p)$ is called a zero-cycle if $w(v_k, v_{k+1}) = 0$ for all $k = 1, \dots \dots, p - 1$, since w stands for the weight rather than length. Given $A = (a_{ij}) \in \mathbb{R}_{\max}^{n\times n}$ the symbol D_A will denote the weighted digraph (V, E, w) where $F_A = (V, E)$ and $w(i, j) = a_{ij}$ for all $(i, j) \in E$. If $\pi = (i_1, i_2, \dots \dots, i_p)$ is a path in D_A, then we denote $w(\pi, A) = w(\pi)$ and it now follows from the definitions that $w(\pi, A) = w(\pi) = a_{i_1 i_2} + a_{i_2 i_3} + \dots + a_{i_{p-1} i_p}$ if $p > 1$ and ε if $p = 1$.

Chapter 2

Max-plus Linear Equations

This chapter discusses max-plus linear algebra. We will see that many concepts of ordinary linear algebra have a max-plus version. Cuninghame-Green [9] and Gaubert [4, 5], are all contributors to the development of max-plus linear algebra. Specifically, we will consider the solvability of linear systems such as $A\otimes x = b$ and linear independence and dependence.

We will also study the eigenvalue and eigenvector problem. The main question is whether these conventional linear algebra concepts have max-plus versions and if so, how they are similar and / or different from conventional algebra results.

In max-plus algebra, the lack of additive inverses also causes difficulty when solving linear systems of equations such as $A\otimes x = b$. As in conventional algebra the solution to $A\otimes x = b$ does not always exist in max-plus algebra and if it does, it is not necessarily unique. We will explore other linear systems in max-plus algebra as well.

2.1 Solution of $A\otimes x = b$

In this section, we are mainly interested in systems of linear max-plus equations for which we can obtain a general solution that consists of the systems $A\otimes x = b$. However, we must first consider the problem in $\mathbb{R}_{\max}^{n\times n}$ rather than in $\mathbb{R}_{\max}$, and second, we must somewhat weaken the notion of 'solution'. A subsolution of $A\otimes x = b$ is an x which satisfies $A\otimes x \leq b$, where the order relation on the vectors can also be defined by

$$x \leq y \text{ if } x\oplus y = y.$$

Theorem 2.1[1] Given an $n \times n$ matrix A and an n-vector b with entries in $\mathbb{R}_{\max}$, the greatest subsolution of $A\otimes x = b$ exists and is given by

$$-x_j = \max_i(-b_i + a_{ij})$$

Proof: We have that

$$A \otimes x \leq b \Leftrightarrow \{\oplus_j \; a_{ij} \otimes x_j \leq b_i, \text{for all } i\} \Leftrightarrow \left\{\max_{1 \leq j \leq n} (a_{ij} + x_j) \leq b_i \text{ , for all } i\right\}$$

$$\Leftrightarrow \{a_{ij} + x_j \leq b_i \text{ , for all } i, j\} \Leftrightarrow \{x_j \leq b_i - a_{ij} \text{ , for all } i, j\}$$

$$\Leftrightarrow \left\{x_j \leq \min_i (b_i - a_{ij}) \text{ , for all } j\right\} \Leftrightarrow \left\{-x_j \geq \max_i (-b_i + a_{ij}) \text{ , for all } j\right\}$$

Conversely, it can be checked similarly that the vector x defined by $-x_j = \max_i(-b_i + a_{ij}), \; \forall j$ is a subsolution. Therefore, it is the greatest one.

Another alternative notation is given in *J.R.S. Dias et al*. [40] for solving the linear systems by using the residuation theory to determine the greatest solution (with respect to the natural order of the dioid) of the inequality $f(a) \leq b$.

Theorem 2.2 [40] We consider the complete dioid $\mathbb{R}_{\max}$ and $\mathbb{R}_{\max}^{m \times n}$, the dioid of matrices with elements in $\mathbb{R}_{\max}$, as well as the matrices $A \in \mathbb{R}_{\max}^{m \times n}$ and $B \in \mathbb{R}_{\max}^{m \times p}$. The least upper bound of the inequation $A \otimes X \leq B$ exists and is given by A\B. The elements of this matrix are calculated by:

$$(\text{A}\backslash\text{B})_{ij} = \bigwedge_{l=1}^{n} A_{li} \setminus B_{lj}$$

- $\wedge$ denotes the greatest lower bound of the elements
- $\backslash$ denotes a residuation operation
- $a \backslash b = b - a$

Example 2.1 Consider the relation $A \otimes X \leq B$ where

$A = \begin{pmatrix} 1 & 2 \\ 2 & 5 \\ 1 & 0 \end{pmatrix}, X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ and $B = \begin{pmatrix} 5 \\ 6 \\ 8 \end{pmatrix}$ which are $A = (a_{ij}) \in \mathbb{R}_{\max}^{m \times n}, \; X = (x_1, \ldots, x_n)^T \in \mathbb{R}_{\max}^{n \times p}$ and $B = (b_1, \ldots, b_m)^T \in \mathbb{R}_{\max}^{m \times p}$.

The max-plus multiplication corresponds to classical addition, so its residual corresponds to classical subtraction, i.e. $1 \otimes x \leq 4$ admits the solution set $X = \{x | x \leq 1\backslash 4\}$, where $1\backslash 4 = 4 - 1 = 3$ is the greatest solution of this set. Applying the rules of residuation in max-plus algebra to the relation $A \otimes X \leq B$ yields [40]

$$(\text{A}\backslash\text{B}) \begin{pmatrix} (1\backslash 5) & \wedge & (2\backslash 6) & \wedge & (1\backslash 8) \\ (2\backslash 5) & \wedge & (5\backslash 6) & \wedge & (0\backslash 8) \end{pmatrix} = \begin{pmatrix} 4 \\ 1 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

The matrix $A\backslash B = (4 \quad 1)^T$ is the greatest solution for X that ensures $A \otimes X \leq B$.

Thus, $A \otimes (A\backslash B) = \begin{pmatrix} 1 & 2 \\ 2 & 5 \\ 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 4 \\ 1 \end{pmatrix} = \begin{pmatrix} 5 \\ 6 \\ 5 \end{pmatrix} \leq \begin{pmatrix} 5 \\ 6 \\ 8 \end{pmatrix} = B$

and if we use Theorem 2.1 then we will obtain:

$-x_1 = \max(-5+1, -6+2, -8+1) = -4$

$-x_2 = \max(-5+2, -6+5, -8+0) = -1$

Hence, $X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$ which is the same as above.

If $A = (a_{ij}) \in \mathbb{R}_{\max}^{m \times n}$ and $b = (b_1, \ldots, b_m)^T \in \mathbb{R}^m$,

then the max-algebraic linear system $A \otimes x = b$ written in the conventional notation is the nonlinear system; $\max_{j=1,\ldots,n} (a_{ij} + x_j) = b_i \qquad (i = 1, \ldots, m)$

By subtracting the right-hand side values, we obtain

$\max_{j=1,\ldots,n} (a_{ij} - b_j + x_j) = 0 \qquad (i = 1, \ldots, m)$

A linear system whose right hand-side constants are all zero will be called normalized and the above process will be called normalization.

Example 2.2 Consider the system of linear equations

$$1 \otimes x_1 \oplus 2 \otimes x_2 \oplus 3 \otimes x_3 = 5$$

$$2 \otimes x_1 \oplus 5 \otimes x_2 \oplus 3 \otimes x_3 = 3$$

$$1 \otimes x_1 \oplus 0 \otimes x_2 \oplus 8 \otimes x_3 = 17$$

First, we write the set of equations in matrix form with max-plus as $A \otimes x = b$

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{pmatrix} \otimes \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 5 \\ 3 \\ 17 \end{pmatrix}, \text{ where:}$$

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{pmatrix}, x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \text{ and } b = \begin{pmatrix} 5 \\ 3 \\ 17 \end{pmatrix}.$$

After normalization, it becomes:

$$\begin{pmatrix} 1-5 & 2-5 & 3-5 \\ 2-3 & 5-3 & 3-3 \\ 1-17 & 0-17 & 8-17 \end{pmatrix} \otimes \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} -4 & -3 & -2 \\ -1 & 2 & 0 \\ -16 & -17 & -9 \end{pmatrix} \otimes \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Normalization is nothing else than multiplying the system by the matrix

$B = \operatorname{diag}(b_1^{-1}, b_2^{-1}, \dots, b_m^{-1})$ from the left, that is

$B \otimes A \otimes x = B \otimes b = 0.$

Notice that the first equation of the normalized system above reads

$\max(x_1 - 4, x_2 - 3, x_3 - 2) = 0.$

Thus, if $\bar{x_j} = (\bar{x}_1, \bar{x}_2, \bar{x}_3)$ is a solution to this system, then

$\bar{x}_1 \leq 4,\ \bar{x}_2 \leq 3,\ \bar{x}_3 \leq 2$ and at least one of these inequalities will be satisfied with equality.

Thus,

$\max(x_1 - 4, x_2 - 3, x_3 - 2) = 0$

$\max(x_1 - 1, x_2 + 2, x_3 + 0) = 0$

$\max(x_1 - 16, x_2 - 17, x_3 - 9) = 0$

For x_1 we obtain from the equations:

$x_1 \leq 4,\ x_1 \leq 1, x_1 \leq 16$

$\Longrightarrow \bar{x}_1 \leq \min(4,1,16) \Longrightarrow \bar{x}_1 \leq 1$

For x_2 we obtain it from the above equations

$x_2 \leq 3,\ x_2 \leq -2, x_2 \leq 17$

$\Longrightarrow \bar{x}_2 \leq \min(3,-2,17) \Longrightarrow \bar{x}_2 \leq -2$

For x_3 we obtain it from the above equations

$x_3 \leq 2,\ x_3 \leq 0, x_3 \leq 9$

$\Longrightarrow \bar{x}_3 \leq \min(2,0,9) \Longrightarrow \bar{x}_3 \leq 0$

Hence, $\bar{x} = (1, -2, 0)^T$ is the largest possible solution to these inequalities.

Example 2.3: Find the solution of the system $A \otimes x = b$, where

$$A = \begin{pmatrix} -1 & 1 & 1 \\ -5 & -3 & -2 \\ -5 & -2 & 3 \\ -2 & -2 & 2 \\ -4 & -1 & 1 \end{pmatrix}, x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \text{ and } b = \begin{pmatrix} 2 \\ -2 \\ 1 \\ 0 \\ 3 \end{pmatrix}.$$

First, we write the set of equations in matrix form with max-plus as $A \otimes x = b$

$$\begin{pmatrix} -1 & 1 & 1 \\ -5 & -3 & -2 \\ -5 & -2 & 3 \\ -2 & -2 & 2 \\ -4 & -1 & 1 \end{pmatrix} \otimes \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2 \\ -2 \\ 1 \\ 0 \\ 3 \end{pmatrix}.$$

After normalization, it becomes

$$\begin{pmatrix} -3 & -1 & -1 \\ -3 & -1 & 0 \\ -6 & -3 & 2 \\ -2 & -2 & 2 \\ -7 & -4 & -2 \end{pmatrix} \otimes \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

Thus,

$\max(x_1 - 3, x_2 - 1, x_3 - 1) = 0$

$\max(x_1 - 3, x_2 - 1, x_3 + 0) = 0$

$\max(x_1 - 6, x_2 - 3, x_3 + 2) = 0$

$\max(x_1 - 2, x_2 - 2, x_3 + 2) = 0$

$\max(x_1 - 7, x_2 - 4, x_3 - 2) = 0$

For x_1 we obtain from the equations:

$x_1 \leq 3\,,\;\; x_1 \leq 3\,,\;\; x_1 \leq 6\,,\;\; x_1 \leq 2,\;\; x_1 \leq 7$

$\Rightarrow \bar{x}_1 \leq \min\,(3, 3, 6, 2, 7) \Rightarrow \bar{x}_1 \leq 2$

For x_2 we obtain it from the above equations

$x_2 \leq 1\,,\;\; x_2 \leq 1\,,\;\; x_2 \leq 3\,,\;\; x_2 \leq 2,\;\; x_2 \leq 4$

$\Rightarrow \bar{x}_2 \leq \min\,(1, 1, 3, 2, 4) \Rightarrow \bar{x}_2 \leq 1$

For x_3 we obtain it from the above equations

$x_3 \leq 1\,,\;\; x_3 \leq 0\,,\;\; x_3 \leq -2\,,\;\; x_3 \leq -2,\;\; x_3 \leq 2$

$\Rightarrow \bar{x}_3 \leq \min(1, 0, -2, -2, 2) \Rightarrow \bar{x}_3 \leq -2$

$\Rightarrow \bar{x}_j = (\,\bar{x}_1, \bar{x}_2, \bar{x}_3)^T$, hence $\bar{x} = (2, 1, -2)^T$

Clearly for all j then $x_j \leq \bar{x_j}$ where $-\bar{x_j}$ is the column j maximum. However, equality must be attained in some of these inequalities so that in every row there is at least one column maximum which is attained by x_j. This was noticed in Zimmermann [36], and is accurately formulated in the theorem below in which it is supposed that we study a system

$A \otimes x = 0$, where $A = (a_{ij}) \in \mathbb{R}_{\max}^{m \times n}$, and we denote

$S = \{x \in \mathbb{R}^n;\ A \otimes x = 0\}$

$\bar{x}_j = -\max_i a_{ij}$ for all $j \in N$ where $N = \{1, 2,3, \dots, n\}$,

$\bar{x}_j = (\bar{x}_1, \dots \dots, \bar{x}_n)^T$

$M_j = \left\{k \in M;\ a_{kj} = \max_i a_{ij}\right\}$ for all $j \in N$

Note that $A \otimes x = b$ has a solution if and only if $A \otimes \bar{x} = b$

Theorem 2.3 (Combinatorial method) [16]

$x \in S$ if and only if

1. $x \leq \bar{x}$ and
2. $\bigcup_{j \in N_x} M_j = M$, where $N_x = \{j \in N; x_j = \bar{x_j}\}$.

It follows that $A \otimes x = 0$ has no solution if $\bar{x}$ is not a solution. Therefore, $\bar{x}$ is called principal subsolution.

Corollary 2.1 The following three statements are equivalent:

1. $S \neq \emptyset$
2. $\bar{x} \in S$
3. $\bigcup_{j \in N} M_j = M$

Corollary 2.2 [20] $S = \{\bar{x}\}$ if and only if

$i)$ $\bigcup_{j \in N} M_j = M$ and

$ii)$ $\bigcup_{j \in N'} M_j \neq M$ for any $N' \subseteq N, N' \neq N$

<u>Note</u>: Given that $A = (a_{ij}) \in \mathbb{R}_{\max}^{m \times n}$ and $b \in \mathbb{R}_{\max}^{m}$ the system $A \otimes x = b$ has a solution iff

$$\bigcup_{j=1}^{n} M_j = M$$

Example 2.4 Find the solution of the system

$$\begin{pmatrix} -3 & 1 & 0 \\ 1 & -4 & 2 \\ 0 & 3 & 1 \end{pmatrix} \otimes \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 6 \\ 5 \\ 2 \end{pmatrix}$$

Normalization gives

$$\begin{pmatrix} -9 & -5 & -6 \\ -4 & -9 & -3 \\ -2 & 1 & -1 \end{pmatrix} \otimes \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\bigcup_{j=1,2,3} M_j = \{3\} \neq M$$

Thus, there is no solution of this system.

Hence $\bar{x} = [2, -1, 1]^T$ is a subsolution

$A \otimes \bar{x} = [1, 3, 2]^T \leq [6, 5, 2]^T$

Example 2.5 Find the solution of

$$\begin{pmatrix} -2 & 2 & 2 \\ -5 & -3 & -2 \\ \varepsilon & \varepsilon & 3 \\ -3 & -3 & 2 \\ 1 & 4 & \varepsilon \end{pmatrix} \otimes \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 3 \\ -2 \\ 1 \\ 0 \\ 5 \end{pmatrix}$$

Normalization gives

$$\begin{pmatrix} -5 & -1 & -1 \\ -3 & -1 & 0 \\ \varepsilon & \varepsilon & 2 \\ -3 & -3 & 2 \\ -4 & -1 & \varepsilon \end{pmatrix} \otimes \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

By finding M_j where $j = 1,2,3$

$M_j = \{k : \overline{x_j} = -a_{kj} + b_k\}$, where $k = 1,2,3,4,5$

$M_1 =$ the rows where $\overline{x_1} = -a_{k1} + b_k = 0$

$M_2 =$ the rows where $\overline{x_2} = -a_{k2} + b_k = 0$

$M_3 =$ the rows where $\overline{x_3} = -a_{k3} + b_k = 0$

Thus, $M_1 = \{2,4\}, M_2 = \{1,2,5\}$, and $M_3 = \{3,4\}$

$$\bigcup_{j=1,2,3} M_j = \{1,2,3,4,5\} = M$$

Hence, $\bar{x} = [3, 1, -2]^T$ a solution is because $\bigcup_{j=1,2,3} M_j = M$, and it is a non-unique solution because $[4, 1, -2]^T$ and $[\varepsilon, 1, -2]^T$ are also solutions.

2.2 Eigenvalues and Eigenvectors

2.2.1 Existence and Uniqueness

Given a matrix A with entries in $\mathbb{R}_{\max}^{n\times n}$, we consider the problem of the existence of eigenvalues and eigenvectors, that is, the existence of λ and x such that

$$A \otimes x = \lambda \otimes x, \text{where } x \neq \varepsilon.$$

If $A = (a_{ij}) \in \mathbb{R}_{\max}^{n\times n}$ and the graph $G(A) = (V, E)$ where V is the set of vertices and arc sets

$E = \{(i,j);\ a_{ij} \neq \varepsilon\}$, and associate an element $A_{ij} \in \mathbb{R}_{\max}$ with each arc $(j, i) \in E$, then G is called a digraph. We will define the maximum cycle mean and prove the existence of eigenvalues and eigenvectors. The numerical value of A_{ij} is the weight of the arc from j to i and if no such arc exists, then $A_{ij} = \varepsilon$. Let $\rho = (i_1, i_2, \ldots\ldots, i_k)$ be a path in a weighted graph of A, then the weight of this path is denoted by $|\rho|_w$is product $A_{i_1 i_2} \otimes A_{i_2 i_3} \otimes \ldots\ldots \otimes A_{i_{k-1} i_k}$ and the length of this path, $|\rho|_l$, is $k-1$. $(A^k)_{ij}$ is the maximum weight with respect to all paths of length k from j to i.

Thus, $(A^k)_{ij} = \oplus\ A_{i_1 i_2} \otimes A_{i_2 i_3} \otimes \ldots\ldots \otimes A_{i_{k-1} i_k}$ where $i_k = j$. If no such path exists, then $(A^k)_{ij} = \varepsilon$. $(i\,, j)$ the entry of $A^* = I \oplus A \oplus \ldots\ldots \oplus A^n \oplus A^{n+1} \oplus \ldots\ldots\ldots$denotes the maximum weight of all paths of any length from vertex (node) j to vertex (node) i.

Definition 2.1 The mean weight of a path is defined as the sum of weights of the individual arcs of this path, divided by the length of this path. If the path is denoted by ρ, then mean weight equals $\frac{|\rho|_w}{|\rho|_l}$. If such a path is a cycle, then we are talking about the mean weight of the cycle, or simply the **cycle mean**.

Theorem 2.4 [1] If A is irreducible, or equivalently if $G(A)$ is strongly connected, there exists one and only one eigenvalue (but possibly several eigenvectors). This eigenvalue is equal to the maximum cycle mean of the graph $G\ (A)$:

$$\lambda = \max_{\rho} \frac{|\rho|_w}{|\rho|_l}$$

where ρ ranges over the set of cycles of $G\ (A)$.

__**Proof: Existence of x and λ**__

Consider matrix $B = -\lambda \otimes A = (0 \otimes -\lambda) \otimes A$, where $\lambda = \max_{\rho} \frac{|\rho|_w}{|\rho|_l}$, where ρ is a cycle in $G(A)$.

The maximum cycle weight of $G(B)$ is 0. Hence B^* and $B^+ = B \otimes B^*$ are in $\mathbb{R}_{\max}^{n\times n}$. Matrix B^+ has some columns with diagonal entries equal to 0. Suppose a vertex k is in the maximum cycle of $G(B)$, then the maximum weight of paths from k to k is 0. Therefore, we have $0 = B_{kk}^+$.

Let B_k denote the k^{th} column of B. Then, since $B = -\lambda \otimes A$, $B^+ = B \oplus B^2 \oplus B^3 \oplus B^4 \oplus \ldots\ldots$..

$B^+ = B \otimes B^*$ and $B^* = 0 \oplus B^+$, for k^{th} column,

$B_k^+ = B_k^* \Rightarrow B \otimes B_k^* = B_k^+ = B_k^* \Rightarrow -\lambda \otimes A \otimes B_k^* = B_k^* \Rightarrow A \otimes B_k^* = \lambda \otimes B_k^*$.

Hence $x = B_k^* = B_k^+$ is an eigenvector of A corresponding to the eigenvalue λ.

■

The set of vertices of $G(A)$ corresponding to the entries of x, where $x_j \neq \varepsilon$ is called the support of x.

__**Graph interpretation of λ**__

If λ satisfies equation $A \otimes x = \lambda \otimes x$, there is a component of x, say $x_{i_1} \neq \varepsilon$. Then we have $(A \otimes x)_{i_1} = \lambda \otimes x_{i_1}$ and there is an index i_2 such that $A_{i_1 i_2} \otimes x_{i_2} = \lambda \otimes x_{i_1}$. Hence, $x_{i_2} \neq \varepsilon$ and $A_{i_1 i_2} \neq \mathcal{E}$. We can repeat this argument and obtain a sequence $\{i_j\}$ such that

$A_{i_{j-1} i_j} \otimes x_{i_j} = \lambda \otimes x_{i_{j-1}}, x_{i_{j-1}} \neq \varepsilon$ and $A_{i_{j-1} i_j} \neq \varepsilon$.

At some stage, we must reach an index i_l already encountered in the sequence, since the number of vertices is finite. Therefore, we obtain a cycle $\beta = (i_l, i_m, \ldots, i_l + 1, i_l)$.

By multiplication along this cycle, we obtain

$A_{i_l i_{l+1}} \otimes A_{i_{l+1} i_{l+2}} \otimes \cdots \otimes A_{i_m i_l} \otimes x_{i_{l+1}} \otimes x_{i_{l+2}} \otimes \cdots \otimes x_{i_m} \otimes x_{i_l} = \lambda^{\otimes m-l+1} \otimes x_{i_l} \otimes x_{i_{l+1}} \otimes \cdots \otimes x_{i_m}$.

Since $x_{ij} \neq \varepsilon$ for all i_j, we may simplify the equation above, which shows that $\lambda^{\otimes m-l+1} = (m-l+1)\lambda$ is the weight of the cycle of length $m-l+1$, or, otherwise stated, λ is the average weight of cycle β.

If A is irreducible, all the components of x are different from ε

Assume that the support of x does not cover the whole graph. Then, there are arcs going from the support of x to other vertices, because the graph $G(A)$ has only one strongly connected component. Therefore, the support of $A \otimes x$ is larger than the support of x, which contradicts Equation
$A \otimes x = \lambda \otimes x$.

Uniqueness in the irreducible case

Consider any cycle $\gamma = (i_1, i_2, \ldots, i_p, i_1)$ such that its vertices belong to the support of x (here any node of $G(A)$).
We have $A_{i_2 i_1} \otimes x_{i_1} \leq \lambda \otimes x_{i_2}, \ldots. A_{i_p i_{p-1}} \otimes x_{i_{p-1}} \leq \lambda \otimes x_{i_p}$, $A_{i_1 i_p} \otimes x_{i_p} \leq \lambda \otimes x_{i_1}$.
Hence, by the same argument as in the paragraph on the graph interpretation of λ, we see that λ is greater or equal to the average weight of γ. Therefore, λ is the maximum cycle mean and, thus, it is unique.
It is important to understand the role of the support of x in the previous proof. If $G(A)$ is not strongly connected, the support of x is not necessarily the whole set of vertices and, in general, there is no unique eigenvalue (see Example 2.7 below).

Note: The part of the proof on the graph interpretation of λ indeed showed that, for a general matrix A, any eigenvalue is equal to a cycle mean. Therefore, the maximum cycle mean is equal to the maximum eigenvalue of the matrix.

Example 2.6 (No unique eigenvector) with the only assumption of Theorem 2.4 on irreducibility, the uniqueness of the eigenvector is not guaranteed, as is shown by the following example:

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 1 \otimes \begin{pmatrix} 0 \\ -1 \end{pmatrix} \quad , \text{ here } \lambda = 1$$

and

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = 1 \otimes \begin{pmatrix} -1 \\ 0 \end{pmatrix}.$$

The two eigenvectors are obviously not 'proportional'.

Example 2.7 (A not irreducible)

- The following example is a trivial counterexample to the uniqueness of the eigenvalue when $G(A)$ is not connected:

  $$\begin{pmatrix} 1 & \varepsilon \\ \varepsilon & 2 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ \varepsilon \end{pmatrix} = 1 \otimes \begin{pmatrix} 0 \\ \varepsilon \end{pmatrix} \quad , \text{ here } \lambda = 1$$

  and

  $$\begin{pmatrix} 1 & \varepsilon \\ \varepsilon & 2 \end{pmatrix} \otimes \begin{pmatrix} \varepsilon \\ 0 \end{pmatrix} = 2 \otimes \begin{pmatrix} \varepsilon \\ 0 \end{pmatrix} \quad \text{but here } \lambda = 2$$

- In the following example $G(A)$ is connected but not strongly connected. Nevertheless, there is only one eigenvalue:

  $$\begin{pmatrix} 1 & 0 \\ \varepsilon & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ \varepsilon \end{pmatrix} = 1 \otimes \begin{pmatrix} 0 \\ \varepsilon \end{pmatrix} \quad , \text{ here } \lambda = 1$$

  but

  $$\begin{pmatrix} 1 & 0 \\ \varepsilon & 0 \end{pmatrix} \otimes \begin{pmatrix} a \\ 0 \end{pmatrix} = \lambda \otimes \begin{pmatrix} a \\ 0 \end{pmatrix}$$

has no solutions because the second equation implies $\lambda = e$, and then the first equation has no solutions for the unknown a.

- In the following example $G(A)$ is connected but not strongly connected and there are two eigenvalues:

  $$\begin{pmatrix} 0 & 0 \\ \varepsilon & 1 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ \varepsilon \end{pmatrix} = 0 \otimes \begin{pmatrix} 0 \\ \varepsilon \end{pmatrix} \quad , \text{ here } \lambda = 0$$

  and

  $$\begin{pmatrix} 0 & 0 \\ \varepsilon & 1 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} = 1 \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \text{but here } \lambda = 1$$

Example 2.8 Let $A = \begin{pmatrix} -3 & -2 & 8 \\ 1 & 0 & 4 \\ 2 & 3 & -6 \end{pmatrix}$ and the precedence graph of A is shown in Figure 2.1.

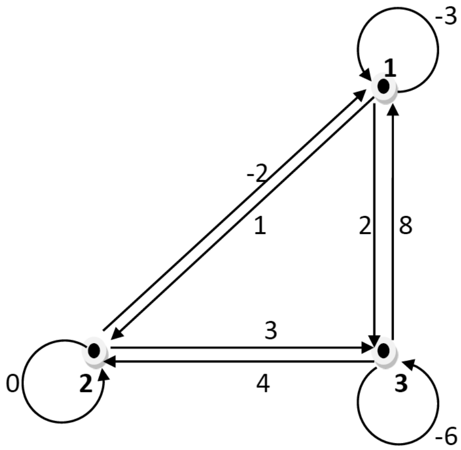

Figure 2.1: The precedence graph of A

Then the cycle means of A are

1. Cycles of length (1): $A_{11} = -3$, $A_{22} = 0$ and $A_{33} = -6$
2. Cycles of length (2):

$$A_{12} = \frac{a_{12} + a_{21}}{2} = \frac{-2+1}{2} = \frac{-1}{2}$$

$$A_{13} = \frac{a_{13} + a_{31}}{2} = \frac{8+2}{2} = 5 \ , \qquad A_{23} = \frac{a_{23} + a_{32}}{2} = \frac{7}{2}$$

3. Cycles of length (3):

$$A_{13} = \frac{a_{12} + a_{23} + a_{31}}{3} = \frac{-2+4+2}{3} = \frac{4}{3}$$

$$A_{12} = \frac{a_{13} + a_{32} + a_{21}}{3} = \frac{8+3+1}{3} = 4$$

$$\lambda = \max\left\{\frac{a_{i_1i_2} + a_{i_2i_3} + \ldots\ldots\ldots\ldots + a_{i_ki_1}}{k}; i_1, \ldots\ldots\ldots\ldots i_k \in N\right\}$$

Note: Obviously, it is enough to consider cycles of length ≤ 3.

$$\lambda = \max\{A_{11}, A_{22}, \ldots\} = \max\{-3, 0, -6, \frac{-1}{2}, 5, \frac{7}{2}, \frac{4}{3}, 4\}$$

Hence the eigenvalue $\lambda = 5$.

In order to find the eigenvector, we need to determine $G(B)$ where

$$B = -\lambda \otimes A = -5 \otimes \begin{pmatrix} -3 & -2 & 8 \\ 1 & 0 & 4 \\ 2 & 3 & -6 \end{pmatrix}$$

$$= \begin{pmatrix} -5 \otimes -3 & -5 \otimes -2 & -5 \otimes 8 \\ -5 \otimes 1 & -5 \otimes 0 & -5 \otimes 4 \\ -5 \otimes 2 & -5 \otimes 3 & -5 \otimes -6 \end{pmatrix} = \begin{pmatrix} -8 & -7 & 3 \\ -4 & -5 & -1 \\ -3 & -2 & -11 \end{pmatrix}$$

Suppose that $x = (x_1, x_2, x_3)^T$ is an eigenvector. Since vertex 1 and 3 determine the critical cycle in Figure 2.2., we may choose either x_1 or x_3 to be 0.

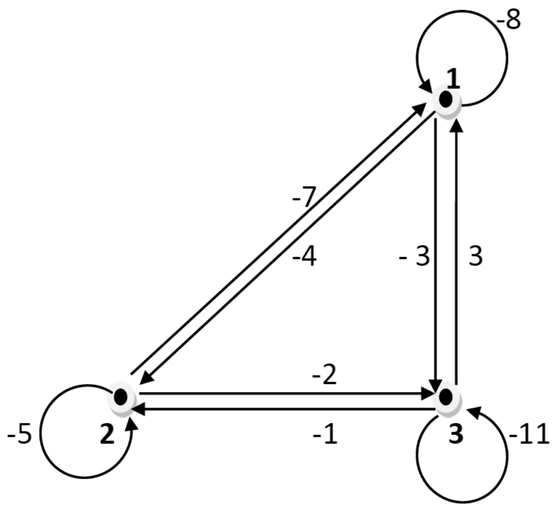

Figure 2.2: The precedence graph of B

The maximum weight of a path from vertex 1 to vertex 2 is -4 and -3 from vertex1 to vertex 3 in $G(B)$**.** Hence, $x = B_1^+ = (0, -4, -3)^T$ is the eigenvector of A and $(3, -1, 0)$ is also an eigenvector of A.

Check: $\begin{pmatrix} -3 & -2 & 8 \\ 1 & 0 & 4 \\ 2 & 3 & -6 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ -4 \\ -3 \end{pmatrix} = \begin{pmatrix} 5 \\ 1 \\ 2 \end{pmatrix} = 5 \otimes \begin{pmatrix} 0 \\ -4 \\ -3 \end{pmatrix}$ and

$$\begin{pmatrix} -3 & -2 & 8 \\ 1 & 0 & 4 \\ 2 & 3 & -6 \end{pmatrix} \otimes \begin{pmatrix} 3 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 8 \\ 4 \\ 5 \end{pmatrix} = 5 \otimes \begin{pmatrix} 3 \\ -1 \\ 0 \end{pmatrix}.$$

Note: It is important to note that an eigenvector is never unique, in the sense that if x is an eigenvector then $\alpha\otimes x$ is also always an eigenvector. The two eigenvectors shown in Example 2.8 are actually the same in the above sense: the second one is $3 \otimes x$.

2.2.2 Power Method [12]

We now describe the power method in ordinary linear algebra for computing the dominant eigenvalue and eigenvector. Its extension to the inverse power method is practical for finding any eigenvalue, if a good initial approximation is known. Some schemes for finding eigenvalues use other methods that converge quickly, but the accuracy is limited. To discuss the situation, we shall need the following definitions.

Definition 2.2 If λ_1 is an eigenvalue of $\boldsymbol{A}$ that is larger in absolute value than any other eigenvalue, it is called the **dominant eigenvalue**. An eigenvector $\boldsymbol{V}_1$ corresponding to λ_1 is called a **dominant eigenvector**.

Definition 2.3 An eigenvector $\boldsymbol{V}$ is said to be **normalized** if the coordinate of the largest magnitude is equal to unity (i.e., the largest coordinate in the vector $\boldsymbol{V}$ is the number 1). It is easy to normalize an eigenvector $[v_1\ v_n \dots\dots\dots\ v_n]^T$ by forming a new vector $V = ({}^1/_c)[v_1\ v_n \dots\dots\dots\ v_n]^T$, where

$$c = V_j \ \text{ and } \ |V_j| = \max_{1\le i\le n}\{|V_i|\}.$$

Suppose that the matrix A has a dominant eigenvalue λ and that there is a unique normalized eigenvector $\boldsymbol{V}$ that corresponds to λ. This eigenvalue λ and the eigenvector V can be found by the following iterative procedure called the **power method**. Start with the vector

$$X_0 = [1\ \ 1\ \ \dots\dots 1]^T. \tag{2.1}$$

Generate the sequence $\{X_k\}$ recursively, using

$$Y_k = AX_k \tag{2.2}$$

$$X_{k+1} = \frac{1}{c_{k+1}} Y_k,$$

where c_{k+1} is the coordinate of Y_k of largest magnitude (in the case of a tie, choose the coordinate that comes first). The sequences $\{X_k\}$ and $\{c_k\}$ will converge to V and λ, respectively:

$$\lim_{k\to\infty} X_k = V \ \text{ and } \ \lim_{k\to\infty} C_k = \lambda \tag{2.3}$$

In max-plus algebra, the power method can also be used for finding the largest eigenvalue. First, we assume that the matrix A has an eigenvalue with corresponding eigenvectors. Then, we choose an initial guess $x_0 \in \mathbb{R}_{\max}^n$ of one of the dominant eigenvectors of A. Finally, we form the sequence given by

$$x_1 = A \otimes x_0$$
$$x_2 = A \otimes x_1 = A \otimes A \otimes x_0 = A^{\otimes 2} \otimes x_0$$
$$x_3 = A \otimes x_2 = A \otimes (A^{\otimes 2} \otimes x_0) = A^{\otimes 3} \otimes x_0$$
$$\vdots$$
$$x_t = A^{\otimes t} \otimes x_0$$

where $t = 1,2, \dots \dots$, $x \in \mathbb{R}_{\max}^n$ and $A = (a_{ij}) \in \mathbb{R}_{\max}^{n \times n}$ and by properly scaling this sequence, we will see that we obtain the eigenvector of A [12].

Assume that x_m is an eigenvector of A. Then, $x_{m+1} = A \otimes x_m = \lambda \otimes x_m$, $x_{m+2} = A \otimes x_{m+1} = A \otimes \lambda \otimes x_m = \lambda^{\otimes 2} \otimes x_m, \dots$ where λ is the corresponding eigenvalue. More generally,

$$x_{t+m} = \lambda^{\otimes t} \otimes x_m \text{ where } t = 1,2, \dots \tag{2.4}$$

Equation (2.4) yields

$(x_{t+1})_i - (x_{t+1})_j = (x_t)_i - (x_t)_j$ and $(x_{t+1})_i = \lambda \otimes (x_t)_i$ for $i, j = 1, \dots \dots, n$.

Thus, the solution of these equations exhibits a kind of periodicity. In particular, $x_{k+\rho} = \lambda^{\otimes \rho} \otimes x_k$, where ρ is a period.

Thus, $\lambda = \frac{(x_{k+\rho})_i - (x_k)_i}{\rho}$, for any $i = 1,2, \dots, n$.

Recall that this is the maximum cycle mean of A. Now, assume that we have $\frac{(x_{k+\rho})_i - (x_k)_i}{\rho}$ a constant λ. An eigenvector $\boldsymbol{x}$ can be represented as:

$$x_{k+\rho-1} \oplus \lambda \otimes x_{k+\rho-2} \oplus \lambda^{\otimes 2} \otimes x_{k+\rho-3} \oplus \dots \dots \oplus \lambda^{\otimes \rho-1} \otimes x_k \text{ [12].}$$

Example 2.9 Find the eigenvalue and the eigenvector by using the power method for the matrix A where $A = \begin{pmatrix} 3 & 7 \\ 2 & 4 \end{pmatrix}$.

Let $x_0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

Thus, $x_1 = A \otimes x_0 = \begin{pmatrix} 3 & 7 \\ 2 & 4 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 7 \\ 4 \end{pmatrix}$

$$x_2 = A \otimes x_1 = \begin{pmatrix} 3 & 7 \\ 2 & 4 \end{pmatrix} \otimes \begin{pmatrix} 7 \\ 4 \end{pmatrix} = \begin{pmatrix} 11 \\ 9 \end{pmatrix}$$

$$x_3 = A \otimes x_2 = \begin{pmatrix} 3 & 7 \\ 2 & 4 \end{pmatrix} \otimes \begin{pmatrix} 11 \\ 9 \end{pmatrix} = \begin{pmatrix} 16 \\ 13 \end{pmatrix}$$

$$\Rightarrow x_3 - x_1 = \begin{pmatrix} 9 \\ 9 \end{pmatrix}.$$

This gives the eigenvalue $\lambda = \dfrac{9}{3-1} = 4.5$ (2 is the period) and the eigenvector is

$$x = x_2 \oplus (\lambda \otimes x_1) = \begin{pmatrix} 11 \\ 9 \end{pmatrix} \oplus \left(4.5 \otimes \begin{pmatrix} 7 \\ 4 \end{pmatrix} \right) = \begin{pmatrix} 11.5 \\ 9 \end{pmatrix}$$

$$A \otimes x = \begin{pmatrix} 3 & 7 \\ 2 & 4 \end{pmatrix} \otimes \begin{pmatrix} 11.5 \\ 9 \end{pmatrix} = \begin{pmatrix} 16 \\ 13.5 \end{pmatrix}$$

and $\lambda \otimes x = 4.5 \otimes \begin{pmatrix} 11.5 \\ 9 \end{pmatrix} = \begin{pmatrix} 16 \\ 13.5 \end{pmatrix}.$

Hence $A \otimes x = \lambda \otimes x$

and $x = \begin{pmatrix} 11.5 \\ 9 \end{pmatrix}$ is the eigenvector of A.

Example 2.10 Find the eigenvalue and the eigenvector by using the power method for the matrix *A,*

where $A = \begin{pmatrix} 3 & 0 \\ 8 & -1 \end{pmatrix}.$

Solution: If $x_0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, then

$$x_1 = A \otimes x_0 = \begin{pmatrix} 3 & 0 \\ 8 & -1 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ 8 \end{pmatrix}$$

$$x_2 = A \otimes x_1 = \begin{pmatrix} 3 & 0 \\ 8 & -1 \end{pmatrix} \otimes \begin{pmatrix} 3 \\ 8 \end{pmatrix} = \begin{pmatrix} 8 \\ 11 \end{pmatrix}$$

$$x_3 = A \otimes x_2 = \begin{pmatrix} 3 & 0 \\ 8 & -1 \end{pmatrix} \otimes \begin{pmatrix} 8 \\ 11 \end{pmatrix} = \begin{pmatrix} 11 \\ 16 \end{pmatrix}$$

$$\Longrightarrow x_3 - x_1 = \begin{pmatrix} 11 \\ 16 \end{pmatrix} - \begin{pmatrix} 3 \\ 8 \end{pmatrix} = \begin{pmatrix} 8 \\ 8 \end{pmatrix}.$$

Thus, the eigenvalue is $\lambda = \frac{8}{3-1} = 4$

and the eigenvector $x = x_2 \oplus \lambda \otimes x_1 = \begin{pmatrix} 8 \\ 11 \end{pmatrix} \oplus \left(4 \otimes \begin{pmatrix} 3 \\ 8 \end{pmatrix}\right) = \begin{pmatrix} 8 \\ 12 \end{pmatrix}$

$$A \otimes x = \begin{pmatrix} 3 & 0 \\ 8 & -1 \end{pmatrix} \otimes \begin{pmatrix} 8 \\ 12 \end{pmatrix} = \begin{pmatrix} 12 \\ 16 \end{pmatrix}$$

$$\lambda \otimes x = 4 \otimes \begin{pmatrix} 8 \\ 12 \end{pmatrix} = \begin{pmatrix} 12 \\ 16 \end{pmatrix}.$$

Hence $A \otimes x = \lambda \otimes x.$

Next, we will take matrices of size 3×3 and 4×4 and try to find out the number of iterations that is needed to find the eigenvalue.

Example 2.11 Find the eigenvalue and the eigenvector by using the power method for the matrix *A,*

where $A = \begin{pmatrix} -3 & -2 & 8 \\ 1 & 0 & 4 \\ 2 & 5 & -6 \end{pmatrix}$.

Solution: Let, $x_0 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$.

Thus, $x_1 = A \otimes x_0 = \begin{pmatrix} -3 & -2 & 8 \\ 1 & 0 & 4 \\ 2 & 5 & -6 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 8 \\ 4 \\ 5 \end{pmatrix}$

$$x_2 = A \otimes x_1 = \begin{pmatrix} -3 & -2 & 8 \\ 1 & 0 & 4 \\ 2 & 5 & -6 \end{pmatrix} \otimes \begin{pmatrix} 8 \\ 4 \\ 5 \end{pmatrix} = \begin{pmatrix} 13 \\ 9 \\ 10 \end{pmatrix}$$

$$x_3 = A \otimes x_2 = \begin{pmatrix} -3 & -2 & 8 \\ 1 & 0 & 4 \\ 2 & 5 & -6 \end{pmatrix} \otimes \begin{pmatrix} 13 \\ 9 \\ 10 \end{pmatrix} = \begin{pmatrix} 18 \\ 14 \\ 15 \end{pmatrix}$$

$$x_4 = A \otimes x_3 = \begin{pmatrix} -3 & -2 & 8 \\ 1 & 0 & 4 \\ 2 & 5 & -6 \end{pmatrix} \otimes \begin{pmatrix} 18 \\ 14 \\ 15 \end{pmatrix} = \begin{pmatrix} 23 \\ 19 \\ 20 \end{pmatrix}$$

Hence, $x_4 - x_1 = \begin{pmatrix} 23 \\ 19 \\ 20 \end{pmatrix} - \begin{pmatrix} 8 \\ 4 \\ 5 \end{pmatrix} = \begin{pmatrix} 15 \\ 15 \\ 15 \end{pmatrix}$

The eigenvalue $\lambda = \dfrac{15}{4-1} = 5$

Eigenvector $x = x_3 \oplus \lambda \otimes x_2 \oplus \lambda^{\otimes 2} \otimes x_1$

$$= \begin{pmatrix} 18 \\ 14 \\ 15 \end{pmatrix} \oplus 5 \otimes \begin{pmatrix} 13 \\ 9 \\ 10 \end{pmatrix} \oplus 5^{\otimes 2} \otimes \begin{pmatrix} 8 \\ 4 \\ 5 \end{pmatrix} = \begin{pmatrix} 18 \oplus 18 \oplus 18 \\ 14 \oplus 14 \oplus 14 \\ 15 \oplus 15 \oplus 15 \end{pmatrix} = \begin{pmatrix} 18 \\ 14 \\ 15 \end{pmatrix}$$

$$\Rightarrow A \otimes x = \begin{pmatrix} 23 \\ 19 \\ 20 \end{pmatrix} = \lambda \otimes x$$

Hence $x = \begin{pmatrix} 18 \\ 14 \\ 15 \end{pmatrix}$ is the eigenvector of A.

Example 2.12 Find the eigenvalue and the eigenvector by using the power method for the matrix *A,*

where $A = \begin{pmatrix} \varepsilon & 1 & \varepsilon & \varepsilon \\ 8 & \varepsilon & \varepsilon & 5 \\ \varepsilon & 2 & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & 7 & \varepsilon \end{pmatrix}$.

Solution: The precedence graph of the matrix A is shown in Fig.2.3, and

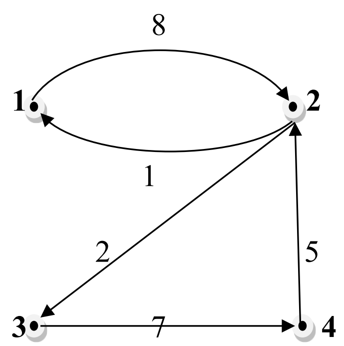

Figure 2.3: The precedence graph of *A*

A is irreducible $\Longrightarrow G(A)$ strongly connected.

Let $x_0 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$

Thus, $x_1 = A \otimes x_0 = \begin{pmatrix} 1 \\ 8 \\ 2 \\ 7 \end{pmatrix}$, $x_2 = A \otimes x_1 = \begin{pmatrix} 9 \\ 12 \\ 10 \\ 9 \end{pmatrix}$, $x_3 = A \otimes x_2 = \begin{pmatrix} 13 \\ 17 \\ 14 \\ 17 \end{pmatrix}$,

$x_4 = A \otimes x_3 = \begin{pmatrix} 18 \\ 22 \\ 19 \\ 21 \end{pmatrix} x_5 = A \otimes x_4 = \begin{pmatrix} 23 \\ 26 \\ 24 \\ 26 \end{pmatrix}$, $x_6 = A \oplus x_5 = \begin{pmatrix} 27 \\ 31 \\ 28 \\ 31 \end{pmatrix}$, $x_7 = A \otimes x_6 = \begin{pmatrix} 32 \\ 36 \\ 33 \\ 35 \end{pmatrix}$

$\therefore\ x_7 - x_4 = \begin{pmatrix} 14 \\ 14 \\ 14 \\ 14 \end{pmatrix}$

The eigenvalue of A is $\lambda = \frac{14}{3} \approx 4.7$

Eigenvector $x = x_6 \oplus \lambda \otimes x_5 \oplus \lambda^{\otimes 2} \otimes x_4 \oplus \lambda^{\otimes 3} \otimes x_3 \oplus \lambda^{\otimes 4} \otimes x_2 \oplus \lambda^{\otimes 5} \otimes x_1$

$$x = \begin{pmatrix} 27 \\ 31 \\ 28 \\ 31 \end{pmatrix} \oplus 4.7 \otimes \begin{pmatrix} 23 \\ 26 \\ 24 \\ 26 \end{pmatrix} \oplus 4.7^{\otimes 2} \otimes \begin{pmatrix} 18 \\ 22 \\ 19 \\ 21 \end{pmatrix} \oplus 4.7^{\otimes 3} \otimes \begin{pmatrix} 13 \\ 17 \\ 14 \\ 17 \end{pmatrix} \oplus 4.7^{\otimes 4} \otimes \begin{pmatrix} 9 \\ 12 \\ 10 \\ 9 \end{pmatrix}$$

$$\oplus 4.7^{\otimes 5} \otimes \begin{pmatrix} 1 \\ 8 \\ 2 \\ 7 \end{pmatrix}$$

$$= \begin{pmatrix} 27 \oplus 27.7 \oplus 27.4 \oplus 27.1 \oplus 27.8 \oplus 24.5 \\ 31 \oplus 30.7 \oplus 31.4 \oplus 31.1 \oplus 30.8 \oplus 31.5 \\ 28 \oplus 28.7 \oplus 28.4 \oplus 28.1 \oplus 28.8 \oplus 25.5 \\ 31 \oplus 30.7 \oplus 30.4 \oplus 31.1 \oplus 27.8 \oplus 30.5 \end{pmatrix} = \begin{pmatrix} 27.8 \\ 31.5 \\ 28.8 \\ 31.1 \end{pmatrix}$$

$$\therefore A \otimes x = \begin{pmatrix} 32.5 \\ 36.1 \\ 33.5 \\ 35.8 \end{pmatrix} = \lambda \otimes x$$

Thus, by using the power method it sometimes takes long, especially when we have large-sized matrices to find the eigenvalue and eigenvector, if we use the maximum cycle mean method, then A is irreducible $\Longrightarrow G(A)$ strongly connected

$\Longrightarrow \lambda(A)$ exists and

$\lambda(A)=\max\left(\frac{1+8}{2}\ ,\ \frac{5+7+2}{3}\right) = \frac{14}{3}$ the eigenvalue of *A*.

Let *B* be a matrix where $B = -\lambda \otimes A$

$$B = \frac{-14}{3} \otimes \begin{pmatrix} \varepsilon & 1 & \varepsilon & \varepsilon \\ 8 & \varepsilon & \varepsilon & 5 \\ \varepsilon & 2 & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & 7 & \varepsilon \end{pmatrix} = \begin{pmatrix} \varepsilon & \frac{-11}{3} & \varepsilon & \varepsilon \\ \frac{10}{3} & \varepsilon & \varepsilon & \frac{1}{3} \\ \varepsilon & \frac{-8}{3} & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \frac{7}{3} & \varepsilon \end{pmatrix}$$

The precedence graph of the matrix B is shown in Fig.2.4, and the maximum cycle weight of $G\,(B) = 0$.

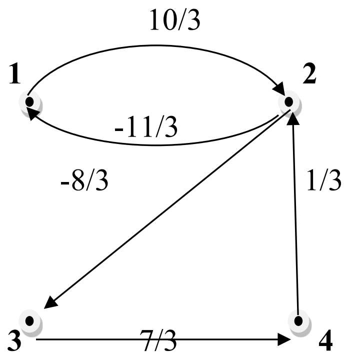

Figure 2.4: The precedence graph of *B*

Hence, B^* and B^+ are in $\mathbb{R}_{\max}^{4\times4}$ and $B^* = I \oplus B \oplus B^{\otimes 2} \oplus B^{\otimes 3} \oplus \ldots . \oplus B^{\otimes r}$

where *r* is the length of the longest path in *G*(*B*) and *r = 3*.

$$B^* = \begin{pmatrix} 0 & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & 0 & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & 0 & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & 0 \end{pmatrix} \oplus \begin{pmatrix} \varepsilon & \frac{-11}{3} & \varepsilon & \varepsilon \\ \frac{10}{3} & \varepsilon & \varepsilon & \frac{1}{3} \\ \varepsilon & \frac{-8}{3} & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \frac{7}{3} & \varepsilon \end{pmatrix} \oplus \begin{pmatrix} \frac{-1}{3} & \varepsilon & \varepsilon & \frac{-10}{3} \\ \varepsilon & \frac{-1}{3} & \frac{8}{3} & \varepsilon \\ \frac{2}{3} & \varepsilon & \varepsilon & \frac{-7}{3} \\ \varepsilon & \frac{-1}{3} & \varepsilon & \varepsilon \end{pmatrix} \oplus \begin{pmatrix} \varepsilon & -4 & -1 & \varepsilon \\ 3 & 0 & \varepsilon & 0 \\ \varepsilon & -3 & 0 & \varepsilon \\ 3 & \varepsilon & \varepsilon & 0 \end{pmatrix}$$

$$\therefore B^* = \begin{pmatrix} 0 & \frac{-11}{3} & -1 & \frac{-10}{3} \\ \frac{10}{3} & 0 & \frac{8}{3} & \frac{1}{3} \\ \frac{2}{3} & \frac{-8}{3} & 0 & \frac{-7}{3} \\ 3 & \frac{-1}{3} & \frac{7}{3} & 0 \end{pmatrix}$$

$$B^+ = B \otimes B^* = \begin{pmatrix} \frac{-1}{3} & \frac{-11}{3} & -1 & \frac{-10}{3} \\ \frac{10}{3} & 0 & \frac{8}{3} & \frac{1}{3} \\ \frac{2}{3} & \frac{-8}{3} & 0 & \frac{-7}{3} \\ 3 & \frac{-1}{3} & \frac{7}{3} & 0 \end{pmatrix}$$

From these two matrices, B^* and B^+, we can note that:

$B_2^* = B_2^+, B_2^* = B_2^+$

Taking B_2^* and $B_2^+ = \begin{pmatrix} \frac{-11}{3} \\ 0 \\ \frac{-8}{3} \\ \frac{-1}{3} \end{pmatrix}$.

Then we can verify the eigenvector by using the equation $A \otimes x = \lambda \otimes x$

$$A \otimes x = \begin{pmatrix} 1 \\ \frac{14}{3} \\ 2 \\ \frac{13}{3} \end{pmatrix} = \lambda \otimes x = \frac{14}{3} \otimes \begin{pmatrix} \frac{-11}{3} \\ 0 \\ \frac{-8}{3} \\ \frac{-1}{3} \end{pmatrix} = \begin{pmatrix} 1 \\ \frac{14}{3} \\ 2 \\ \frac{13}{3} \end{pmatrix}$$

2.3 Modeling Issue

Example 2.13 [2] Let us assume that the department of the railway company in section 1.1, Example 1.2, decides to buy an additional train in order to reach the speed-up to the network's behavior (i.e., to obtain a timetable with an average time less than λ =4).

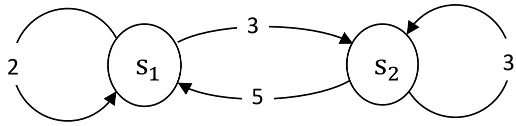

Figure 1.1: Train problem

On which cycle should this additional train run? Assume that the additional train is set on the track from S_1 to S_2, just outside the station S_1 and at the time when train number *k* has already left in the direction S_2. Hence, train number k is in front of the new train. If train *k* is re-numbered as the $(k-1)^{st}$ and the new train is given the number k, then the form that produce the smallest possible departure times is given by

$x_1(k+1) = \max\,(x_1(k)+2, x_2(k)+5)$

$x_2(k+1) = \max\,(x_1(k-1)+3, x_2(k)+3)$

which can be written as a first-order difference equation by introducing an auxiliary variable x_3 with $x_3(k+1) \overset{\text{def}}{=} x_1(k)$ as follows:

$$\begin{pmatrix} x_1(k+1) \\ x_2(k+1) \\ x_3(k+1) \end{pmatrix} = \begin{pmatrix} 2 & 5 & \varepsilon \\ \varepsilon & 3 & 3 \\ 0 & \varepsilon & \varepsilon \end{pmatrix} \otimes \begin{pmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \end{pmatrix} \qquad (2.5)$$

In order to interpret this in a different way, one can think of the auxiliary variable x_3 as the departure time at an auxiliary station S_3 situated on the track from S_1 to S_2, just outside S_1, such that the travel time between S_1 and S_3 equals 0 and the travel time from S_3 to S_2 is 3.

There are, of course, other places where the auxiliary station could be located, for example, somewhere on the inner cycle or on one of the two outer cycles.

If, instead of having S_3 nearby S_1 as above, one could locate S_3 just before S_2, still on the track from S_1 to S_2, and then the equations become:

$$\begin{pmatrix} x_1(k+1) \\ x_2(k+1) \\ x_3(k+1) \end{pmatrix} = \begin{pmatrix} 2 & 5 & \varepsilon \\ \varepsilon & 3 & 0 \\ 3 & \varepsilon & \varepsilon \end{pmatrix} \otimes \begin{pmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \end{pmatrix} \tag{2.6}$$

However, with S_3 just after S_2 on the track towards S_1,

$$\begin{pmatrix} x_1(k+1) \\ x_2(k+1) \\ x_3(k+1) \end{pmatrix} = \begin{pmatrix} 2 & \varepsilon & 5 \\ 3 & 3 & \varepsilon \\ \varepsilon & 0 & \varepsilon \end{pmatrix} \otimes \begin{pmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \end{pmatrix} \tag{2.7}$$

Each of the three models (2.5), (2.6), and (2.7) essentially describes the same speed-up of the network's behavior. It is not astonishing, then, that the eigenvalues of the three matrices are identical. A little exercise shows that these eigenvalues all equal 3. That the eigenvalues cannot be smaller than 3 is easy to understand, since the outer cycle at S_2 has one train and the travel time equals 3. On the inner cycle, the average interdeparture time cannot be smaller than 8/3. Evidently, the outer cycle at S_2 has now become the predicament. A small calculation will provide that eigenvectors corresponding to models (2.5), (2.6), and (2.7) will be

$$\begin{pmatrix} 0 \\ -2 \\ -3 \end{pmatrix}, \begin{pmatrix} 0 \\ -2 \\ 0 \end{pmatrix} \text{ and } \begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix}$$

respectively. Since eigenvectors are determined up to the addition of a constant, the eigenvectors given above are scaled in such a way that the first element is equal to zero. For the first (and second) model, the departure times at S_1 are $0,3,6,9,\ldots$, and for S_2 they are $-2,1,4,7,\ldots$.for the third model these sequences are $0,3,6,9,\ldots$and $1,4,7,10,\ldots$, respectively.

Thus, one notices that the k^{th} departure time at S_2 of model (2.6) coincides with the $(k-1)^{st}$ departure time at S_2 of model (2.7).

Apparently, the geographical shift of station S_3 on the inner cycle, from just before S_2 to just after it, causes a shift in the counting of the departures and their times.

2.4 Max-plus Linear Discrete Event Systems

2.4.1 Max-plus Linear State Space Models

A Discrete Event System (DES) only with synchronization and no concurrency can be modeled by a max-plus-algebraic model as:

$$\begin{cases} x(k) = A \otimes x(k-1) \oplus B \otimes u(k) \\ y(k) = C \otimes x(k) \end{cases} \tag{2.8}$$

with $A \in \mathbb{R}_{\max}^{n\times n}$, $B \in \mathbb{R}_{\max}^{n\times m}$, and $C \in \mathbb{R}_{\max}^{l\times n}$ where m is the number of inputs and l the number of outputs. The vector x represents the state, u is the input vector, and y is the output vector of the system. It is important to note that in (2.8) the components of the input, the output, and the state are event times, and that the counter k in (2.8) is an event counter. For a manufacturing system, $u(k)$ would typically represent the time instants at which raw material is fed to the system for the k^{th} time, $x(k)$ the time instants at which the machines start processing the k^{th} batch of intermediate products, and $y(k)$ the time instants at which the k^{th} batch of finished products leaves the system. Due to the analogy with conventional linear time-invariant systems, a DES that can be modeled by (2.8) will be called a max-plus linear time-invariant DES system.

Typical examples of systems that can be modeled as max-plus linear DES are production systems, railway networks, urban traffic networks, and queuing systems.

We will now illustrate in detail how a max-plus linear model of the form (2.8) can describe the behavior of a simple manufacturing system. Consider the production system of Figure 2.5 [3].

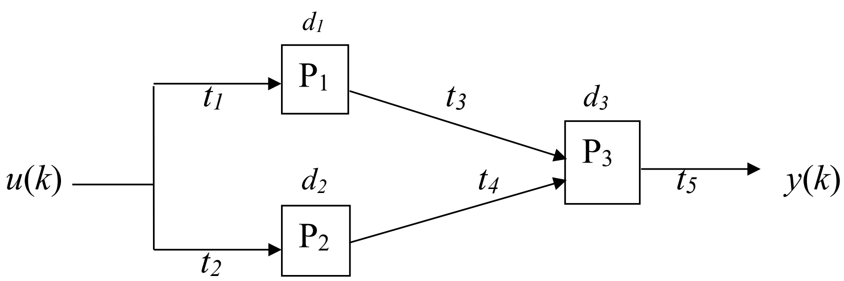

Figure 2.5: A simple manufacturing system

This manufacturing system consists of three processing units: P_1, P_2 and P_3, and works in batches (one batch for each finished product). Raw material is fed to P_1 and P_2, processed and sent to P_3, where assembly takes place. Note that each input batch of raw material is split into two parts: one part of the batch goes to P_1 and the other part goes to P_2.

The processing times for P_1, P_2 and P_3 are respectively d_1, d_2 and d_3 time units. We assume that it takes t_1 time units for the raw material to get from the input source to P_1, and t_3 time units for a finished product of P_1 to get to P_3. The other transportation times (t_2, t_4, and t_5) are assumed to be negligible.

At the input of the system and between the processing units, there are buffers with a capacity that is large enough to ensure that no buffer overflow occurs. A processing unit can only start working on a new product if it has finished processing the previous one. We assume that each processing unit starts working as soon as all parts are available.

Now we write down the max-plus-algebraic state space model of this DES. First, we determine $x_1(k)$, i.e., the time instant at which the processing unit P_1 starts working for the k^{th} time. If we feed raw material to the system for the k^{th} time, then this raw material is available at the input of the processing unit P_1 at time $t = u(k) + t_1$.

However, P_1 can only start working on the new batch of raw material as soon as it has finished processing the current, i.e. the$(k-1)^{th}$, batch.

Since the processing time on P_1 is d_1 time units, the $(k-1)^{th}$ intermediate product will leave P_1 at time $t = x_1(k-1) + d_1$.

Since P_1 starts working on a batch of raw material as soon as the raw material is available and the current batch has left the processing unit, this implies that we have

$x_1(k) = \max\,(x_1(k-1) + d_1, u(k) + t_1)$.

Using a similar reasoning, we find

$x_2(k) = \max\,(x_2(k-1) + d_2, u(k) + t_2)$

$x_3(k) = \max\,(x_1(k) + d_1 + t_3, x_2(k) + d_2 + t_4, x_3(k-1) + d_3, u(k) + \max\,(d_1 + t_1 + t_3, d_2 + t_2 + t_4)$

$y(k) = x_3(k) + d_3 + t_5$.

If we rewrite the above evolution equations as a max-plus-linear discrete event systems state space model of the form (2.8), we obtain

$$x(k) = \begin{pmatrix} d_1 & \varepsilon & \varepsilon \\ \varepsilon & d_2 & \varepsilon \\ 2d_1 + t_3 & 2d_2 + t_4 & d_3 \end{pmatrix} \otimes x(k-1) \oplus \begin{pmatrix} t_1 \\ t_2 \\ \max(d_1 + t_1 + t_3, d_2 + t_2 + t_4) \end{pmatrix} \otimes u(k)$$

$$y(k) = (\varepsilon \quad \varepsilon \quad d_3 + t_5) \otimes x(k)$$

2.4.2 Example of Simple Production System

Example 2.14 Consider the simple production system of Figure 2.6. [3].

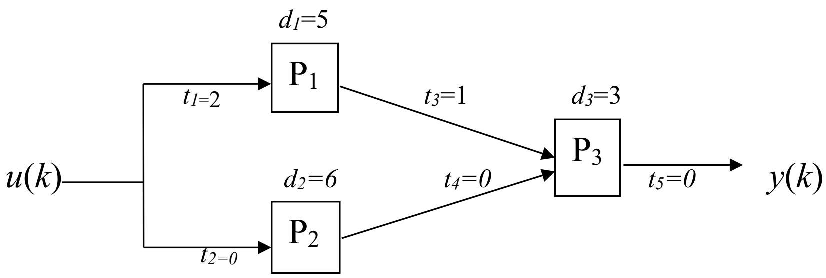

Figure 2.5: A simple manufacturing system

This system consists of three processing units: P_1, P_2, and P_3. Raw material is fed to P_1 and P_2, processed, and sent to P_3 where assembly takes place. The processing times for P_1, P_2, and P_3 are respectively d_1 =5, d_2 =6, and d_3 =3 time units.

We assume that it takes t_1 = 2 time units for the raw material to get from the input source to P_1 and that it takes t_3 = 1 time unit for the finished products of processing unit P_1 to reach P_3.

The other transportation times (t_2, t_4, and t_5) are assumed to be negligible. At the input of the system and between the processing units, there are buffers with a capacity that is large enough to ensure that no buffer overflow will occur. Initially all buffers are empty and none of the processing units contains raw material or intermediate products.

A processing unit can only start working on a new product if it has finished processing the previous one. We assume that each processing unit starts working as soon as all parts are available. Define

- $u(k)$: time instant at which raw material is fed to the system for the k^{th} time,
- $x_i(k)$: time instant at which the i^{th} processing unit starts working for the k^{th} time,
- $y\ (k)$: time instant at which the k^{th} finished product leaves the system.

Let us now determine the time instant at which the processing unit P_1 starts working for the k^{th} time. If we feed raw material to the system for the k^{th} time, then this raw material is available at the input of processing unit P_1 at time $t = u(k) + 2$. However, P_1 can only start working on the new batch of raw material as soon as it has finished processing the previous, i.e., the $(k-1)^{st}$, batch. Since the processing time on P_1 is d_1 = 5 time units, the $(k-1)^{st}$ intermediate product will leave P_1 at time $t = x_1(k-1)\ + 5$. Since P_1 starts working on a batch of raw material as soon as the raw material is available and the current batch has left the processing unit, this implies that we have

$$x_1(k) = \max(x_1(k-1) + 5, u(k) + 2) \tag{2.9}$$

for k =1,2, . . . The condition that initially the processing unit P_1 is empty and idle corresponds to the initial condition $x_1(0)= \varepsilon$ since then it follows from (2.9) that $x_1(1) = u(1) + 2$, i.e., the first batch of raw material that is fed to the system will be processed immediately (after a delay of 2 time units needed to transport the raw material from the input to P_1).

Using a similar reasoning, we find the following expressions for the time instants at which P_2 and P_3 start working for the $(k+1)^{st}$time and for the time instant at which the k^{th}-finished product leaves the system:

$$x_2(k) = \max(x_2(k-1) + 6, u(k) + 0) \tag{2.10}$$

$$\begin{aligned} x_3(k) &= \max(x_1(k) + 5 + 1, x_2(k) + 6 + 0, x_3(k-1) + 3) \\ &= \max(x_1(k-1) + 11, x_2(k-1) + 12, x_3(k-1) + 3, u(k) + 8) \end{aligned} \tag{2.11}$$

$$y(k) = x_3(k) + 3 + 0 \tag{2.12}$$

For $k = 1, 2 \ldots$ The condition that initially all buffers are empty corresponds to the initial condition $x_1(0) = x_2(0) = x_3(0) = \varepsilon$.

Let us now rewrite the equations of the production system using the symbols ⊕ and ⊗; in that case, (2.9) can be written as

$$x_1(k) = 5 \otimes x_1(k-1) \oplus 2 \otimes u(k) .$$

If we do the same for (2.4) – (2.6), the resulting equations in max-plus-algebraic matrix notation, we obtain

$$x(k) = \begin{pmatrix} 5 & \varepsilon & \varepsilon \\ \varepsilon & 6 & \varepsilon \\ 11 & 12 & 3 \end{pmatrix} \otimes x(k-1) \oplus \begin{pmatrix} 2 \\ 0 \\ 8 \end{pmatrix} \otimes u(k)$$

$$y(k) = [\varepsilon \quad \varepsilon \quad 3] \otimes x(k),$$

where $x(k) = [x_1(k) \quad x_2(k) \quad x_3(k)]^T$.

Example 2.15 Consider the more complex manufacturing system in Figure 2.7, which also can be described by using the form (2.8). This manufacturing system consists of five processing units: P_1, P_2, P_3, P_4 and P_5 and works in batches (one batch for each finished product).

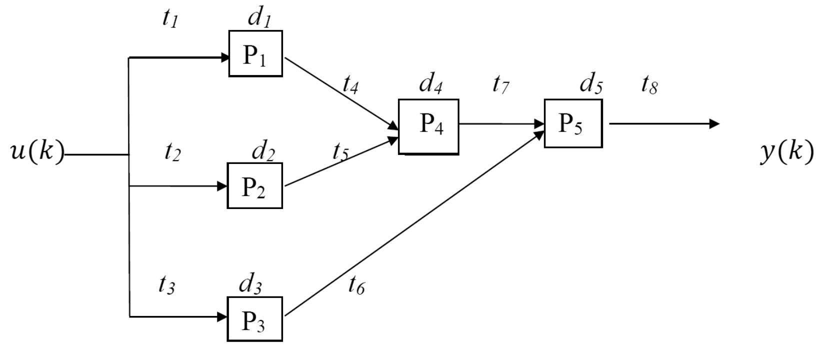

Figure 2.7: A manufacturing system

Raw material is fed to P_1, P_2 and P_3. P_1 and P_2 are processed and sent to P_4. P_3 and P_4 are processed and sent to P_5, where assembly takes place. Note that each input batch of raw material is split into three parts: one part of the batch goes to P_1, the second part goes to P_2 and the third part goes to P_3. The processing times for P_1, P_2, P_3 P_4 and P_5 are respectively d_1, d_2, d_3, d_4, d_5 time units.

We assume that it takes t_1 time units for the raw material to get from the input source to P_1, and t_4 time units for a finished product of P_1 to get to P_4, and t_7 time units for a finished product of P_4 to get to P_5. The other transportation times are assumed negligible.

At the input of the system and between the processing units, there are buffers with a capacity that is large enough to ensure that no buffer overflow occurs. A processing unit can only start working on a new product if it has finished processing the previous one. We assume that each processing unit starts working as soon as all parts are available.

Now we write down the max-plus-algebraic state space model of this DES. First, we determine $x_1(k)$, i.e., the time instant at which processing unit P_1 starts working for the k^{th} time. If we feed raw material to the system for the k^{th} time, then this raw material is available at the input of processing unit P_1 at time $t = u(k) + t_1$.

However, P_1 can only start working on the new batch of raw material as soon as it has finished processing the current, i.e. the $(k - 1)^{th}$, batch.

Since the processing time on P_1 is d_1 time units, the $(k - 1)^{th}$ intermediate product will leave P_1 at time $t = x_1(k - 1) + d_1$.

Since P_1 starts working on a batch of raw material as soon as the raw material is available and the current batch has left the processing unit, this implies that we have

$x_1(k) = \max(x_1(k-1) + d_1, u(k) + t_1)$

Using a similar reasoning, we find

$$x_2(k) = \max(x_2(k-1) + d_2, u(k) + t_2)$$

$$x_3(k) = \max(x_3(k-1) + d_3, u(k) + t_3)$$

$$\begin{aligned} x_4(k) &= \max(x_1(k) + d_1 + t_4\,,\ x_2(k) + d_2 + t_5,\ x_4(k-1) + d_4\,) \\ &= \max(x_1(k-1) + 2d_1 + t_4\,,\ x_2(k-1) + 2d_2 + t_5,\ x_4(k-1) + d_4, u(k) \\ &\qquad + \max(d_1 + t_1 + t_4\,, d_2 + t_2 + t_5)) \end{aligned}$$

$$\begin{aligned} x_5(k) &= \max(x_3(k) + d_3 + t_6\,, x_4(k) + d_4 + t_7, x_5(k-1) + d_5) \\ &= \max(x_3(k-1) + d_3 + d_3 + t_6, u(k) + t_3 + d_3 + t_6, x_1(k-1) + 2d_1 + t_4 + d_4 + \\ &\quad t_7\,,\ x_2(k-1) + 2d_2 + t_5 + d_4 + t_7,\ x_4(k-1) + d_4 + d_4 + t_7, u(k) + \\ &\quad \max(d_1 + t_1 + t_4 + d_4 + t_7\,, d_2 + t_2 + t_5 + d_4 + t_7), x_5(k-1) + d_5) \end{aligned}$$

$$\begin{aligned} &= \max\,(x_1(k-1) + 2d_1 + t_4 + d_4 + t_7,\ x_2(k-1) + 2d_2 + t_5 + d_4 + t_7,\ \ x_3(k-1) + 2d_3 \\ &\qquad + t_6\,,\ x_4(k-1) + 2d_4 + t_7,\ x_5(k-1) + d_5\,, u(k) + \max(d_3 + t_6 + t_3\,, d_1 + t_1 + t_4 \\ &\qquad\quad + d_4 + t_7, d_2 + t_2 + t_5 + d_4 + t_7)) \end{aligned}$$

$$y(k) = x_5(k) + d_5 + t_8$$

If we rewrite the above equations as a Max-plus-linear discrete event systems state space model of the form (2.8), we obtain

$$x(k) = \begin{pmatrix} d_1 & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & d_2 & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & d_3 & \varepsilon & \varepsilon \\ 2d_1 + t_4 & 2d_2 + t_5 & \varepsilon & d_4 & \varepsilon \\ 2d_1 + t_4 + d_4 + t_7 & 2d_2 + t_5 + d_4 + t_7 & 2d_3 + t_6 & 2d_4 + t_7 & d_5 \end{pmatrix}$$

$$\otimes x(k-1) \oplus \begin{pmatrix} t_1 \\ t_2 \\ t_3 \\ \max(d_1 + t_1 + t_4, d_2 + t_2 + t_5) \\ \max(d_3 + t_3 + t_6, t_1 + d_1 + d_4 + t_4 + t_7, t_2 + d_2 + t_5 + d_4 + t_7) \end{pmatrix} \otimes u(k)$$

$y(k) = (\varepsilon \quad \varepsilon \quad \varepsilon \quad \varepsilon \quad d_5 + t_8)\otimes x(k),$

where $x(k) = (x_1(k) \quad x_2(k) \quad x_3(k) \quad x_4(k) \quad x_5(k))^T$

2.5 An Introduction to Max-plus Algebraic System Theory

In this section, we will introduce the system theory for max-plus linear time for DESs [3, 10]. In Section 2.4, we have shown by some simple examples that time deterministic DESs in which the sequence of the events and the duration of the activities are fixed or can be determined in advance can be described by an n^{th} order state space model of the form

$$\begin{cases} x(k) = A\otimes x(k-1)\oplus B\otimes u(k) \\ \qquad y(k) = C\otimes x(k) \end{cases} \tag{2.8}$$

For all $k \in N_0$ with an intial condition $x(0) = x_0$ and where $A \in \mathbb{R}_{\max}^{n\times n}$, $B \in \mathbb{R}_{\max}^{n\times m}$ and $C \in \mathbb{R}_{\max}^{l\times n}$. Let us introduce some analysis techniques for DESs that can be described by a model of the form (2.8).

$x(1) = A\otimes x(0)\oplus B\otimes u(0)$

$x(2) = A\otimes x(1)\oplus B\otimes u(1)$

$\qquad = A^{\otimes 2}\otimes x(0)\oplus A\otimes B\otimes u(0))\oplus B\otimes u(1)$

Thus, $x(k) = A^{\otimes k}\otimes x(0) \oplus_{i=0}^{k-1} A^{\otimes(k-i-1)}\otimes B\otimes u(i)$

where the empty max-algebraic sum $\oplus_{i=0}^{k-1}$ is equal to $\varepsilon_{n\times 1}$ by definition.

$$y(k) = C \otimes A^{\otimes k} \otimes x(0) \oplus \otimes_{i=0}^{k-1} C \otimes A^{k-i-1} \otimes B \otimes u(i) \tag{2.13}$$

Consider two input sequences $u_1 = \{u_1(k)\}_{k=0}^{\infty}$, $u_2 = \{u_2(k)\}_{k=0}^{\infty}$.

Let $y_1 = \{y_1(k)\}_{k=1}^{\infty}$ be the output sequence that corresponds to the input sequence (with initial condition $x_{1,0}$) and let $y_2 = \{y_2(k)\}_{k=1}^{\infty}$ be the output sequence that corresponds to the input sequence u_2 (with initial condition $x_{2,0}$).

Let $\alpha, \beta \in \mathbb{R}_{\max}$. From (2.13) it follows that the output sequence that corresponds to the input sequence $\alpha \otimes u_1 \oplus \beta \otimes u_2 = \alpha \otimes \{u_1(k)\}_{k=0}^{\infty} \oplus \beta \otimes \{u_2(k)\}_{k=0}^{\infty}$ (with the initial condition $\alpha\otimes x_{1,0}\oplus\beta\otimes x_{2,0}$) is given by $\alpha\otimes y_1\oplus\beta\otimes y_2$.

This explains why DESs that can be described by a model of the form (2.8) are called max-linear.

Now we assume that $x(0) = \varepsilon_{n\times 1}$ and let $p \in N_0$

If we define $Y = \left(y(1) \quad y(2) \quad \dots\dots \ y(p)\right)^T$ and $U = (u(0) \quad u(1) \dots\dots u(p-1))^T$, then (2,13) results in $Y = H \otimes U$

$$H = \begin{pmatrix} C \otimes B & \varepsilon & \dots & \varepsilon \\ C \otimes A \otimes B & C \otimes B & \dots & \varepsilon \\ \vdots & \vdots & & \vdots \\ C \otimes A^{\otimes p-1} \otimes B & C \otimes A^{\otimes p-2} \otimes B & \dots & C \otimes B \end{pmatrix}$$

For the production system of Example 2.14, Section 2.4, this means that if we know the time instants at which raw material is fed to the system, we can compute the time instants at which the finished products will leave the system.

If we know the vector Y of latest times at which the finished products have to leave the system, we can compute the vector U of (latest) time instants at which raw material has to be fed to the system by solving the system of max-plus linear equations $H \otimes U = Y$, if it has a solution, or finding the greatest sub solution of $H \otimes U = Y$.

However, if we have perishable goods it is better to minimize the maximal deviation between the required and the actual finishing times. In this case, we must solve the following problem:

$$\text{minimize } \max_i |(Y - H \otimes U)_i|$$

This problem can be solved using Theorem 2.1. Let $i \in \{1,2,\dots,m\}$, a max-algebraic unit impulse is a sequence that is defined as follows

$$e(k) = \begin{cases} 0 & \text{if } k = 0 \\ \varepsilon & \text{if } k \neq 0 \end{cases}, \text{ for } k = 0, 1, 2, \dots$$

If we apply a max-plus algebra unit impulse to the i^{th} input of the system and if we suppose that $x(0) = \varepsilon_{n \times 1}$ then we obtain

$$y(k) = C \otimes A^{\otimes k-1} \otimes B_i, \text{ for } k = 1, 2, 3, \dots$$

as an output of the DES. This output is called the impulse response due to a max-plus algebraic impulse at the ith input. Note that $y(k)$ corresponds to the ith column of the matrix,

$$G_{k-1} \overset{\text{def}}{=} C \otimes A^{\otimes k-1} \otimes B, \qquad \text{for } k = 1, 2, 3, \dots$$

Therefore, the sequence $\{G_k\}_{k=0}^{\infty}$ is called the impulse response of the DES. The G_k's are called the **impulse response matrices** or **Markov parameters**.

Example 2.14 section 2.4 shows the meaning to the impulse response of the production system. In the beginning all the buffers of the system are empty, then begin to feed the raw materials in the buffer input and continue to feed the raw materials at such a rate that the input buffer is never empty. Moments in time that the final products leave the system are in accordance with the terms of the impulse response.

Consider the autonomous DES described by

$$x(k+1) = A \otimes x(k)$$
$$y(k) = C \otimes x(k)$$

With $x(0) = x_0$. For the production system of Example 2.14, section 2.4, this means that we start from a situation in which some internal buffers and the entire input buffer are not empty in the beginning, if $x_0 \neq \varepsilon_{n\times 1,}$ and that afterwards the raw material is fed to the system at such a rate that the input buffers never become empty.

If the system matrix A is irreducible, then we can calculate the ultimate behavior of the autonomous DES by solving the max-algebraic eigenvalue problem

$A \otimes v = \lambda \otimes v$ there exist integers $k_0 \in N, c \in N$ such that

$x(k+c) = \lambda^{\otimes c} \otimes x(k)$ which means

$x_i(k+c) - x_i(k) = c\lambda$ for $i = 1,2, \dots\dots, n$ for all $k \geq k_0$. (2.14)

This behavior will be called cyclic. From (2.14) it follows that for a production system λ will be the average duration of a cycle of the production process when the system has reached its cyclic behavior. The average production rate will then be equal $\frac{1}{\lambda}$.

Example 2.16 [3] Consider the production system of Example 2.14.

Define $Y = [\,y(1) \quad y(2) \quad y(3) \quad y(4)]^T$ and $U = [u(0) \quad u(1) \quad u(2) \quad u(3)]^T$.

If $x_0 = \varepsilon_{3\times 1}$, then we have $Y = H \otimes U$ with

$$H = \begin{pmatrix} 11 & \varepsilon & \varepsilon & \varepsilon \\ 16 & 11 & \varepsilon & \varepsilon \\ 21 & 16 & 11 & \varepsilon \\ 27 & 21 & 16 & 11 \end{pmatrix}$$

If we feed raw material to the system at time instants $u(0) = 0, u(1) = 9, u(2) = 12, u(3) = 15$,

$$\Rightarrow U = \begin{pmatrix} 0 \\ 9 \\ 12 \\ 15 \end{pmatrix}$$

$$\Rightarrow H \otimes U = \begin{pmatrix} 11 & \varepsilon & \varepsilon & \varepsilon \\ 16 & 11 & \varepsilon & \varepsilon \\ 21 & 16 & 11 & \varepsilon \\ 27 & 21 & 16 & 11 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 9 \\ 12 \\ 15 \end{pmatrix} = \begin{pmatrix} 11 \\ 20 \\ 25 \\ 30 \end{pmatrix} = Y$$

Thus, the finished products will leave the system at time instants $y(1) = 11, y(2) = 20, y(3) = 25$ and $y(4) = 30$.

If the finished parts should leave the system before time instants 17, 19, 24 and 27 and if we want to feed the raw material to the system as late as possible, then we should feed raw material to the system at time instants 0, 6, 11, 16, since $[0\ \ 6\ \ 11\ \ 16]^T$ is the greatest subsolution of

$$H \otimes U = \begin{pmatrix} 17 \\ 19 \\ 24 \\ 27 \end{pmatrix}$$

The impulse response of the system is given by

$$\{G_k\}_{k=0}^{\infty} = 11, 16, 21, 27, 33, 39, 45, 51, 57, 63, 69, 75, \dots \dots \dots$$

Although the system matrix A is not irreducible, the system in the autonomous case does exhibit an ultimately cyclic behavior of the form (2.14) with $\lambda = 6$ and $c = 1$. It is easy to verify that λ corresponds to the largest average cycle weight of the precedence graph of A (See Figure 2.8) and to the largest max-algebraic eigenvalue of A. If we feed the system by raw material at a rate such that the input buffer never becomes empty, then after a finite number of production cycles, the difference between $x_i(k+1)$ and $x_i(k)$ will be equal to 6 for all processing units P_i.

Thus, the average production rate of the system is $\frac{1}{6}$, i.e. every 6 time units a finished part leaves the production system.

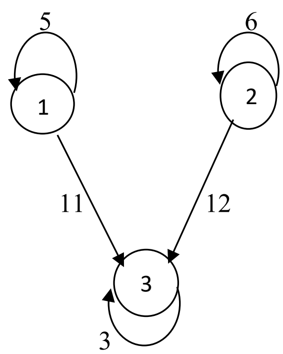

Figure 2.8: The precedence graph of the matrix A of example 2.13 section 2.4

2.6 Characteristic Equation and the Cayley-Hamilton Theorem

This section is organized as follows: In 2.6.1, we introduce some notations and definitions. In Subsection 2.6.2, we give a short introduction to the characteristic equation and the Cayley-Hamilton theorem in conventional algebra [1, 7]. In 2.6.3, we derive the characteristic equation of a matrix in the max-plus algebra. We also include an example in which we compute the max-algebraic characteristic equation of a given matrix [1], [7] and [11].

2.6.1 Notations and Definitions

If A is a set, then $|A|$ denotes the number of elements of A. We use P_n to represent the set of all permutations of n numbers. The set of even permutations of n numbers is denoted by P_n^e. and the set of the odd permutations of n numbers is denoted by P_n^0. An element σ of P_n will be considered as a mapping from $\{1, 2, \ldots, n\}$ to $\{1, 2, \ldots, n\}$.

We say a permutation is even if it can be written as a product of an even number of (usually non-disjoint) transpositions (i.e. 2-cycles). Likewise, a permutation is odd if it can be written as a product of an odd number of transpositions [7].

Let S_n be the symmetric group associated to the bijections of the set $= \{1,2, \ldots, n\}$. A transposition is a 2-cycle $c \in S_n$. It is known that transpositions generate S_n. Equivalently stated, every permutation can be written as a product of transpositions. Notice, though, that unlike the decomposition of σ into disjoint cycles, the decomposition of a permutation as a product of transpositions is not unique! However, the parity of the number of transpositions which appear in any such decomposition is independent from the chosen decomposition. Let s be the number of transpositions which appear in a decomposition of $\sigma \epsilon S_n$.

Definition 2.3 The signature of σ is

$$\mathrm{sgn}(\sigma) = (-1)^s.$$

There is an equivalent way to state the definition of the signature of a permutation σ by considering the canonical decomposition of σ into disjoint cycles.

Let us assume that the decomposition of σ into disjoint cycles is given by $\sigma = c_1 \ldots c_r$.

Definition 2.4 The signature of a *r*-cycle $c \in S_n$ is

$$\operatorname{sgn}(c) = \begin{cases} -1 & \text{if } r \text{ is odd} \\ 1 & \text{if } r \text{ is even} \end{cases}$$

Consequently, we set

Definition 2.5 The signature of the permutation $\sigma = c_1 \dots \dots c_r$ is

$$\operatorname{sgn}(c) = \prod_{i=1}^{r} \operatorname{sgn}(c_i).$$

The signature of the permutation σ is denoted by sgn(σ). We use C_k^n to represent the set of all subsets with k elements of the set $\{1, 2, \dots, n\}$.

Let A be an *n* by *n* matrix and let $\phi \subseteq \{1, 2, \dots, n\}$. The submatrix obtained by removing all rows and columns of *A* except for those indexed by ϕ is denoted by $A_{\varphi\varphi}$. The matrix $A_{\varphi\varphi}$ is called a principal submatrix of the matrix *A*.

Definition 2.6 Let f and g be real functions. The function *f* is asymptotically equivalent to *g* in the neighborhood of ∞, denoted by $f(x) \sim g(x), x \to \infty$,

if $\lim_{x \to \infty} \frac{f(x)}{g(x)} = 1$.

We say that $f(x) \approx 0, x \to \infty$ if there is a real number *K* such that

$f(x) = 0$ for all $x \geq K$.

If F and G are real *m* by *n* matrix-valued functions, then we have $F(x) \sim G(x)$ when

$\boldsymbol{x} \to \infty$ if $f_{ij}(x) \sim g_{ij}(x)$ when $x \to \infty$ for all i, j.

2.6.2 The Characteristic Equation in Ordinary Linear Algebra

In conventional linear algebra, the **Cayley–Hamilton theorem** (named after the mathematicians Arthur Cayley and William Hamilton) states that every square matrix over a commutative ring (including the real or complex field) satisfies its own characteristic equation [1], [7] and [11].

More specifically, if A is a given $n \times n$ matrix and I_n is the $n \times n$ identity matrix, then the characteristic polynomial of *A* is defined as:

$$p(\lambda) = \det(\lambda I_n - A)$$

where "det" is the determinant operation, since the entries of the matrix are (linear or constant) polynomials in λ, the determinant is also a polynomial in λ.

The Cayley–Hamilton theorem declares that ″substituting″ the matrix A for λ in these polynomial results in zero matrices: $\mathrm{P(A)} = 0$.
The powers of λ that have become powers of *A* by the substitution should be computed by repeated matrix multiplication, and the constant term should be multiplied by the identity matrix (the zero$^{\text{th}}$ power of *A*) so that it can be added to the other terms.
The Cayley–Hamilton theorem is equivalent to the statement that the minimal polynomial of a square matrix divides its characteristic polynomial.

Example 2.17 As a concrete example [7].

Let $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ its characteristic polynomial given by

$$p(\lambda) = \det(\lambda \times I_2 - A) = \begin{vmatrix} \lambda - 1 & -2 \\ -3 & \lambda - 4 \end{vmatrix}$$

$$= (\lambda - 1)(\lambda - 4) - (-2)(-3) = \lambda^2 - 5\lambda - 2.$$

The Cayley–Hamilton theorem claims that if we define

$p(X) = X^{\,2} - 5X - 2I_2$, then

$$p(A) = A^{\,2} - 5A - 2I_2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

which one can easily verify.

Note: The characteristic equation in conventional algebra is used to calculate the eigenvalue of a matrix.
Now we need to define a characteristic polynomial, characteristic equation and provide an analogue of the Cayley–Hamilton theorem in max-plus algebra.

2.6.3 The Characteristic Equation in Max-plus Algebra

To derive the max-plus algebraic characteristic equation of a matrix [1],

let $a, b, c \in \mathbb{R}_{\max}$ then we have:

$$a \oplus b = c \Leftrightarrow z^a + z^b \simeq \alpha z^c \quad \text{when } z \to \infty$$

$$a \otimes b = c \Leftrightarrow z^a z^b = z^c \quad \text{for all } z \in \mathbb{R}^+$$

where $\alpha = 1$ if $a \neq b$ and $\alpha = 2$ if $a = b$ and where $z^{\varepsilon} = 0$ for all $z \in \mathbb{R}^+$ by definition.

If $A \in \mathbb{R}_{\max}^{n \times n}$ then z^A is a real *n* by *n* matrix with domain of definition $\mathbb{R}^+$ that is defined by

$(z^A)_{ij} = z^{a_{ij}}$for all i, j.

The dominant of A is defined as follows:

$$\text{dom}_{\oplus}A = \begin{cases} \text{the highest exponent in } \det z^A & \text{if } \det z^A \neq 0, \\ \varepsilon & \text{otherwise.} \end{cases}$$

Theorem 2.5: Suppose that $A \in \mathbb{R}_{\max}^{n\times n}$, if

$$\det(\lambda I_n - z^{\text{A}}) = \lambda^n + c_1\lambda^{n-1} + c_2\lambda^{n-2} + \cdots + c_{n-1}\lambda + c_n \text{ then}$$

$$A^n + c_1A^{n-1} + c_2A^{n-2} + \cdots + c_{n-1}A + c_nI = 0$$

where $c_k = (-1)^k \sum_{\varphi\in C_n^k} \det A_{\varphi\varphi}$

We will show how the Cayley–Hamilton Theorem can be translated into max-plus algebra. We consider the matrix z^A [1], [7]. We also need a few more definitions.

First, we have the characteristic polynomial of the matrix valued function z^A given by:

$$\det(\lambda I - z^A) = \lambda^n + \gamma_1(z)\,\lambda^{n-1} + \ldots\ldots + \gamma_{n-1}(z)\,\lambda + \gamma_n(z) \qquad (2.15)$$

With coefficients:

$$\gamma_k(z) = (-1)^k \sum_{\phi\in C_n^k} \det z^{A_{\phi\phi}} \qquad (2.16)$$

Therefore

$$(z^A)^n + \gamma_1(z)(z^A)^{n-1} + \ldots + \gamma_{n-1}(z)z^A + \gamma_nI_n = 0 \qquad (2.17)$$

For all $z \in \mathbb{R}^+$.

This is just the result of applying Theorem 2.5 to the matrix z^A.

The highest degree in (2.16) is equal to $\max\{dom_{\oplus}A_{\phi\phi}|\phi \in C_n^k\}$ and

$$\gamma_k(z) \sim (-1)^k\, \overline{\gamma}_k\, z^{\max\{dom_{\oplus}A_{\phi\phi}|\phi\in C_n^k\}},\ z \to \infty$$

where $\bar{\gamma}_k$ is equal to the number of permutations that contribute to the highest degree in (2.15) taking away the P_n^o. However, the highest degree in (2.15) is not necessarily equal to $\max\{dom_{\oplus}A_{\phi\phi}|\phi \in C_n^k\}$ since if P_n^e that contribute to $z^{\max\{dom_{\oplus}A_{\phi\phi}|\phi\in C_n^k\}}$ is equal to P_n^o that contribute to $z^{\max\{dom_{\oplus}A_{\phi\phi}|\phi\in C_n^k\}}$ the term that contains $z^{\max\{dom_{\oplus}A_{\phi\phi}|\phi\in C_n^k\}}$ disappears.

Note: Recall that P_n represents the set of all permutations of n number $\sigma: \{1,2,\ldots,n\} \to \{1,2,\ldots,n\}$. The set of the even permutations of n numbers is denoted by P_n^e and the set of the odd permutations of n numbers is denoted by P_n^o.

Example 2.18 Consider the matrix A [7], where

$$A = \begin{pmatrix} -1 & \varepsilon & 2 \\ 0 & 2 & \varepsilon \\ 1 & 0 & 1 \end{pmatrix} \quad \text{so the matrix valued function } z^{A} = \begin{pmatrix} z^{-1} & 0 & z^2 \\ 1 & z^2 & 0 \\ z & 0 & z \end{pmatrix}$$

$$\text{so } \det z^{A\{1,2\},\{1,2\}} = \det \begin{pmatrix} z^{-1} & 0 \\ 1 & z^2 \end{pmatrix} = z$$

$$\det z^{A\{1,3\},\{1,3\}} = \det \begin{pmatrix} z^{-1} & z^2 \\ z & z \end{pmatrix} = -z^3 + 1$$

$$\det z^{A\{2,3\},\{2,3\}} = \det \begin{pmatrix} z^2 & 0 \\ 0 & z \end{pmatrix} = z^3$$

$$\Rightarrow \quad \mathrm{dom}_{\oplus} A_{\{1,2\},\{1,2\}} = 1, \ \mathrm{dom}_{\oplus} A_{\{1,3\},\{1,3\}} = 3 \ \text{ and } \ \mathrm{dom}_{\oplus} A_{\{2,3\},\{2,3\}} = 3$$

and because $\gamma_k(z) = (-1)^k \sum_{\phi \in c_n^k} \det z^{A_{\varphi\varphi}}$

$$\Rightarrow \gamma_2(z) = (-1)^2 (\det z^{A\{1,2\},\{1,2\}} + \det z^{A\{1,3\},\{1,3\}} + \det z^{A\{2,3\},\{2,3\}})$$

$$= z - z^3 + z^3 + 1$$

$$= z + 1$$

Hence, the highest degree of the polynomial $\gamma_2(z)$ is equal to 1 but

$$\max\{ \ \mathrm{dom}_{\oplus} A_{\phi\phi} \mid \phi \in C_3^2 \} = 3 \neq 1.$$

The highest degree term in (2.16) can be determined as follows:

Let us define $\Gamma_k = \left\{ \zeta : \exists \{i_1, i_2, \dots, i_k\} \in C_n^k, \exists\, \sigma \in P_k \text{ such that } \zeta = \sum_{r=1}^{k} a_{\,i_r i_{\sigma(r)}} \right\}$ for $k = 1,2,\dots,n$. For every $k \in \{1,2,\dots,n\}$ and for every $\zeta \in \Gamma_k$ we can define:

$$I_k^e(\zeta) = \left| \left\{ \ \sigma \in P_k^e \mid \exists \{i_1, i_2, \dots, i_k\} \in C_n^k, \text{ such that } \sum_{r=1}^{k} a_{i_r i_{\sigma(r)}} = \zeta \right\} \right|$$

$$I_k^o(\zeta) = \left| \left\{ \ \sigma \in P_k^o \mid \exists \{i_1, i_2, \dots, i_k\} \in C_n^k, \text{ such that } \sum_{r=1}^{k} a_{i_r i_{\sigma(r)}} = \zeta \ \right\} \right|$$

$I_k(\zeta) = I_k^e(\zeta) - I_k^o(\zeta)$

Thus, we can write (2.16) as

$$\gamma_k(z) = (-1)^k \sum_{\zeta \in \Gamma_k} I_k(\zeta)\, z^{\zeta}$$

the highest degree that appears in $\boldsymbol{\gamma_k(z)}$ is given by

$$d_k = \max\{\zeta \in \Gamma k : Ik\ (\zeta) \neq 0\}$$

And the coefficients of the characteristic equation of z^A satisfy:

$\gamma_k(z) \sim (-1)^k I_k(d_k)\, z^{d_k} \quad , z \to \infty$

Define the leading coefficients of $\gamma_k(z)$: $\hat{\gamma}_k = (-1)^k I_k(d_k)$ for $k = 1,2,\ldots,n$.

Let $L = \{k \mid \hat{\gamma}_k > 0\}$ and $J = \{k \mid \hat{\gamma}_k < 0\}$.

Note that we have $I_1^o(\zeta) = 0$ and $I_1^e(\zeta) > 0$ for every $\zeta \in \Gamma_1 = \{a_{ii} | i = 1,2,\ldots,n\}$.

This implies that $I_1(d_1) > 0$ and that $\hat{\gamma}_1 < 0$.

Hence, we always have $1 \in J$

It is easy to verify that if $A \in \mathbb{R}_{\max}^{n\times n}$ then

$$(Z^A)^k \sim Z^{(A^{\otimes k})},\ Z \to \infty \tag{2.18}$$

If we apply (2.18) in (2.17), we get

$$z^{(A^{\otimes n})} + \sum_{k\in I} \hat{\gamma}_k z^{d_k} z^{(A\otimes^{n-k})} \sim \sum_{K\in J} \hat{\gamma}_k\ z^{d_k} z^{(A\otimes^{n-k})}\ ,\ z \to \infty$$

Because all the terms have positive coefficients, comparison of the highest degree terms of corresponding entries on the left-hand and the right-hand side of this expression leads to the following identity in $\mathbb{R}_{\max}$.

$$A^{\otimes n} \oplus_{k\in I} d_k \otimes A^{\otimes n-k} = \bigoplus_{k\in J} d_k \otimes A^{\otimes n-k}$$

This equation can be considered as a max-plus algebraic version of the Cayley–Hamilton theorem if we define the max-plus characteristic equation of *A* as

$$\lambda^{\otimes n} \oplus_{k\in I} d_k \otimes \lambda^{\otimes n-k} = \oplus_{k\in J} d_k \otimes \lambda^{\otimes n-k} \tag{2.19}$$

Example 2.19 Find the max-plus characteristic equation of the matrix A [7], where

$$A = \begin{pmatrix} -2 & 1 & \varepsilon \\ 1 & 0 & 1 \\ \varepsilon & 0 & 2 \end{pmatrix}$$

we have $\Gamma_1 = \{2, 0, -2\}, \Gamma_2 = \{2, 1, 0, -2, \varepsilon\}$, and $\Gamma_3 = \{4, 0, -1, \varepsilon\}$. Now we can see that

$I_1\,(2) = 1, \quad I_1(0) = 1,$

$I_2(2) = 0, \quad I_2(1) = -1, \quad I_2(0) = 1, \quad I_2(-2) = 1, \quad I_2(\varepsilon) = -1,$

$I_3(4) = -1, \quad I_3(0) = 1, \quad I_3(-1) = -1, \quad I_3(\varepsilon) = 1$

Hence, $d_1 = 2$, $d_2 = 1$ and $d_3 = 4$. Note that the maximum value in Γ_2 *is 2* however $I_2(2) = 0$, and thus an even and odd permutation gives us the diagonal value 2 which means the two permutations cancel each other out; hence $d_2 = 1$.

Since $\hat{\gamma}_1 = -1$, $\hat{\gamma}_2 = -1$ and $\hat{\gamma}_3 = 1$, we have $I = \{3\}$ and $J = \{1, 2\}$.

Thus, the max-plus characteristic equation

$$\lambda^{\otimes^3} \oplus 4 = 2 \otimes \lambda^{\otimes^2} \oplus 1 \otimes \lambda$$

And A satisfies its max plus characteristic equation with

$$A^{\otimes^3} \oplus 4 \otimes I_3 = \begin{pmatrix} 4 & 3 & 4 \\ 3 & 4 & 5 \\ 3 & 4 & 6 \end{pmatrix} = 2 \otimes A^{\otimes^2} \oplus 1 \otimes A$$

Where I_3 is the (3 by 3) identity matrix [7].

Chapter 3

Modeling and Scheduling of Train Network

The increasingly saturated European railway infrastructure has, among other concerns, drawn attention to the stability of train schedules, as they may cause domino effect delays across the entire network. A train timetable must be insensitive with regard to small disturbances, so that recovery from such disturbances can occur without external control. After a break of self-regulation, this behavior schedule requires the distribution of accurate recovery times and buffer times to reduce delays and prevent the propagation of delay, respectively. Schedule models for railways are usually based on deterministic process times (running times, and transfer times). Moreover, running times are rounded and train tracks are modified to fit the schedule or constraints. The validity of these decisions and streamline schedules must be evaluated to ensure the viability, stability and durability, with respect to network mutual relations and differences in process times. Train networks can be modeled using max-plus algebra [24]. Stability can be evaluated by calculating the eigenvalue of the matrix in max-plus algebra [2,3,25,26]. This eigenvalue is the minimum cycle time required to satisfy all of the schedule and progress constraints, where the timetable operating with this eigenvalue time is given by the associated eigenvector [1,2]. Thus, if the eigenvalue λ is more than the intended length T of the schedule, then the schedule is unstable. If $\lambda < \mathrm{T}$ the schedule will be stable, and critical if $\lambda = \mathrm{T}$ [25, 26]. If individual trains are delayed, the effect on the whole network is quite difficult to predict. Smaller delays can typically be absorbed by speeding up the trains, and this can be handled by using max-plus algebra. Larger delays are often handled by rescheduling, typically using optimization, see for example De Schutter et al. [27], D'Ariano et al. [28], Corman et al. [9], van den Boom et al. [10], and Kecman et al. [31].

In this chapter, we study the impact of both permanent and dynamic delays in a train network but restrict ourselves to using max-plus algebra and, thus, we do not consider rescheduling. In practice, then, our study is limited to delays up to half of the cycle time. Meeting conditions, including those introduced by having single tracks, are also fully handled using max-plus state-space formalism, by extending the state with delayed states. When constructing a recovery matrix [25] of this extended system, it naturally results in redundancy, as the same physical state appears many times. This redundant recovery information can, however, be incorrect, due to the fact that no constraints are specified for the delayed states, which are only shifted copies of the most recent state. The parts of the recovery matrix corresponding to the most recent states are still valid.

3.1 An Example of Scheduled Max-plus Linear Systems

Consider the train network in Figure 3.1 [32]. This is a simple network consisting of four stations, Helsinki (H), Karjaa (K), Salo (S) and Turku (T). The end stations are modeled with vertices for both arrival (A in front of the city first letter) and departure (D). The stops at the intermediate stations are short and, thus, only the departures are modeled. The weights d_i on the arcs corresponds to the traveling times, while d_1 and d_5 are service times at end stations. The stations between Helsinki and Karjaa are connected by double tracks, and the other connections are single tracks that introduce meeting time conditions. There are five trains available for this, which also introduce some constraints.

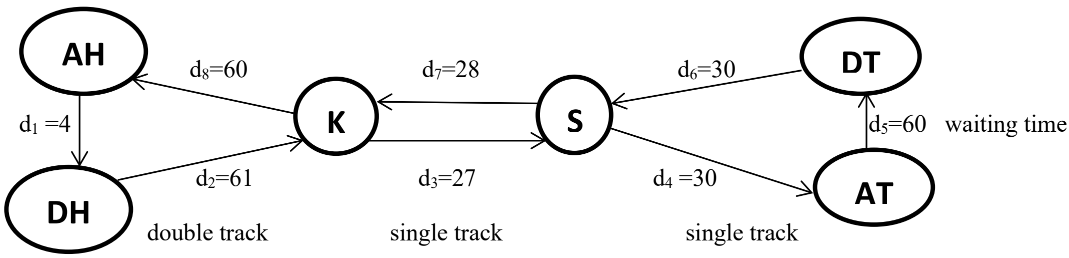

Figure 3.1. The railroad network between Helsinki and Turku in Finland

Table 3.1 provides the schedule [32] of five trains running regularly between Helsinki and Turku and gives the information in connection with the nominal traveling times and the departures.

Table 3.1. Train timetable for trains 1, …, 5 between Turku and Helsinki in hours: minutes. Abbreviations: D=departure, A=arrival, T=Turku, and H=Helsinki.

	1	2	3	4	5
DH	8:02	9:02	10:02	11:02	12:02
AT	10:00	11:00	12:00	13:00	14:00
DT	11:00	12:00	13:00	14:00	15:00
AH	12:58	13:58	14:58	15:58	16:58
DH	13:02	14:02	15:02	16:02	17:02
AT	15:00	16:00	17:00	18:00	19:00
DT	16:00	17:00	18:00	19:00	20:00
AH	17:58	18:58	19:58	20:58	21:58

Now, in order to define the train network as a discrete event system (DES), a state vector is defined as $x = (x_{DH}, x_{KS}, x_{ST}, x_{AT}, x_{DT}, x_{SK}, x_{KH}, x_{AH})^T$ with descriptive subscripts:

DH=	departure from Helsinki	DT=	departure from Turku
KS=	departure from Karjaa to Salo	SK=	Salo to Karjaa
ST=	Salo to Turku	KH=	Karjaa to Helsinki
AT=	arrival to Turku	AH=	arrival to Helsinki

The argument *k* on the states denotes the *k*th departure or when indicated arrival to the end stations. Furthermore, k also indicates the train number, so that $x_{DH}\ (k)$ is the departure time from Helsinki for train *k*, and $x_{AT}\ (k)$ is the arrival of the same train to Turku.

The period of the timetable is T=60 minutes. Due to that, we have only one track between Karjaa and Salo, and between Salo and Turku, we obtain the following meeting conditions:

$x_{KS}(k) \geq x_{KH}(k-3)$ (in Karjaa),

$x_{ST}(k) \geq x_{SK}(k-2)$ (in Salo for the train going towards Turku),

$x_{DT}(k) \geq x_{AT}(k+1)$ (in Turku),

$x_{SK}(k) \geq x_{ST}(k+2)$ (in Salo for the train going towards Karjaa).

Combination of the meeting conditions and the constraints introduced by traveling times gives the following equations (the first one comes from having only five trains):

$x_{DH}\ (k) = x_{AH}(k-5) + d_1,$

$x_{KS}(k) = \max\big(x_{DH}(k) + d_2, x_{KH}(k-3)\big)$

$x_{ST}(k) = \max\big(x_{KS}(k) + d_3\, , x_{SK}(k-2)\big)$

$x_{AT}(k) = x_{ST}(k) + d_4$

$$x_{DT}(k) = \max\big(x_{AT}(k) + d_5, x_{AT}(k+1)\big) \quad (3.1)$$

$$x_{SK}(k) = \max\ \big(x_{DT}(k) + d_6\, , x_{ST}(k+2)\big) \quad (3.2)$$

$x_{KH}(k) = x_{SK}(k) + d_7$

$x_{AH}(k) = x_{KH}(k) + d_8$

In order to obtain an equation of type $x(k) = A \otimes x(k-1)$, the right-hand side expressions containing k or higher indices are substituted with expressions containing index $k-1$ at most:

$x_{DH}\ (k) = x_{AH}(k-5) + d_1,$

$x_{KS}(k) = \max\big(x_{AH}(k-5) + d_1 + d_2, x_{KH}(k-3)\big),$

$x_{ST}(k) = \max(x_{AH}(k-5) + d_1 + d_2\ + d_3, x_{SK}(k-2),\ x_{KH}(k-3) + d_3),$

$x_{AT}(k) = \max(x_{AH}(k-5) + d_1 + d_2\ + d_3 + d_4,\ x_{SK}(k-2) + d_4,\ x_{KH}(k-3) + d_3 + d_4\ ),$

$$x_{DT}(k) = \max(x_{AH}(k-5) + d_1 + d_2 + d_3 + d_4 + d_5, x_{SK}(k-2) + d_4 + d_5, x_{KH}(k-3) + d_3 + d_4 + d_5, x_{AH}(k-4) + d_1 + d_2 + d_3 + d_4, x_{SK}(k-1) + d_4, x_{KH}(k-2) + d_3 + d_4),$$

$$x_{SK}(k) = \max(x_{AH}(k-5) + d_1 + d_2 + d_3 + d_4 + d_5 + d_6, x_{SK}(k-2) + d_4 + d_5 + d_6, x_{KH}(k-3) + d_3 + d_4 + d_5 + d_6, x_{AH}(k-4) + d_1 + d_2 + d_3 + d_4 + d_6, x_{SK}(k-1) + d_4 + d_6, x_{KH}(k-2) + d_3 + d_4 + d_6, x_{AH}(k-3) + d_1 + d_2 + d_3, x_{KH}(k-1) + d_3),$$

$$x_{KH}(k) = \max(x_{AH}(k-5) + d_1 + d_2 + d_3 + d_4 + d_5 + d_6 + d_7, x_{SK}(k-2) + d_4 + d_5 + d_6 + d_7, x_{KH}(k-3) + d_3 + d_4 + d_5 + d_6 + d_7, x_{AH}(k-4) + d_1 + d_2 + d_3 + d_4 + d_6 + d_7, x_{SK}(k-1) + d_4 + d_6 + d_7, x_{KH}(k-2) + d_3 + d_4 + d_6 + d_7, x_{AH}(k-3) + d_1 + d_2 + d_3 + d_7, x_{KH}(k-1) + d_3 + d_7),$$

$$x_{AH}(k) = \max(x_{AH}(k-5) + d_1 + d_2 + d_3 + d_4 + d_5 + d_6 + d_7 + d_8, x_{SK}(k-2) + d_4 + d_5 + d_6 + d_7 + d_8, x_{KH}(k-3) + d_3 + d_4 + d_5 + d_6 + d_7 + d_8, x_{AH}(k-4) + d_1 + d_2 + d_3 + d_4 + d_6 + d_7 + d_8, x_{SK}(k-1) + d_4 + d_6 + d_7 + d_8, x_{KH}(k-2) + d_3 + d_4 + d_6 + d_7 + d_8, x_{AH}(k-3) + d_1 + d_2 + d_3 + d_7 + d_8, x_{KH}(k-1) + d_3 + d_7 + d_8).$$

Define the augmented system $x_j(k)$ where $j = 1,2,3,\dots,40$:

$$\left.\begin{aligned}
x_j(k) &= x_{DH}(k-j+1), \quad j = 1,\dots,5 \\
x_j(k) &= x_{KS}(k-j+6), \quad j = 6,\dots,10 \\
x_j(k) &= x_{ST}(k-j+11), \quad j = 11,\dots,15 \\
x_j(k) &= x_{AT}(k-j+16), \quad j = 16,\dots,20 \\
x_j(k) &= x_{DT}(k-j+21), \quad j = 21,\dots,25 \\
x_j(k) &= x_{SK}(k-j+26), \quad j = 26,\dots,30 \\
x_j(k) &= x_{KH}(k-j+31), \quad j = 31,\dots,35 \\
x_j(k) &= x_{AH}(k-j+36), \quad j = 36,\dots,40
\end{aligned}\right\} \tag{3.3}$$

This means that $x_i(k) = x_{i-1}(k-1)$ for $i = 2,3,\dots\dots\dots\dots,40$ except that $i = 1, 6, 11, 16, 21, 26, 31$ and 36. The main equations using numbers as subscripts then become as follows:

$x_1(k) = x_{40}(k-1) + d_1$,

$x_6(k) = \max\big(x_{40}(k-1) + d_1 + d_2, x_{33}(k-1)\big)$,

$x_{11}(k) = \max(x_{40}(k-1) + d_1 + d_2 + d_3, x_{27}(k-1), x_{33}(k-1) + d_3)$,

$x_{16}(k) = \max(x_{40}(k-1) + d_1 + d_2 + d_3 + d_4,\ x_{27}(k-1) + d_4,\ x_{33}(k-1) + d_3 + d_4)$,

$x_{21}(k) = \max(x_{40}(k-1) + d_1 + d_2 + d_3 + d_4 + d_5,\ x_{27}(k-1) + d_4 + d_5,\ x_{33}(k-1) + d_3 + d_4 + d_5, x_{39}(k-1) + d_1 + d_2 + d_3 + d_4,\ x_{26}(k-1) + d_4,\ x_{32}(k-1) + d_3 + d_4)$,

$x_{26}(k) = \max(x_{40}(k-1) + d_1 + d_2 + d_3 + d_4 + d_5 + d_6, x_{27}(k-1) + d_4 + d_5 + d_6,\ x_{33}(k-1) + d_3 + d_4 + d_5 + d_6, x_{39}(k-1) + d_1 + d_2 + d_3 + d_4 + d_6,\ x_{26}(k-1) + d_4 + d_6,\ x_{32}(k-1) + d_3 + d_4 + d_6, x_{38}(k-1) + d_1 + d_2 + d_3, x_{31}(k-1) + d_3)$,

$x_{31}(k) = \max(x_{40}(k-1) + d_1 + d_2 + d_3 + d_4 + d_5 + d_6 + d_7,\ x_{27}(k-1) + d_4 + d_5 + d_6 + d_7,\ x_{33}(k-1) + d_3 + d_4 + d_5 + d_6 + d_7, x_{39}(k-1) + d_1 + d_2 + d_3 + d_4 + d_6 + d_7,\ x_{26}(k-1) + d_4 + d_6 + d_7,\ x_{32}(k-1) + d_3 + d_4 + d_6 + d_7, x_{38}(k-1) + d_1 + d_2 + d_3 + d_7, x_{31}(k-1) + d_3 + d_7)$, and

$x_{36}(k) = \max(x_{40}(k-1) + d_1 + d_2 + d_3 + d_4 + d_5 + d_6 + d_7 + d_8,\ x_{27}(k-1) + d_4 + d_5 + d_6 + d_7 + d_8,\ x_{33}(k-1) + d_3 + d_4 + d_5 + d_6 + d_7 + d_8, x_{39}(k-1) + d_1 + d_2 + d_3 + d_4 + d_6 + d_7 + d_8,\ x_{26}(k-1) + d_4 + d_6 + d_7 + d_8,\ x_{32}(k-1) + d_3 + d_4 + d_6 + d_7 + d_8,\ x_{38}(k-1) + d_1 + d_2 + d_3 + d_7 + d_8,\ x_{31}(k-1) + d_3 + d_7 + d_8)$.

If we rewrite the above evolution equations as a max-plus-linear discrete event systems state space model of the form

$$x(k) = A \otimes x(k-1), \tag{3.4}$$

we obtain a square matrix A of size 40×40. For example, the 36th row in the matrix A is: $[\varepsilon \ldots\ldots\ldots\ldots \varepsilon\ 148\ \ 208\ \ \varepsilon\ \ \varepsilon\ \ \varepsilon\ \ 115\ \ 175\ \ 235\ \ \varepsilon\ \varepsilon\ \varepsilon\ \varepsilon\ \ 180\ \ 240\ \ 300]$, where the entry 148 has column index 26.

The power method [1, 2, 3] is used for finding the eigenvalue λ of the matrix A. The method means repetitive multiplications $x(k) = A \otimes x(k-1) = A^{\otimes k} \otimes x(0)$, and it stops when there are integers $i > j \geq 0$ and a real number c for which $x(i) = x(j) \otimes c$. The eigenvalue is then given by $\lambda(A) = \frac{c}{i-j}$. In this case, using $x(0) = \mathbf{0}$, iteration according Equation (3.4) gives

$x(12) = A \otimes x(11)$

$= [\,664\ \ 604\ \ 544\ \ 484\ \ 424\ \ 725\ \ 665\ \ 605\ \ 545\ \ 485\ \ 752\ \ 692\ \ 632\ \ 572\ \ 512\ \ 782\ \ 722\ \ 662\ \ 602\ \ 542\ \ 842\ \ 782\ \ 722\ \ 662\ \ 602\ \ 872\ \ 812\ \ 752\ \ 692\ \ 632\ \ 900\ \ 840\ \ 780\ \ 720\ \ 660\ \ 960\ \ 900\ \ 840\ \ 780\ \ 720\,]^T$,

$x(13) = A \otimes x(12)$

$= [\,724\ \ 664\ \ 604\ \ 544\ \ 484\ \ 785\ \ 725\ \ 665\ \ 605\ \ 545\ \ 812\ \ 752\ \ 692\ \ 632\ \ 572\ \ 842\ \ 782$

$722\ 662\ \ 602\ \ 902\ \ 842\ \ 782\ \ 722\ \ 662\ \ 932\ 872\ \ 812\ \ 752\ \ 692\ \ 960\ \ 900\ \ 840\ \ 780\ 720$ $1020\ \ 960\ \ 900\ \ 840\ \ 780\ ]^T$, and

$x(13) = x(12) \otimes 60.$

Thus, the eigenvalue is $\lambda(A) = \frac{60}{13-12} = 60$. The eigenvalue represents the cycle of the schedule, which means that the trains start from each station every 60 minutes.

This also means that $x(13)$ is an eigenvector, and $(x(13) - c)$, where c is any constant, is also an eigenvector. One eigenvector of A is v, where

$$v = [0\ \ -60\ \ -120\ \ -180\ \ -240\ \ 61\ \ 1\ \ -59\ \ -119\ -179\ \ 88\ \ 28\ \ -32\ \ -92\ \ -152\ \ -118\ \ 58\ \ -2\ \ -62\ -122\ \ 178\ \ 118\ \ 58\ -2\ \ -6\ \ 208\ \ 148\ \ 88\ \ 28\ \ -32\ \ 236\ \ 176\ \ 116\ \ 56\ -2\ \ 296\ \ 236\ \ 176\ \ 116\ \ 56]^T. \quad (3.5)$$

This eigenvector v includes the schedule of the trains, relative to the last departure from Helsinki (the first element of v). Therefore, the element -240 means that five departures back a train from Helsinki left 240 minutes ago, and the element 296 means that it takes 296 minutes for a train to come back to Helsinki.

3.2 Delay Sensitivity Analysis

All the travel times d_i introduced in Section 3.1 consist of a minimal travel time and a slack time. Here it is assumed that the minimal travel time is 90% of the nominal time, and the slack is thus 10%. For the small waiting time $d_1$in Helsinki, it is assumed that there is no slack.

Handling delays is a relevant and common problem in train networks, and the sensitivity of delays can be analyzed using max-plus models. A permanent delay means that the nominal travel times are increased, which is compensated for by decreasing the other travel times to their minimal values. This gives a slightly different system, for which a new eigenvalue can be calculated. The relative and absolute limits for increasing the different traveling times individually without violation of the roundtrip time (i.e. $\lambda > \mathrm{T}$) are presented in Table 3.2.

Table 3.2: Delay sensitivity of the different traveling times.

Traveling time with delay	d_1	d_2	d_3	d_4	d_5	d_6	d_7	d_8
Relative limit	440%	18%	28%	10%	10%	10%	27.5%	19.3%
Absolute limit (minutes)	17.6	11.5	7.8	3	6	3	7.7	11.6

Table 3.2 show the maximal value that a single traveling time d_i can be increased, and still get the nominal roundtrip time (given by the eigenvalue of the modified matrix) by decreasing all the other traveling times to their minimal values. For example, if we increase d_7 by 27.5%, which is equal to

7.7 minutes, and reduce all the other traveling times to their minimal values, we will still obtain the eigenvalue $\lambda = 60$.
A limitation with the analysis is that it assumes a permanent change in the delays, and results concerns only steady state. It does not give information about dynamic delay propagation, which is the theme of the following section.

3.3 Dynamic Delay Propagation

The delay sensitivity analysis in Section 3.2 assumed that we had permanent changes in the traveling times. A more normal situation is that the delay only concerns one single travel time, which means that the corresponding max-plus system matrix becomes time varying, due to the fact that the travel times d_i become time varying (indicated by an index *k*). This is so due to the meeting conditions that are Equations (3.1) and (3.2), where future states $x_{AT}(k+1)$ and $x_{ST}(k+2)$ appear. These are expanded to $\max(x_{AH}(k-4) + d_1(k+1) + d_2(k+1) + d_3(k+1) + d_4(k+1),\ x_{SK}(k-1) + d_4(k+1),\ x_{KH}(k-2) + d_3(k+1) + d_4(k+1))$ and $\max(x_{AH}(k-3) + d_1(k+2) + d_2(k+2) + d_3(k+2),\ x_{KH}(k-1) + d_3(k+2))$ respectively. As indicated with iteration indices newer versions of travel times are needed in these equations. Speeding up can also only be done after the delay has appeared, which in our case means that after a delay in $d_i(k)$, only the traveling times $d_j(k)$, with $j > i$, can be decreased in the same iteration *k*. In the next iteration, all the traveling times can be decreased.
In Table 3.3, it has been tested how long it takes for a delay of 10, 20 and 30 minutes respectively in a certain travel time, to disappear from the system.

Table 3.3: Times expressed in minutes that it takes for a delay in a certain traveling time to disappear from the system.

Travel Time	**Delay 10** min	**Delay 20** min	**Delay 30** min
d_1	89.2	182.4	301.3
d_2	88.3	182.4	300.4
d_3	93.2	182.4	300.4
d_4	91	185.1	303.1
d_5	91	185.1	303.1
d_6	93.2	182.4	300.4
d_7	68	184.2	305.3
d_8	89.2	182.4	301.3

The calculation of the disappearance of a delay can be done as follows. Let M_n denote a matrix with the nominal timetables, that is $M_n = [v, v \otimes T, v \otimes T^{\otimes 2}, \ldots]$, and M_d is a matrix with the delayed arrival and departure times at corresponding times. The part of the timetables that can be used for selecting the part of the timetable that is affected by a delay using the logical expression$(M_d - M_n) > 0$. This means that the time instant of the last delay t_d can be found using

$$t_d = \max\left[M_d\big((M_d - M_n) > 0\big) - M_n(i,j)\right],$$

where i and j are the timetable indices when the actual first delay takes place. For example 88.3 in Table 3 means that if the single traveling time d_2 is increased by 10 minutes, and the traveling times d_3, d_4, d_5, d_6, d_7 and d_8 are speeded up to their minimal values, then the time instant of the last deviation from the timetable is 88.3 minutes after the delay.

3.4 Recovery Matrix

In Goverde [25], max-plus linear systems are written in polynomial form,

$$x(k) = A_0 \otimes x(k) \oplus A \otimes x(k-1) \oplus w(k), \tag{3.6}$$

where A is defined as in Equation (3.4), A_0 is the matrix describing the direct connections from $x(k)$ to $x(k)$, and $w(k)$ is the nominal departure times in period k.

A_0 is in this case given by all the direct traveling times d_i, including all delayed states, such that $A_0(m+5, m) = d_i$, for $m = (i-1)5 + n$, for all $n = 1,2, \ldots 5$, and for all $i = 2, 3, \ldots 8$.

All the other elements of A_0 are ε, as there are no direct connections. The departure times are given by the eigenvector v in Equation (3.5), and the period T according to $w(k) = T^{\otimes k} \otimes v$. The polynomial equation can be written using a single matrix A_p, according to

$$x(k) = A_p \otimes x(k-1) \oplus w(k), \tag{3.7}$$

where $A_p = A_0 \oplus A \otimes T^{\otimes -1}$.

Definition: Consider the max-plus linear system in equation (3.7). The entry r_{ij} of the recovery matrix R is defined as the maximum delay of $x_j(m)$ such that $x_i(k)$ is not delayed for any $k > m$ [25]. The following equation [2,4] defines the elements of the recovery matrix,

$$r_{ij} = w_i - w_j - \left[A_p^+\right]_{ij},$$

where the w_i and w_j are elements of vector w, $A_p^+ = \oplus_{k=1}^{\infty} A_p^{\otimes k}$, and the notation $\left[A_p^+\right]_{ij}$ refers to the ij^{th} element of the matrix A_p^+. If in the graph of A_p^+ no path exists from node j to node i, then $r_{ij} = \infty$. The recovery matrix, thus, takes values from the extended set $\overline{\mathbb{R}}_{\max} = \mathbb{R}_{\max} \cup \{\infty\}$.

In the studied train network between Helsinki and Turku, constructed from Table 3.1 presented in Figure 3.1, the recovery matrix R is of size 40×40, with $T = 60$. A 20×20 submatrix of that matrix is given in Table 3.4.

Table 3.4: The upper left quadrant of the recovery matrix, with diagonal element shaded.

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
1	29.6	29.6	29.6	29.6	29.6	23.5	23.5	23.5	23.5	23.5	20.8	20.8	20.8	20.8	20.8	17.8	17.8	17.8	17.8	17.8
2	0	29.6	29.6	29.6	29.6	23.5	23.5	23.5	23.5	23.5	20.8	20.8	20.8	20.8	20.8	17.8	17.8	17.8	17.8	17.8
3	0	0	29.6	29.6	29.6	23.5	23.5	23.5	23.5	23.5	20.8	20.8	20.8	20.8	20.8	17.8	17.8	17.8	17.8	17.8
4	0	0	0	29.6	29.6	23.5	23.5	23.5	23.5	23.5	20.8	20.8	20.8	20.8	20.8	17.8	17.8	17.8	17.8	17.8
5	0	0	0	0	29.6	23.5	23.5	23.5	23.5	23.5	20.8	20.8	20.8	20.8	20.8	17.8	17.8	17.8	17.8	17.8
6	6.1	28.6	28.6	35.7	35.7	22.5	22.5	22.5	29.6	29.6	19.8	19.8	19.8	26.9	26.9	16.8	16.8	16.8	23.9	23.9
7	6.1	6.1	28.6	35.7	35.7	0	22.5	22.5	29.6	29.6	19.8	19.8	19.8	26.9	26.9	16.8	16.8	16.8	23.9	23.9
8	6.1	6.1	6.1	35.7	35.7	0	0	22.5	29.6	29.6	19.8	19.8	19.8	26.9	26.9	16.8	16.8	16.8	23.9	23.9
9	6.1	6.1	6.1	6.1	35.7	0	0	0	29.6	29.6	19.8	19.8	19.8	26.9	26.9	16.8	16.8	16.8	23.9	23.9
10	6.1	6.1	6.1	6.1	6.1	0	0	0	0	29.6	19.8	19.8	19.8	26.9	26.9	16.8	16.8	16.8	23.9	23.9
11	8.8	20.8	31.3	38.4	38.4	2.7	14.7	25.2	32.3	32.3	12	12	22.5	29.6	29.6	9	9	19.5	26.6	26.6
12	8.8	8.8	31.3	38.4	38.4	2.7	2.7	25.2	32.3	32.3	0	12	22.5	29.6	29.6	9	9	19.5	26.6	26.6
13	8.8	8.8	8.8	38.4	38.4	2.7	2.7	2.7	32.3	32.3	0	0	22.5	29.6	29.6	9	9	19.5	26.6	26.6
14	8.8	8.8	8.8	8.8	38.4	2.7	2.7	2.7	2.7	32.3	0	0	0	29.6	29.6	9	9	19.5	26.6	26.6
15	8.8	8.8	8.8	8.8	8.8	2.7	2.7	2.7	2.7	2.7	0	0	0	0	29.6	9	9	19.5	26.6	26.6
16	11.8	23.8	34.3	41.4	41.4	5.7	17.7	28.2	35.3	35.3	3	15	25.5	32.6	32.6	12	12	22.5	29.6	29.6
17	11.8	11.8	34.3	41.4	41.4	5.7	5.7	28.2	35.3	35.3	3	3	25.5	32.6	32.6	0	12	22.5	29.6	29.6
18	11.8	11.8	11.8	41.4	41.4	5.7	5.7	5.7	35.3	35.3	3	3	3	32.6	32.6	0	0	22.5	29.6	29.6
19	11.8	11.8	11.8	11.8	41.4	5.7	5.7	5.7	5.7	35.3	3	3	3	3	32.6	0	0	0	29.6	29.6
20	11.8	11.8	11.8	11.8	11.8	5.7	5.7	5.7	5.7	5.7	3	3	3	3	3	0	0	0	0	29.6

According to Goverde [25], the j^{th} column of the recovery matrix R gives the recovery time from event j to all other events in the timetable and, thus, represents the impact a delay of event j has on future train events, and the i^{th} row of the recovery matrix R gives the recovery time from event i from all other events in the timetable and, thus, represents the sensitivity of event i on delays of preceding events. The diagonal elements of R again represent recovery times to the event itself. In our example, most of our states are delayed versions of previous states. As can be noted in Table 3.4, not all diagonal elements representing the same departure at different times are the same.

For example $r_{16,16} = 12$, $r_{18,18} = 22.5$ and $r_{19,19} = 29.6$, although these elements all correspond to the event "arrival in Turku" at times k, $k - 2$ and $k - 3$ respectively. As k is arbitrary, all these recovery elements should logically be the same. This is not so because the delayed versions are just memory variables, for which no other constraints than the back shifting according Equation 3.3 is present and, thus, the recovery matrix is not correct for these. Thus, in our example only every fifth row in the recovery matrix shows true recovery times, and these are shown in Table 3.5.

Table 3.5: The relevant parts of the recovery matrix. Diagonal elements highlighted with green, and recovery times related to a full cycle is highlighted with orange

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
1	29.6	29.6	29.6	29.6	29.6	23.5	23.5	23.5	23.5	23.5	20.8	20.8	20.8	20.8	20.8	17.8	17.8	17.8	17.8	17.8
6	6.1	28.6	28.6	35.7	35.7	22.5	22.5	22.5	29.6	29.6	19.8	19.8	19.8	26.9	26.9	16.8	16.8	16.8	23.9	23.9
11	8.8	20.8	31.3	38.4	38.4	2.7	14.7	25.2	32.3	32.3	12	12	22.5	29.6	29.6	9	9	19.5	26.6	26.6
16	11.8	23.8	34.3	41.4	41.4	5.7	17.7	28.2	35.3	35.3	3	15	25.5	32.6	32.6	12	12	22.5	29.6	29.6
21	17.8	29.8	40.3	41.4	47.4	11.7	23.7	34.2	35.3	41.3	9	21	31.5	32.6	38.6	6	18	28.5	29.6	35.6
26	20.8	32.8	38.4	44.4	50.4	14.7	26.7	32.3	38.3	44.3	12	24	29.6	35.6	41.6	9	21	26.6	32.6	38.6
31	23.6	35.6	41.2	47.2	53.2	17.5	29.5	35.1	41.1	47.1	14.8	26.8	32.4	38.4	44.4	11.8	23.8	29.4	35.4	41.4
36	29.6	41.6	47.2	53.2	59.2	23.5	35.5	41.1	47.1	53.1	20.8	32.8	38.4	44.4	50.4	17.8	29.8	35.4	41.4	47.4

	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40
1	11.8	11.8	11.8	11.8	11.8	8.8	8.8	8.8	8.8	8.8	6	6	6	6	6	0	0	0	0	0
6	10.8	10.8	10.8	17.9	17.9	7.8	7.8	7.8	14.9	14.9	5	5	5	12.1	12.1	6.1	6.1	6.1	6.1	6.1
11	3	3	13.5	20.6	20.6	0	0	10.5	17.6	17.6	7.7	7.7	7.7	14.8	14.8	8.8	8.8	8.8	8.8	8.8
16	6	6	16.5	23.6	23.6	3	3	13.5	20.6	20.6	10.7	10.7	10.7	17.8	17.8	11.8	11.8	11.8	11.8	11.8
21	6	12	22.5	23.6	29.6	3	9	19.5	20.6	26.6	10.7	10.7	16.7	17.8	23.8	11.8	11.8	11.8	11.8	17.8
26	3	15	20.6	26.6	32.6	6	12	17.6	23.6	29.6	7.7	13.7	14.8	20.8	26.8	8.8	8.8	8.8	14.8	20.8
31	5.8	17.8	23.4	29.4	35.4	2.8	14.8	20.4	26.4	32.4	10.5	16.5	17.6	23.6	29.6	11.6	11.6	11.6	17.6	23.6
36	11.8	23.8	29.4	35.4	41.4	8.8	20.8	26.4	32.4	38.4	6	22.5	23.6	29.6	35.6	17.6	17.6	17.6	23.6	29.6

The recovery matrix takes into consideration only one train, not the whole system, and it gives all the information for the delay of one train only. A 0 in the recovery matrix means a tight schedule, with no slack.

For example, the first row in the reduced recovery matrix is easy to interpret; the first value is 29.6, which is the total slack for a single train. After that the slack is reduced by the slack in corresponding travel time, up to the final value 0, which corresponds to that no slack is present in the 4-minute waiting time in Helsinki (d_1). All the other traveling times are assumed to have 10% slack. The other zero (row 11, column 26) is due to a meeting condition (in Salo).

The results shown in Table 3.2 can also be calculated using recovery matrix calculations. In Table 3.2, it was assumed that we have a permanent delay in one travel time. The maximum tolerance for a permanent delay in one travel time can be obtained by increasing the corresponding travel time in the recovery matrix, until we start obtaining negative entries on the relevant diagonal elements in the recovery matrix (the ones indicated by green in Table 3.5).

The results in Table 3.5 can only partially be calculated using recovery matrix calculations. In Table 3.3, certain temporary delays (10, 20 and 30 minutes) were considered. In Table 3.3, it can be seen that the time it takes for the system to catch up after delays of 30 minutes are all slightly more than 300 minutes. This is not a coincidence, as in most cases it is the delayed train itself that uses most time to catch up, and the recovery time 29.6 in positions highlighted with orange means that if we have a delay which is larger than 29.6, it will take more than 300 minutes (i.e. a full cycle) for the system to catch up.

3.5 Conclusion

This chapter described how a max-plus model for a train system can be constructed. Meeting conditions caused by having a single track, and other physical constrains, was handled by extending the state space with delayed states, which has enabled rewriting the state update equation in the form $x(k) = A \otimes x(k-1)$. Static and dynamic delay sensitivity of the network has been analyzed by modifying the A-matrix and using eigenvalue calculations. The obtained results were compared to standard recovery matrix-based calculations. A recovery matrix for the chosen extended state space becomes large and contains even irrelevant information. Guidelines for finding and interpreting the relevant information from the recovery matrix have been discussed. Max-plus formalism was used throughout this chapter.

Chapter 4

Modeling and Scheduling of Production Systems

Scheduling of manufacturing systems is a difficult task, since they consist of many units with complex relations and interdependences. In order to deal with this complexity, modeling and scheduling techniques are used to guarantee that the whole production process is executed in a more dynamic and reliable way than producing decisions manually. The questions of production scheduling in manufacturing processes is becoming more important considering the increasing demand of economic and environmental constraints. Therefore, in this chapter, we consider industrial production scheduling problems for a manufacturing system consisting of parallel batch processes. These processes can interact with each other and, therefore, influence the production of different batches in these processes. These kinds of problems can be represented using discrete event systems, which in general lead to a nonlinear description when using the conventional linear algebra. Therefore, we suggest using the max-plus technique which results in systems that are "linear" in the max-plus algebra. Scheduling of production systems is a common engineering problem, see for example Giffler and Thompson [34]. Scheduling using max-plus algebra has also been studied in Baccelli et al. [1]. Because scheduling systems are linear in max-plus algebra, we can use more effective methods that are available for modeling, simulation, and analysis of such systems. The production scheduling in a manufacturing process consisting of 6 stages and 6 units done in parallel batches is consider as a case study in this chapter.

The chapter is organized as follows. First, the scheduling problem is produced, formulation and scheduling of a small production system consisting of 6 stages and 6 units is used as a case study and modeling and simulation results for this problem are analyzed in Section 4.1. Section 4.2 presents the analysis of the schedule of the production system. Asymptotic cases are studied in Sections 4.3 and 4.4, and Section 4.5 gives concluding remarks.

4.1 Manufacturing System

Consider the production system shown in Figure 4.1, the scheduling of which has been previously studied by Björkqvist et al. [33] using optimization.

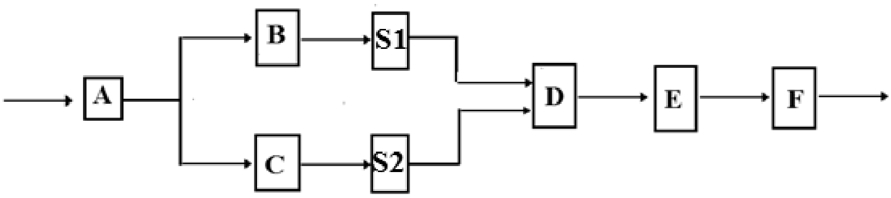

Figure 4.1: A manufacturing system

This manufacturing system consists of six processing stages A, B, C, D, E and F. Out of these, B and C are performed in parallel, and D and E can overlap by 6 hours, otherwise the stages are performed in order. Some of the stages have smaller capacities than the others, which means that they must be performed more often, see Table 4.1.

Table 4.1: Specifications for the production stages in the manufacturing system.

Stages	Processing time (h)	Cleaning time (h)	Durability (h)	Amount/batch (kg), repetitions/300kg in parenthesis
A	7	4	48	300 (1)
B	4	2	60	50 (3)
C	7	2	60	75 (2)
D	10	4	24	150 (2)
E	8	4	24	150 (2)
F	1	0	48	150 (2)

For the stages, there are six different units, out of which units 1 and 2 can be used for stages A and D, and units 3 and 4 can be used for stages B and C. The last two units can only be used for E and F respectively. The reactors used for stage A and D are also used for temporary storage, which means that the reactors cannot be used for another task before they are emptied. Half of the output from A goes through B, and the other half goes through C. After stages B and C, temporary storages S1 and S2 are used, gathering all intermediates needed for stage D. Stage E is performed overlapping with stage D, so that stage D and E are performed for 6 hours in parallel. The final stage F must be performed immediately after stage E has finished. The processing and cleaning times for stage N are denoted P_N and c_N respectively and given in Table 4.1.

A central limiting factor for the production scheduling is the durability of the products produced in stages B and C, which can only be stored for 60 hours. This means that the unit producing A should be switched to production of D and, thus, the temporary storages S1 and S2 will be emptied.

4.1.1 A Max-plus Model for the Production System

Some simplifications were found necessary and/or appropriate. In particular, the following assumptions were made:

1. Unit 3 is used only for stage B, and unit 4 is used only for stage C.
2. The temporary storages after B and C respectively are only limited by the storage time [33], not by their size.
3. Scheduling of stage D is done based on the preliminary schedule obtained from the max-plus model. The model is constructed so that it gives all the alternatives for stage D, and the final schedule is obtained based on a set of simple rules.
4. Stages E and F are directly dependent on D, so the schedule for these are constructed based on the selected schedule for D, and these are left out from the max-plus model. Normally E and F simply follow D as they are faster than D, but when D is performed using both unit 1 and 2, E becomes the limiting factor. However, this can also be simply taken into account when selecting the schedule for D.
5. Durability of the output from each stage is first ignored, as it can be shown that it does not limit production of batches smaller than 4200kg. The durability constraints will play a major role later, in the asymptotic analysis in Section 4.3.

The goal is to perform production as fast as possible, subject to all the constraints that are present. This will now be formulated using a max-plus model, and for that, we need 10 states x, listed and described below (where U_i = Unit i):

1. U_1 doing A
2. U_2 doing D, step 1
3. U_2 doing D, step 2
4. U_3 doing B, step 1
5. U_3 doing B, step 2
6. U_3 doing B, step 3
7. U_4 doing C, step 1
8. U_4 doing C, step 2

9. U_1 doing D, step 1
10. U_1 doing D, step 2

Now we write down the max-plus-algebraic state space model of this production, with all the constraints from the production and cleaning times included, in the equations that follow.

$x_1(k) = \max(x_1(k-1) + P_A, x_5(k-1) + P_B, x_7(k-1) + P_C)$
$x_2(k) = \max(x_3(k-1) + P_D, x_5(k) + P_B, x_7(k) + P_C)$
$x_3(k) = \max(x_2(k) + P_D, x_6(k) + P_B, x_8(k) + P_C)$
$x_4(k) = \max(x_1(k) + P_A, x_6(k-1) + P_B)$
$x_5(k) = \max(x_4(k) + P_B)$
$x_6(k) = \max(x_5(k) + P_B)$
$x_7(k) = \max(x_8(k-1) + P_C, x_1(k) + P_A)$
$x_8(k) = \max(x_7(k) + P_C)$
$x_9(k) = \max(x_6(k) + c_A, x_8(k) + c_A)$
$x_{10}(k) = \max(x_9(k) + P_D)$

In order to obtain an equation of type

$$x(k) = A \otimes x(k-1), \tag{4.1}$$

the right-hand-side expressions containing k or higher indices are substituted with expressions containing index $k-1$ at most. After some straightforward (but tedious) algebraic manipulations and simplifications, we obtain the following equations:

$x_1(k) = \max(x_1(k-1) + P_A, x_5(k-1) + P_B, x_7(k-1) + P_C)$

$$x_2(k) = \max(x_1(k-1) + 2P_A + k_1, x_3(k-1) + P_D, x_5(k-1) + P_A + P_B + k_1, x_6(k-1) + 3P_B, x_7(k-1) + P_A + P_C + k_1, x_8(k-1) + 2P_C)$$

$$x_3(k) = \max(x_1(k-1) + 2P_A + k_2, x_3(k-1) + 2P_D, x_5(k-1) + P_A + P_B + k_2, x_6(k-1) + 3P_B + k_3, x_7(k-1) + P_A + P_C + k_2, x_8(k-1) + 2P_C + k_4)$$

$$x_4(k) = \max(x_1(k-1) + 2P_A, x_5(k-1) + P_A + P_B, x_6(k-1) + P_B, x_7(k-1) + P_A + P_C)$$

$$x_5(k) = \max(x_1(k-1) + 2P_A + P_B, x_5(k-1) + P_A + 2P_B, x_6(k-1) + 2P_B, x_7(k-1) + P_A + P_B + P_C)$$

$$x_6(k) = \max(x_1(k-1) + 2P_A + 2P_B, x_5(k-1) + P_A + 3P_B, x_6(k-1) + 3P_B, x_7(k-1) + P_A + 2P_B + P_C)$$

$$x_7(k) = \max\left(x_1(k-1) + 2P_A, x_5(k-1) + P_A + P_B, x_7(k-1) + P_A + P_C, x_8(k-1) + P_C\right)$$

$$x_8(k) = \max\left(x_1(k-1) + 2P_A + P_C, x_5(k-1) + P_A + P_B + P_C, x_7(k-1) + P_A + 2P_C, x_8(k-1) + 2P_C\right)$$

$$x_9(k) = \max\left(x_1(k-1) + 2P_A + c_A + k_1, x_5(k-1) + P_A + P_B + c_A + k_1, x_6(k-1) + 3P_B + c_A, x_7(k-1) + P_A + P_C + c_A + k_1, x_8(k-1) + 2P_C + c_A\right)$$

$$x_{10}(k) = \max\left(x_1(k-1) + 2P_A + P_D + c_A + k_1, x_5(k-1) + P_A + P_B + P_D + c_A + k_1, x_6(k-1) + 3P_B + P_D + c_A, x_7(k-1) + P_A + P_C + P_D + c_A + k_1, x_8(k-1) + 2P_C + P_D + c_A\right).$$

For simplicity, the following constants have been introduced in the above equations: $k_1 = \max(2P_B, P_C)$, $k_2 = \max(2P_B + P_D, 3P_B, P_C + P_D, 2P_C)$, $k_3 = \max(P_B, P_D)$ and $k_4 = \max(P_C, P_D)$. After introduction of numerical values from Table 4.1, the *A*-matrix of the system becomes

$$A = \begin{pmatrix} 7 & \varepsilon & \varepsilon & \varepsilon & 4 & \varepsilon & 7 & \varepsilon & \varepsilon & \varepsilon \\ 22 & \varepsilon & \varepsilon & \varepsilon & 19 & 12 & 22 & 14 & \varepsilon & \varepsilon \\ 32 & \varepsilon & 10 & \varepsilon & 29 & 22 & 32 & 30 & \varepsilon & \varepsilon \\ 14 & \varepsilon & 20 & \varepsilon & 11 & 4 & 14 & \varepsilon & \varepsilon & \varepsilon \\ 18 & \varepsilon & \varepsilon & \varepsilon & 15 & 8 & 18 & \varepsilon & \varepsilon & \varepsilon \\ 22 & \varepsilon & \varepsilon & \varepsilon & 19 & 12 & 22 & \varepsilon & \varepsilon & \varepsilon \\ 14 & \varepsilon & \varepsilon & \varepsilon & 11 & \varepsilon & 14 & 7 & \varepsilon & \varepsilon \\ 21 & \varepsilon & \varepsilon & \varepsilon & 18 & \varepsilon & 21 & 14 & \varepsilon & \varepsilon \\ 26 & \varepsilon & \varepsilon & \varepsilon & 23 & 16 & 26 & 18 & \varepsilon & \varepsilon \\ 36 & \varepsilon & \varepsilon & \varepsilon & 33 & 26 & 36 & 28 & \varepsilon & \varepsilon \end{pmatrix} \quad (4.2)$$

Normally, the cycle time of the discrete event system could be obtained from the eigenvalue of the *A*-matrix, but this is not the case here. The reason for this is that the D-stages have a cycle time of 20, while the others have a cycle time of 15. The graph corresponding to *A* is not strongly connected, i.e., *A* is not irreducible. It is, thus, not possible to calculate an eigenvector based on the matrix *A* in Equation (4.2) This is due to the fact that there are no limits on the storage between production stages B/C and D, so there is nothing synchronizing them. One can see this in Equation (4.2), columns 2, 3, 9 and 10: The states related to stages A, B and C, that is 1 and 4-8, are not constrained by the other states related to stage D. Moreover, as the states related to D have a longer period, they will lag behind, and are, thus, not constrained by the faster states.

4.2 Production Schedule from Iteration of the State Equation

The max-plus model is still useful for production scheduling by iteration of the state equation (4.1). The initial state $x(0)$ is set to 0, and $x(k)$ is given by Equation (4.1) using A from Equation (4.2), and with no B or u. As mentioned in Section 4.1, the results need to be interpreted in order to obtain a close to optimal schedule:

1. The production is of batch type, so a certain number of repetitions of each stage are needed every time. For all stages but D and E, this means that only the necessary number of steps is used from the start of the schedule.
2. The schedule includes all possible D stages, most of them need to be discarded, based on the following:
    a) Unit 1 should finish all A-stages before switching to doing stage D. Thus, all scheduled D-events based on states 9 and 10 prior to this should be discarded.
    b) Out of the remaining D-stages, only the fastest up to the necessary number should be chosen.
3. The model does not include the limitation of the fact that the storages of B and C cannot be negative. This is possible in the beginning when the storages are empty when A is started. In that case, the first start of D cannot be earlier than $P_A + \max(2P_B, P_C)$ (=15 hours in this case), and the second start of D cannot be earlier than $P_A + \max(3P_B, 2P_C)$ (=21 hours). In practice, this is relevant in the case of a 300kg batch size for Unit 1, step 1 case ($19 \longrightarrow 21$ hours) and in the case of a 600kg batch size for the second round of Unit 2, step 1 ($35 \longrightarrow 36$ hours).
4. The model does not contain limitations of stage E. E is only related to stage D, and it is necessary that that previous E-stage needs to finish before contents of the D-stage can be moved to E. The easy way to handle this constraint is to require that there is P_E between each start of D. This constraint becomes active after both unit 1 and 2 start doing stage D.
5. The schedule of E (unit 5) is always $P_D - P_O$ after the end of the corresponding stage D.

In [33], four different batch sizes were considered: 300, 600, 900 and 1200 kg. The schedule for the stages that are not related to stage D is obtained from the eigenvalue and the eigenvector of the matrix where the states related to D (2, 3, 9 and 10) are left out.

This results in the eigenvalue of 15 and eigenvector of $[0 \quad 7 \quad 11 \quad 15 \quad 7 \quad 14]^T$. This means that each unit should be started according to the eigenvector plus a multiple of 15. The schedule for the D-stages is obtained from the preliminary schedule using point 2a and 2b. The capacity of stage D is

150 kg, so 2, 4, 6 and 8 stages of D respectively are needed in the four considered cases. The relevant states are 2, 3, 9 and 10, and the schedule for these are shown in Table 4.2.

Table 4.2: Preliminary production schedule for stage D. Numbers in parenthesis and red are discarded as unit 1 is still needed for stage A, and numbers in blue and with a * are discarded as they are excessive.

Batch size	300kg	600kg		900kg			1200kg			
Unit 2, step 1	15	15	35	15	35	55	15	35	55	75*
Unit 2, step 2	25*	25	45*	25	45	65*	25	45	65	85*
Unit 1, step 1	19	(19)	34	(19)	(34)	49	(19)	(34)	(49)	64
Unit 1, step 2	29*	(29)	44*	(29)	(44)	59*	(29)	(44)	(59)	74

On top of that the constraint related to stage E, that is the time between the start of a unit producing D should not be less than 8, is enforced. This affects all the events starting from the first-time producing D using unit 1, which can be seen in Table 4.3a, where the final schedule for D is given.

Table 4.3a. Final production schedule for stage D including the constraint for E, meaning in this case that there must be 8 hours between each start of D. Unit 2 in black, unit 1 in red and marked with a *. The constraints from B and C storages do not become active for so small batches.

Batch size	stage 1	stage 2	stage 3	stage 4	stage 5	stage 6	stage 7	stage 8
300kg	15	23*						
600kg	15	25	34*	42				
900kg	15	25	35	45	53*	61		
1200kg	15	25	35	45	55	64*	72	80*

Production times and rates for the different batches are given in Table 4.3b.

Table 4.3b. Production rate for different productions

Production	300	600	900	1200	1500	1800	2100	2400	2700	3000	3300	3600	3900	4200
Prod. time	42	61	80	99	118	137	156	175	194	213	232	251	270	289
Prod. rate	7.14	9.84	11.25	12.12	12.71	13.14	13.46	13.71	13.92	14.08	14.22	14.34	14.44	14.53

As can be seen from Table 4.3b, the production time in increased with 19 hours for each addition of 300 kg of production, so one can conjecture that the production rate will converge towards $\frac{300}{19}$ kg/h.

4.3 Asymptotic Case

With the asymptotic case we mean scheduling of very large (unlimited) batches so that the durability constraints need to be considered. The asymptotic production consists of two stages related to the storage of B and C: the filling stage and the emptying stage. The asymptotic production cases are introduced below with a graph of each case. A typical process involves A, B, C, D stages, as already mentioned in Section 4.1.

In the filling stage, the initial production schedule, presented in Section 4.2, accumulates B and C in storages. Accumulation is due to the fact that A is produced at a rate of 300kg/15h, and D at a minimum production rate of $300\mathrm{kg}/20$ hours. A simple but suboptimal strategy is to delay the production of A, so that it also produces at a rate $300\mathrm{kg}/20\mathrm{h} = 15\mathrm{kg}/\mathrm{h}$ and, thus, avoid the accumulation of B and C storages. However, it is beneficial to clean the unit used for production of A and produce D using two units, resulting in a production rate of $300\mathrm{kg}/16\mathrm{h}$. Thus, the scheduling of the production can be seen as switching between the mode where only one unit is used for production of D, and the mode where two units are used for production of D. In the first mode, the total production is limited by the production of D, and the production rate is $r_1 = 150/10$. In the second mode, the total production will be limited by the production of E, when the production rate is $r_2 = 150/8$. The actual production rate r_p will be a with production times weighted average of r_1 and r_2, that is

$$r_p = (t_{d1} \cdot r_1 + t_{d2} \cdot r_2) \cdot \frac{1}{p},$$

where $p = t_{d1} + t_{d2} + 2t_c$ is the period of production, and t_{d1} is the time for production using only one unit for D, t_{d2} is the time for production using two units for D, and t_c is the cleaning time. Now $t_{d2} = \frac{t_{d1} \cdot r_a}{r_2}$ asymptotically, where r_a is the accumulation rate.

The production rate will be

$$r_p = \frac{t_{d1} \cdot r_1 + (\frac{t_{d1} \cdot r_a}{r_2}) \cdot r_2}{p} = \frac{t_{d1} \cdot r_1 + t_{d1} \cdot r_a}{p},$$

which can be differentiated with respect to t_{d1}, and

$$\frac{\partial r_p}{\partial t_{d1}} = \frac{(r_1 + r_a) \cdot t_{d2} + (r_1 + r_a) \cdot 2t_c}{p^2} > 0.$$

The last inequality follows as all the times and rates are positive, which means that t_{d1} should be chosen as large as possible, that is at the durability constraint. Both B and C have a durability of 60 hours, and this is the durability constraint that should be the target for the schedule.
Production schedules can be obtained using max-plus for both the filling and the emptying phase. Switching between these phases cannot be done using max-plus, but it is easily done by keeping track of the B and C storages. Switching can be carried out after a certain number of repetitions of A in the filling stage and after a certain number of repetitions of D in the emptying stage. The schedule after the switch can be initialized based on the previous schedule.

The long-term production consists of two different modes:

1. One of the units 1 or 2 is used for production of A, and the other is used for production of D. In this mode B and C are accumulated in the storages. The length of this mode is characterized by the number of repetitions of A, denoted n_A.
2. Both units 1 and 2 are used for production of D, when the storages of B and C are emptied. The length of this mode is characterized by the number of repetitions of D, denoted n_D.

Scheduling of the long-term production essentially consists of the choice of these repetitions n_A and n_D. The choice of n_A affects the storage times, the more repetitions the more accumulation in the storages, and the longer time for emptying it. The choice of n_D is bound to the choice of n_A, as the storages need to be sufficiently emptied before restart of production of A. It can be shown that n_A should be about twice the n_D. The storages will build up during production of A at a rate

$$\frac{300\text{kg}}{15\text{h}} - \frac{300\text{kg}}{20\text{h}} = 5\ \text{kg/h}$$

The storages will be emptied at a rate $150/8$ kg/h during usage of two D-units. An A-repetition takes 15 hours, and a D-repetition takes 8 hours, and putting equality for buildup and emptying gives

$$n_A \cdot 15\text{h} \cdot 5\text{kg/h} = n_D \cdot 8\text{h} \cdot \frac{150}{8}\text{kg/h}$$

$$\frac{n_A}{n_D} = \frac{150}{75} = 2$$

One can choose to repeat the D-production one time more than necessary, as this reduces the storage times, with a cost of having the D-production unit idle for a while at the beginning of the next mode 1. However, this increases the period length and reduces the overall production rate. Table 4.4 lists the results for different choices of n_A and n_D.

3. The $n_A - n_D$ cycle must be eventually periodic. It is uniquely determined by the storage in B and C at the beginning of the n_A stage, and there are only a finite number of possibilities for those.

For example, if the storages are empty when the n_A stage starts, the process is periodic from the beginning (as when n_A = 15 and n_D= 8, cf. Fig. 4.5).

Table 4.4. Periods, storage times and production rates for long-term strategies for different number of A- and D-repetitions.

n_A	n_D	Period (h)	Maximum storage times for B and C (h)	Production rate (kg/h)	Figure
12	6	230	50	15.65	
12	7	238	48	15.13	
13	6	249	54	15.66	
13	7	257	52	15.18	
14	7	268	58	15.67	4.2 and 4.3
14	8	276	56	15.22	
15	7	287	62	15.68	4.4
15	8	295	60	15.25	4.5
16	8	306	66	15.69	
16	9	314	64	15.29	

As can be seen in Table 4.4, the production rate does not increase monotonically as a function of n_A, which depends on the fact that the switches are better timed in the cases with an even number of A-steps. As can be seen from the table, the number of D-steps is the same in odd numbered A-steps as in the previous even numbered case. In addition, the production is larger during the D-steps (when both units are used for production of D, with a rate 150/8=18.75kg/h) than during A-steps (when only one unit is used for D, with a production of 15kg/h). In our case, the limit for the storage times is 60 hours, and the most relevant schedules are illustrated in Figures 4.2-4.4.

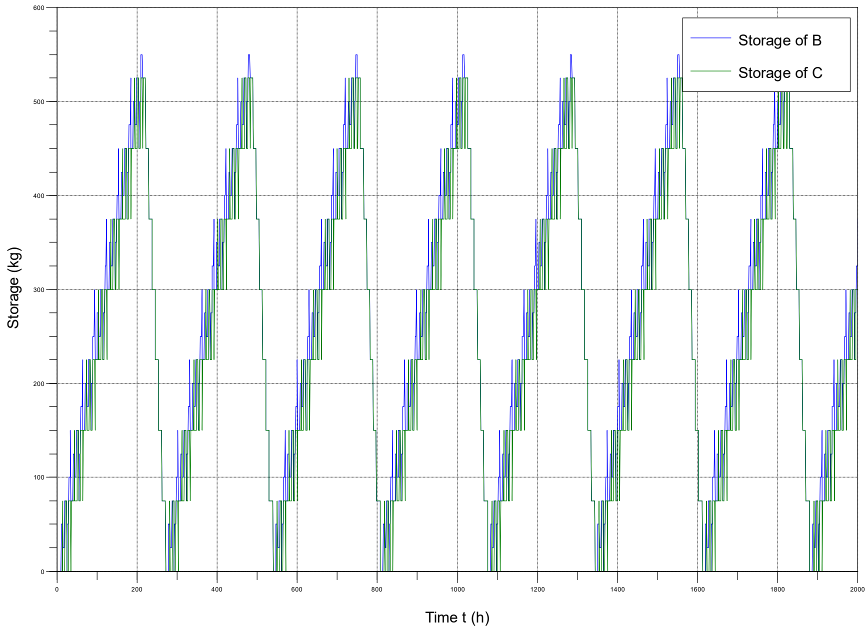

Figure 4.2. Optimal schedule with maximum storage time 58 hours from the start. The period is 268 hours, and the periodicity starts at $t = 3$ hours. Production rate 15.67kg/hours.

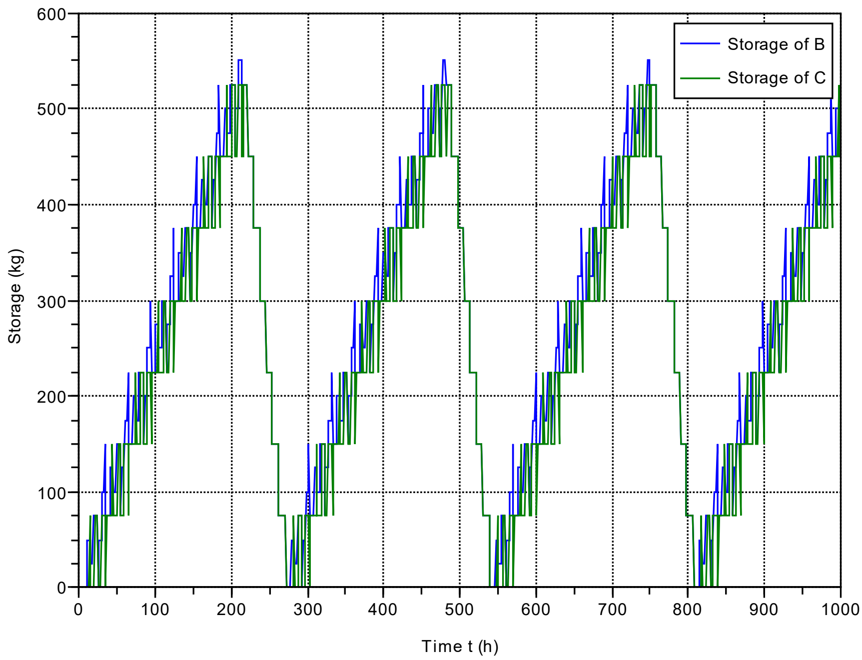

Figure 4.3. The first 1000 hours of the schedule in Figure 4.2.

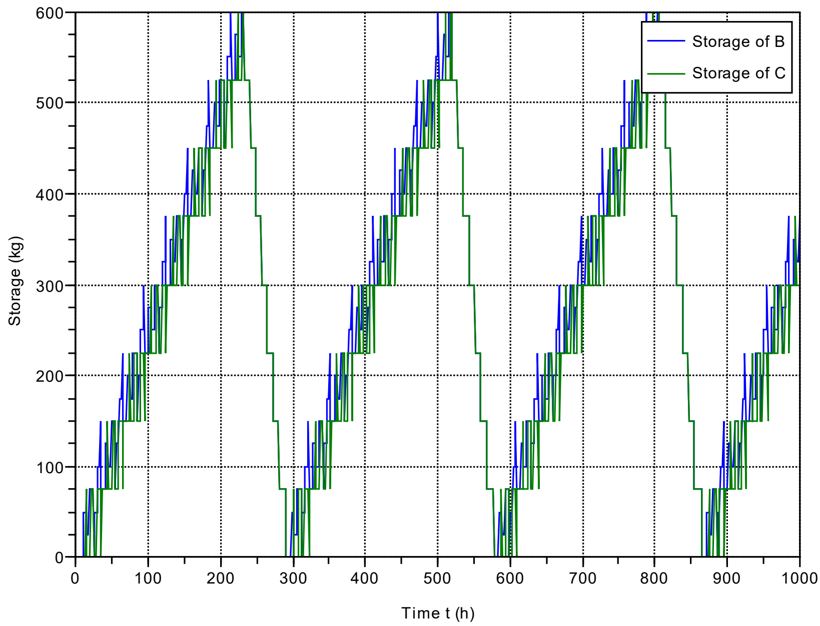

Figure 4.4. Optimal schedule with maximum storage time 62 hours from the start. The period is 287 hours, and the periodicity starts at $t = 3$ hours. Production rate 15.68kg/hour.

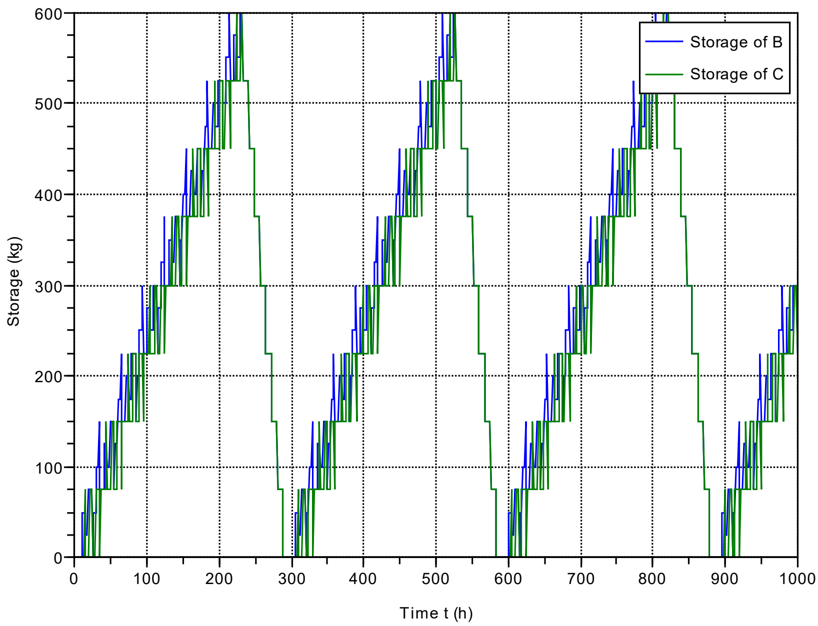

Figure 4.5. Schedule with delayed switch from D to A, resulting in eleven hours waiting for the other D producing unit at each cycle. The period is the 295 hours, and the periodicity starts in the beginning. Production rate 15.25kg/hours

Figure 4.5 results in a worse production rate than in Figure 4.3, and it is not possible to improve on this by increasing the A-steps. As can be seen from Table 4.4, it will result in violation of the durability constraint.

Increasing the storage time seems to converge to the production rate 300/19 as conjectured after Table 4.3b, which can be seen in Table 4.5 and Figure 4.6.

Table 4.5. Numerical test of upper limit on production rate.

Storage time	Production rate
58	15.67
114	15.73
230	15.76
462	15.775
926	15.782
1854	15.786
3710	15.7877
7422	15.78886
14846	15.789026
29694	15.789250
59390	15.789362
∞	300/19 ≈ 15.789474

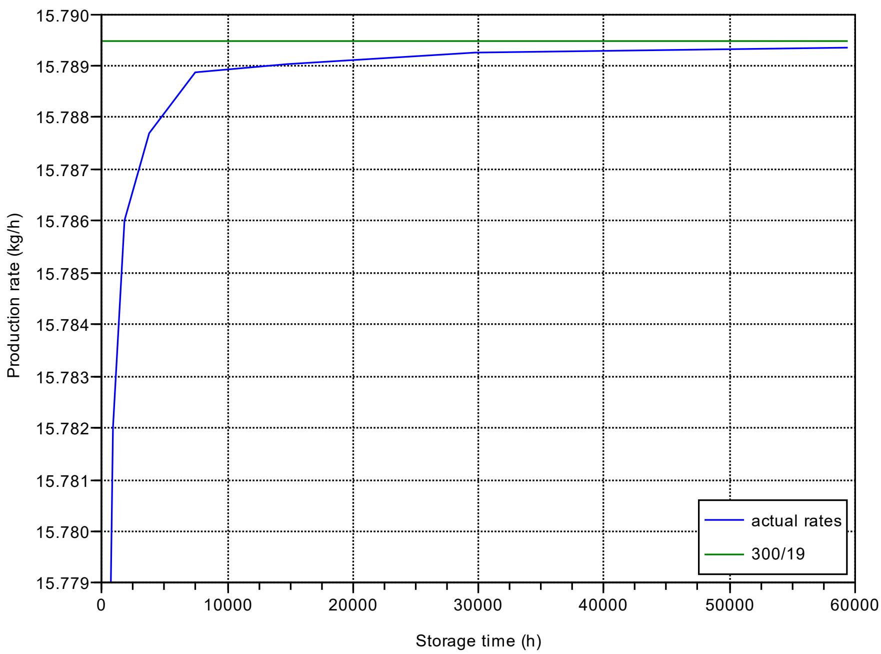

Figure. 4.6. Actual production rates as function of allowed storage times, and the presumed limit 300/19 for the production rate.

4.4 Asymptotic Case Using Max-plus

The asymptotic production consists of two stages, one filling stage and one emptying stage. Three different max-plus models are formulated, describing different parts of the system:

M1. The production of A, B, and C, used in the filling stage

M2. The production of D and E using only one of the units for D, during the filling stage

M3. The production of D and E using both units for D, during the emptying stage

Between the filling and the emptying stage there is a switch, where production of A, B, and C is ended, and the unit producing A is cleaned and used for production of D. This means that the production of D is switched from M2 to M3, and that the state of M3 is initialized using M1 and/or M2 (depending on if the cleaned A-producing unit from M1 or the D-producing unit from M2 is the first available to start M3). Between emptying there is also a switch, where one of the units used for production of D is cleaned, and production of A, that is M1, is started. Furthermore, M3 is switched to M2. The states of M1 and M2 are both initialized using states of M3.

4.4.1 Production of A, B, and C, M1

This system consists of three processing stages A, B, and C. Out of these B and C are performed in parallel. For the stages, there are three different units, out of which unit 1 can be used for stage A and unit 3 can be used for stages B with three steps. Unit 4 can only be used for C with two steps. This will now be formulated using a max-plus model, and for that, we need 6 states x_i, listed and described below:

<u>Filling stages</u>

1. U_1 doing A
2. U_3 doing B, step 1
3. U_3 doing B, step 2
4. U_3 doing B, step 3
5. U_4 doing C, step 1
6. U_4 doing C, step 2

Now we write down the max-plus-algebraic state space model of this DES.

$x_1(k) = \max\,(x_1(k-1) + P_A, x_3(k-1) + P_B, x_5(k-1) + P_C)$

$x_2(k) = \max\,(x_1(k-1) + 2P_A, x_3(k-1) + P_A + P_B, x_4(k-1) + P_B, x_5(k-1) + P_A + P_C)$

$$x_3(k) = \max\left(x_1(k-1) + 2P_A + P_B, x_3(k-1) + P_A + 2P_B, x_4(k-1) + 2P_B, x_5(k-1) + P_A + P_B + P_C\right)$$

$$x_4(k) = \max\left(x_1(k-1) + 2P_A + 2P_B, x_3(k-1) + P_A + 3P_B, x_4(k-1) + 3P_B, x_5(k-1) + P_A + 2P_B + P_C\right)$$

$$x_5(k) = \max\left(x_1(k-1) + 2P_A, x_3(k-1) + P_A + P_B, x_5(k-1) + P_A + P_C, x_6(k-1) + P_C\right)$$

$$x_6(k) = \max\left(x_1(k-1) + 2P_A + P_C, x_3(k-1) + P_A + P_B + P_C, x_5(k-1) + P_A + 2P_C, x_6(k-1) + 2P_C\right)$$

After introduction of numerical values from Table 4.1, the A-matrix of the system becomes as follows:

$$A = \begin{pmatrix} 7 & \varepsilon & 4 & \varepsilon & 7 & \varepsilon \\ 14 & \varepsilon & 11 & 4 & 14 & \varepsilon \\ 18 & \varepsilon & 15 & 8 & 18 & \varepsilon \\ 22 & \varepsilon & 19 & 12 & 22 & \varepsilon \\ 14 & \varepsilon & 11 & \varepsilon & 14 & 7 \\ 21 & \varepsilon & 18 & \varepsilon & 21 & 14 \end{pmatrix} \tag{4.3}$$

The eigenvalue of the A-matrix is 15 and the eigenvector is $[0 \quad 7 \quad 11 \quad 15 \quad 7 \quad 14]^T$.

4.4.2 Production of D and E Using Only One of the Units for D, M2

This system consists of two processing stages D and E. For the stages, there are two different units, out of which unit 1 can be used for stage D, and unit 5 can be used for stages E. In this case, unit 2 is used for production of A. This will now be formulated using a max-plus model, and for that, we need 2 states x_i, listed and described below.

Filling stages

1. U_1 doing D
2. U_5 doing E

Now we write down the max-plus-algebraic state space model of this DES.

$$x_1(k) = \max\left(x_1(k-1) + P_D, x_2(k-1) + P_E - P_D + P_o\right)$$
$$x_2(k) = \max(x_1(k-1) + 2P_D - P_o), x_2(k-1) + P_E)$$

After introduction of numerical values from Table 4.1, the A-matrix of the system becomes

$$A = \begin{pmatrix} 10 & 4 \\ 14 & 8 \end{pmatrix}. \tag{4.4}$$

The eigenvalue of the A-matrix is 10 and the eigenvector is $[0 \quad 4]^T$

4.4.3 Production of D and E Using Both Units for D, M3

This system consists of two processing stages D, E, and F (see Figure 4.7). Stage F is not modeled, as it follows directly after E, and is so fast that it does not constrain anything. For the other stages, there are three different units, out of which units 1 and 2 can be used for stage D, and unit 5 can be used for stages E with two steps.

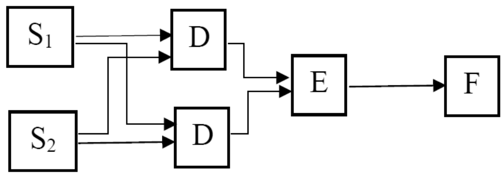

Figure 4.7: System in emptying mode

This will now be formulated using a max-plus model, and for that, we need 4 states x_i, listed and described below:

Stages for emptying cycle

1. U_1 doing D
2. U_2 doing D
3. U_5 doing E, step 1
4. U_5 doing E, step 2

The production of E is modeled using two states, modeling which D production unit is currently served.

Now we write down the max-plus-algebraic state space model of this DES.

$x_1(k) = \max\,(\,x_1(k-1) + P_D, x_4(k-1) + P_E - P_D + P_o)$
$x_2(k) = \max(x_1(k) + P_D - P_o)$
$x_3(k) = \max\,(x_3(k-1) + P_D, x_2(k) + P_E - P_D + P_o)$
$x_4(k) = \max(\,x_3(k) + P_D - P_o)$

In order to obtain an equation of type $x(k) = A \otimes x(k-1)$, the right-hand-side expressions containing k or higher indices are substituted with expressions containing index $k-1$ at most. After some straightforward algebraic manipulations and simplifications, we obtain the following equations:

$x_1(k) = \max\,(\,x_1(k-1) + P_D, x_4(k-1) + P_E - P_D + P_o)$

$x_2(k) = \max(x_1(k-1) + 2P_D - P_o, x_4(k-1) + P_E)$

$x_3(k) = \max\,(x_1(k-1) + P_D + P_E, x_3(k-1) + P_D, x_4(k-1) + 2P_E - P_D + P_o)$

$x_4(k) = \max(\, x_1(k-1) + 2P_D + P_E - P_o, x_3(k-1) + 2P_D - P_o, x_4(k-1) + 2P_E)$

After introduction of numerical values from Table 4.1, the A-matrix of the system becomes

$$A = \begin{pmatrix} 10 & \varepsilon & \varepsilon & 4 \\ 14 & \varepsilon & \varepsilon & 8 \\ 18 & \varepsilon & 10 & 12 \\ 22 & \varepsilon & 14 & 16 \end{pmatrix}. \tag{4.5}$$

The eigenvalue of the A-matrix is 16 where the eigenvector is $[0 \quad 4 \quad 8 \quad 12]^T$. Schedules produced by using simulations of the three models are shown in Tables 4.6 to 4.8. As can be seen in Tables 4.6 to 4.8, we use the choice of $n_A = 14$ and $n_D = 7$ and the period length is 268 hours.

Table 4.6: Schedule for units producing A, B and C according to model M1. In the last round, the start of B3 (highlighted with yellow) liberates A for cleaning (which takes 4 hours) and gives the starting time of D1 in Table 4.8. The cells highlighted with orange are the ones initialized by the orange cells in Table 4.8

Step index	1	2	3	…	14	15	16	…	28	29	30	…	42	43	44	…
A	0	15	30	…	195	268	283	…	463	536	551	…	731	804	819	…
B1	7	22	37	…	202	275	290	…	470	543	558	…	738	811	826	…
B2	11	26	41	…	206	279	294	…	474	547	562	…	742	815	830	…
B3	15	30	45	…	210	283	298	…	478	551	566	…	746	819	834	…
C1	7	22	37	…	202	275	290	…	470	543	558	…	738	811	826	…
C2	14	29	44	…	209	282	297	…	477	550	565	…	745	818	833	…

Table 4.6 shows that the production, also seen in Figure 4.3, is periodic from the start. The period length is 268 hours, which is the difference between the numbers in the orange cells, which indicates the restarting times for production of A. Table 4.7 gives the schedule for the production of D and E, and the same period 268 hours can be seen between the switching time instances highlighted with green cells. Table 4.8 gives the schedule for the production of D and E, using two units for D. The schedule becomes periodic starting from iteration 21, and the period length of 268 hours can again be seen between the switching time instances highlighted with yellow and green respectively.

Table 4.7: Schedule for units producing D and E according to model M2. The cells highlighted with green are the ones initialized by the green cells in Table 4.8

Step index	1	2	3	4	…	20	21	22	…	41	42	43	…
D	15	25	35	45	…	205	272	282	…	472	540	550	…
E	19	29	39	49	…	209	276	286	…	476	544	554	…

Table 4.8: Schedule for units producing D and E according to model M3. The cells highlighted with yellow are the ones initialized by the yellow cells in Table 4.6. The start of the last round of D2, highlighted with orange, is after completion (10 hours) and cleaning (4 hours) used for starting time of A in Table 4.6. The start of the last round of D1, highlighted with green, is after completion (10 hours) used for starting time of D in Table 4.7.

Step index	1	2	3	4	5	6	7	8	9	10	11	12	13	…
D1	214	230	246	262	482	498	514	530	750	766	782	798	1018	…
E1	218	234	250	266	486	502	518	534	754	770	786	802	1022	…
D2	222	238	254		490	506	522		758	774	790		1026	…
E2	226	242	258		494	510	526		762	778	794		1030	…

As can be seen from Table 4.7, the production schedule becomes periodic after the initial phase, where the empty storages introduce constraints that distort the schedule. For example, in the first round A starts at time 0 and D starts at time 15, while in the second round A starts at 268 and D at 272. This faster start of D is possible because one can use raw material from the storages of B and C. It is clear that the schedule will continue as periodic, as the storages are also changing using the same period, and the system starts at the same initial condition at each start of a new period. Tables 4.6 and 4.8 are periodic from the start.

4.5 Conclusion

This chapter described how a max-plus model for a manufacturing system can be constructed. The scheduling of production systems consisting of many stages and different units is considered, where some of the units can be used for various stages. Production unit is used for various stages where cleaning is needed in between, while no cleaning is needed between stages of the same type. The state update equation was in this case obtained in the form $x(k) = A \otimes x(k-1)$ just by several cross-substitutions. Structural decisions such as using a unit for different tasks were found difficult to formulate in max-plus algebra. Only a part of the schedule was obtained from the max-plus model; for the most critical stages the final optimal schedule could be extracted from the max-plus schedule using a couple of simple rules. Still, an optimal schedule was obtained without any optimization.

Chapter 5
Stochastic Max-plus Systems

This chapter specializes in the study of sequences $\{x(k): k \in N\}$, satisfying the difference equation

$$x(k + 1) = A(k) \otimes x(k), \qquad k \geq 0 \tag{5.1}$$

where $x(0) = x_0 \in \mathbb{R}^n_{\max}$ is the initial value and $\{A(k): k \in N\}$ is a sequence of $n \times n$ matrices over $\mathbb{R}_{\max}$ [2]. In order to develop an expressive mathematical theory, we need some extra assumptions on $\{A(k): k \in N\}$.The approach submitted in this chapter supposes that $\{A(k) : k \in N\}$ is a sequence of independent identically distributed random matrices in $\mathbb{R}^{n\times n}_{\max}$, defined in a common probability space. Specifically, we address the case where $\{A(k): k \in N\}$ consists of random matrices. The theory is in fact also available for the more general case of $\{A(k): k \in N\}$ being stationary. We focus on the asymptotic growth rate of $x(k)$. Note that $x(k)$ and, thus, $x(k)/k$ are random variables. We have to be careful with how to interpret the asymptotic growth rate. The key result of this chapter will be that under appropriate conditions the asymptotic growth rate of $x(k)$ defined in 5.1 is, with probability one, a constant [1,2]. The chapter is organized as follows: In Section 5.1, basic concepts are introduced for the stochastic max-plus systems. Moreover, examples of stochastic max-plus systems are given. Section 5.2 is devoted to the subadditive ergodic theory for stochastic sequences. The limit theory for matrices whose communication graph is fixed and has cyclicity one is presented in Section 5.3. Possible relaxations of the rather restrictive conditions needed for the analysis in the latter section are provided in Section 5.4.

5.1 Basic Definitions and Examples

For a sequence of square matrices $\{A(k) : k \in N\}$, we set

$$\bigotimes_{k=l}^{m} A(k) \overset{\text{def}}{=} \begin{cases} A(m) \otimes A(m-1) \otimes \ldots\ldots\ldots\ldots \otimes A(l+1) \otimes A(l), \text{ where } m \geq l \\ I \quad \text{Otherwise} \end{cases}$$

A couple of words based on the stochastic setup are in order here.

Let X be a random element in the definition of $\mathbb{R}_{\max}$ on a probability space (Ω, F, P) modeling the random potential. In determining the expected value of X, denoted by $E[X]$, one must take care of the fact that X may take value ε $(= -\infty)$ with positive probability. This is reflected in the extension of the following for $\mathbb{R}_{\max}$ of the usual definition of integration of the random variable on $\mathbb{R}$. Thus, $X \in \mathbb{R}_{\max}$ is integrable if $E[|X|1_{X\in\mathbb{R}}]$ is finite, where $1_{X\in\mathbb{R}}$ equals one if X is finite and zero otherwise [1,2].
A random matrix A in $\mathbb{R}_{\max}^{n\times n}$ is called integrable if its elements a_{ij} are integrable for $i \in n, j \in m$.
Stochasticity occurs naturally in real-life railway networks. For example, travel times become stochastic due to, for example, weather conditions or the individual behavior of the driver. Another source of randomness is the time for periods of ascent or descent of passengers. In addition, the lack of information about the specifics of the future of the railway system, such as the type of rolling stock, the capacity of certain tracks, and so forth, can be modeled by randomness [2].

Example 5.1 Consider the production system of Figure 5.1

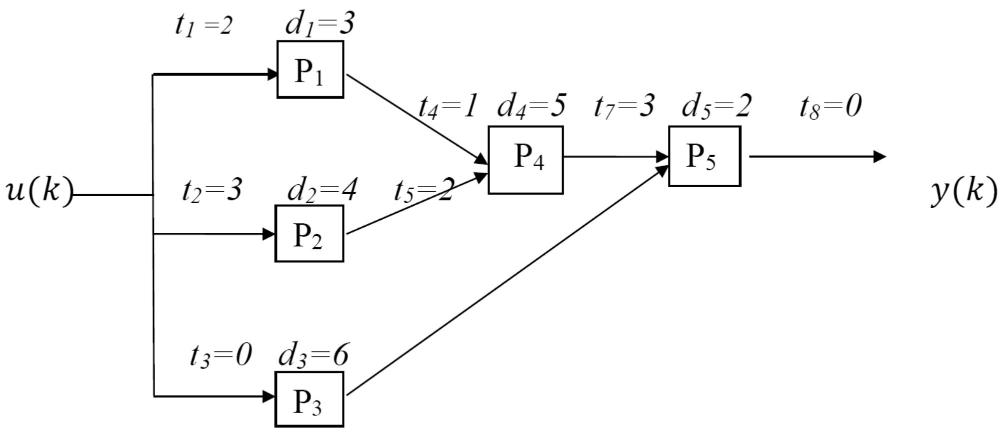

Figure 5.1: A manufacturing system

This manufacturing system consists of five processing units: P_1, P_2, P_3, P_4 and P_5 and works in batches (one batch for each finished product).

Raw material is fed to P_1, P_2 and P_3. P_1 and P_2, processed and sent to P_4. P_3 and P_4 are processed and sent to P_5, where assembly takes place. Note that each input batch of raw material is split into three parts: one part of the batch goes to P_1, the second part goes to P_2 and the third part goes to P_3.

The processing times for P_1, P_2, P_3, P_4, and P_5 are respectively d_1=3, d_2= 4, d_3=6, d_4=5 and d_5=2 time units. We assume that it takes t_1=2 time units for the raw material to get from the input source to P_1, and t_4=1 time units for a finished product of P_1 to get to P_4, and $t_7=3$ time units for a finished product of P_4 to get to P_5.

At the input of the system and between the processing units, there are buffers with a capacity that is large enough to ensure that no buffer overflow occurs. A processing unit can only start working on a new product if it has finished processing the previous one. We assume that each processing unit starts working as soon as all parts are available. Now we write down the max-plus-algebraic state space model of this DES.

$x_1(k+1) = \max\,(x_1(k)+3, u(k)+\ 2)$

$x_2(k+1)\ = \max\,(x_2(k)+4, u(k)+3)$

$x_3(k+1)\ = \max\,(x_3(k)+6, u(k)+0\,)$

$x_4(k+1)\ = \max\,(x_1(k)\ +7, x_2(k)\ +10, x_4(k)\ +5\,, u(k)+\max(3+2+1, 4+3+2))$

$x_4(k+1)\ = \max\,(x_1(k)\ +7, x_2(k)\ +10, x_4(k)\ +5\,, u(k)+9)$

$x_5(k+1) = \max\,(x_1\ +15, x_2(k)\ +18, x_3(k)\ +12, x_4(k)\ +13, x_5(k)+2, u(k)+17).$

In this system, we have buffers with limited capacity. In order to avoid building up of long queues, we need to introduce $x_6, x_7, x_8, x_9,$ and x_{10} , we do not want any of the processing units: P_1, P_2, P_3, and P_4 start working with 11th item before P_5 finishing the 5th item.

<u>Add</u>:

$x_1(k+1)$ also $\geq\ x_5(k-5) = x_{10}(k)$

$x_2(k+1)\qquad \geq\ x_5(k-5) =\ x_{10}(k)$

$x_3(k+1)\qquad \geq\ x_5(k-5) =\ x_{10}(k)$

$x_4(k+1)\qquad \geq\ x_5(k-5) =\ x_{10}(k)$

and

$x_6(k) = x_5(k-1)\ \rightarrow\ x_6(k+1) = x_5(k)$

$x_7(k) = x_5(k-2)\ \rightarrow\ x_7(k+1) = x_5(k-1) = x_6(k)$

$x_8(k) = x_5(k-3) \rightarrow x_8(k+1) =\ x_5(k-2) = x_7(k)$

$x_9(k) = x_5(k-4)\ \rightarrow x_9(k+1) =\ x_5(k-3) = x_8(k)$

$x_{10}(k) = x_5(k-5)\,) \rightarrow x_{10}(k+1) = x_5(k-4) = x_9(k).$

Thus, if $x_6(k) = x_5(k-1)$, then

$x_6(k+1) = x_5(k)$

$x_7(k+1) = x_5(k-1) = x_6(k)$

$x_8(k+1) = x_5(k-2) = x_7(k)$

$x_9(k+1) = x_5(k-3) = x_8(k)$

$x_{10}(k+1) = x_5(k-4) = x_9(k)$.

This has no large buffer, no buildup of long queues. If we rewrite the above evolution equations as a max-plus linear model, and $x(k+1) = A \otimes x(k) \oplus B \otimes u(k+1)$, we obtain:

$$x(k+1) = \begin{pmatrix} 3 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & 0 \\ \varepsilon & 4 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & 0 \\ \varepsilon & \varepsilon & 6 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & 0 \\ 7 & 10 & \varepsilon & 5 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & 0 \\ 15 & 18 & 12 & 13 & 2 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & 0 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & 0 & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & 0 & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & 0 & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & 0 & \varepsilon \end{pmatrix} \otimes x(k) \oplus \begin{pmatrix} 2 \\ 3 \\ 0 \\ 9 \\ 17 \\ \varepsilon \\ \varepsilon \\ \varepsilon \\ \varepsilon \\ \varepsilon \end{pmatrix} \otimes u(k+1)$$

$$\text{where, } A = \begin{pmatrix} 3 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & 0 \\ \varepsilon & 4 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & 0 \\ \varepsilon & \varepsilon & 6 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & 0 \\ 7 & 10 & \varepsilon & 5 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & 0 \\ 15 & 18 & 12 & 13 & 2 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & 0 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & 0 & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & 0 & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & 0 & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & 0 & \varepsilon \end{pmatrix}$$

and the precedence graph of the matrix A is shown in Figure 5.2.

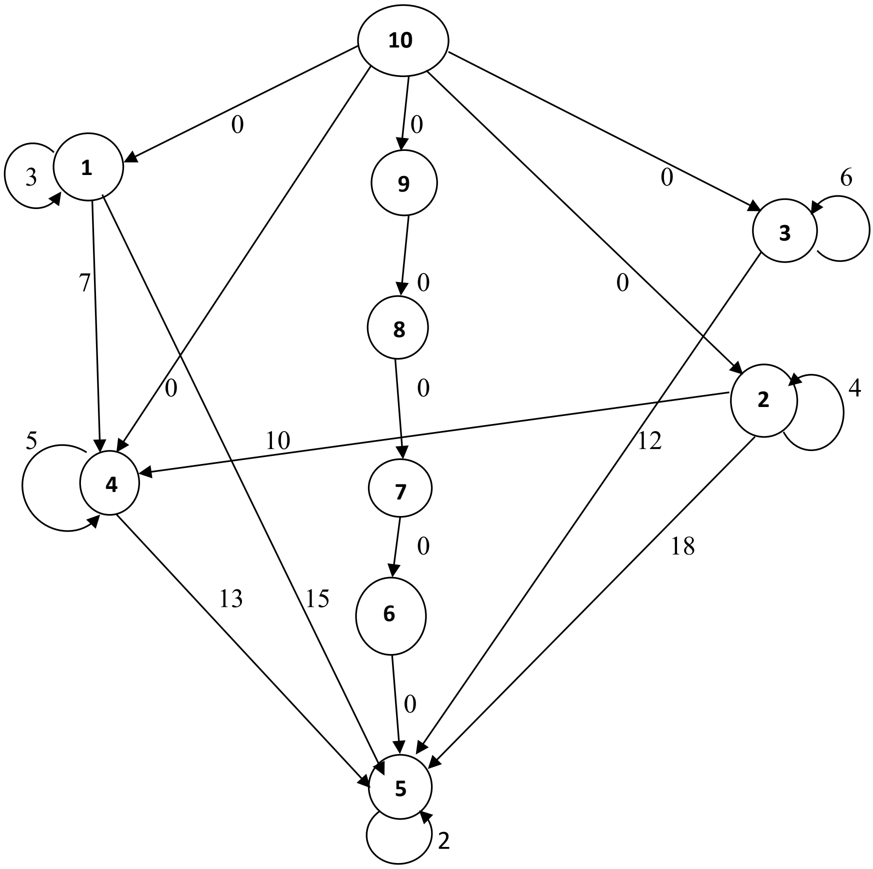

Figure 5.2: Precedence graph of A

The power method is used for finding the eigenvalue λ of the matrix A. The method means repetitive multiplications $x(k) = A \otimes x(k-1) = A^{\otimes k} \otimes x(0)$, and it stops when there are integers $i > j \geq 0$ and a real number c for which $x(i) = x(j) \otimes c$. The eigenvalue is then given by $\lambda(A) = \frac{c}{i-j}$.

In this case, using $x(0) = \mathbf{0}$,

i.e. $x_1(0) = x_2(0) = x_3(0) = x_4(0) = x_5(0) = x_6(0) = x_7(0) = x_8(0) = x_9(0) = x_{10}(0) = 0$.

As can be seen on the next page from $x(1)$ to $x(6)$ a few buffers are empty and in $x(6)$ all buffers are filled.

Iterations according Equation $x(k+1) = A \otimes x(k)$ gives

$x(1) = A \otimes x(0) = [3 \quad 4 \quad 6 \quad 10 \quad 18 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0]^T$

$x(2) = A \otimes x(1) = [6 \quad 8 \quad 12 \quad 14 \quad 23 \quad 18 \quad 0 \quad 0 \quad 0 \quad 0]^T$

$x(3) = A \otimes x(2) = [9 \quad 12 \quad 18 \quad 19 \quad 27 \quad 23 \quad 18 \quad 0 \quad 0 \quad 0]^T$

$x(4) = A \otimes x(3) = [12 \quad 16 \quad 24 \quad 24 \quad 32 \quad 27 \quad 23 \quad 18 \quad 0 \quad 0]^T$

$x(5) = A \otimes x(6) = [15 \quad 20 \quad 30 \quad 29 \quad 37 \quad 32 \quad 27 \quad 23 \quad 18 \quad 0]^T$

$x(6) = A \otimes x(5) = [18 \quad 24 \quad 36 \quad 34 \quad 42 \quad 37 \quad 32 \quad 27 \quad 23 \quad 18]^T$ ……………………..

$x(30) = A \otimes x(29) = [150 \quad 150 \quad 180 \quad 154 \quad 186 \quad 180 \quad 174 \quad 168 \quad 162 \quad 156]^T$

$x(31) = A \otimes x(30) = [156 \quad 156 \quad 186 \quad 160 \quad 192 \quad 186 \quad 180 \quad 176 \quad 168 \quad 162]^T$.

Thus, the eigenvalue is $\lambda(A) = \frac{6}{31-30} = 6$.

The eigenvector of A can be found by using this form:

$v(A) = x(31) \oplus \lambda \otimes x(30) \oplus \lambda^{\otimes 2} \otimes x(29) \oplus \lambda^{\otimes 3} \otimes \ldots \otimes x(2) \oplus \lambda^{\otimes 30} \otimes x(1)$

$\therefore v(A) = (0 \quad 0 \quad 30 \quad 4 \quad 36 \quad 30 \quad 24 \quad 18 \quad 12 \quad 6)^T$ is the eigenvector of A.

Now for the matrix A, we know that $A^{\otimes n} \otimes v = \lambda^{\otimes n} \otimes v$.

If we replace $a_{11} = 3$ by 4 and replace $a_{22} = 4$ by 5 as in matrix A_1, the eigenvalue $\lambda(A) = 6$.

If we replace $a_{44} = 5$ by 4 as in matrix A_2, the eigenvalue $\lambda(A) = 6$.

$$\text{Thus, } A_1 = \begin{pmatrix} 4 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & 0 \\ \varepsilon & 5 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & 0 \\ \varepsilon & \varepsilon & 6 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & 0 \\ 7 & 10 & \varepsilon & 5 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & 0 \\ 15 & 18 & 12 & 13 & 2 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & 0 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & 0 & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & 0 & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & 0 & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & 0 & \varepsilon \end{pmatrix} \Rightarrow \lambda(A_1) = 6$$

$$\text{and } A_2 = \begin{pmatrix} 3 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & 0 \\ \varepsilon & 4 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & 0 \\ \varepsilon & \varepsilon & 6 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & 0 \\ 7 & 10 & \varepsilon & 4 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & 0 \\ 15 & 18 & 12 & 13 & 2 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & 0 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & 0 & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & 0 & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & 0 & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & 0 & \varepsilon \end{pmatrix} \Rightarrow \lambda(A_2) = 6$$

None of the replacements and changes that we have made into the matrices A_1 and A_2 changes the maximum cycle mean of the graph in Figure 5.2.

$$A_1 \otimes A_2 = \begin{pmatrix} 7 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & 0 & 4 \\ \varepsilon & 9 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & 0 & 5 \\ \varepsilon & \varepsilon & 12 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & 0 & 6 \\ 12 & 15 & \varepsilon & 9 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & 0 & 10 \\ 20 & 23 & 18 & 17 & 4 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & 18 \\ 15 & 18 & 12 & 13 & 2 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & 0 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & 0 & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & 0 & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & 0 & \varepsilon & \varepsilon \end{pmatrix} \Rightarrow \lambda(A_1 \otimes A_2) = 12$$

The eigenvector is $v(A_1 \otimes A_2) = [0 \quad 0 \quad 30 \quad 4 \quad 36 \quad 30 \quad 24 \quad 18 \quad 12 \quad 6]^T$

$(A_1 \otimes A_2) \otimes v = [12 \quad 12 \quad 42 \quad 16 \quad 48 \quad 42 \quad 36 \quad 30 \quad 24 \quad 18]^T = \lambda^{\otimes 2} \otimes v$

$\therefore (A_1 \otimes A_2) \otimes v = \lambda^{\otimes 2} \otimes v$

However, if $A^{\otimes 2} = \begin{pmatrix} 6 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & 0 & 3 \\ \varepsilon & 8 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & 0 & 4 \\ \varepsilon & \varepsilon & 12 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & 0 & 6 \\ 12 & 15 & \varepsilon & 10 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & 0 & 10 \\ 20 & 23 & 18 & 18 & 4 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & 18 \\ 15 & 18 & 12 & 13 & 2 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & 0 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & 0 & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & 0 & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & 0 & \varepsilon & \varepsilon \end{pmatrix} \Rightarrow \lambda(A^{\otimes 2}) = 12$

The eigenvector is $v(A^{\otimes 2}) = (0 \quad 0 \quad 30 \quad 4 \quad 36 \quad 30 \quad 24 \quad 18 \quad 12 \quad 6)^T$

However, if we replace $a_{33} = 6$ by 5 as shown in matrix A_3 the eigenvalue $\lambda(A_3) = 5$. If we replace $a_{22} = 4$ by 3 and $a_{44} = 5$ by 8 as shown in matrix $A_{4,}$ the eigenvalue $\lambda(A_6) = 8$.

Therefore, in these cases the matrices are

$$A_3 = \begin{pmatrix} 3 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & 0 \\ \varepsilon & 4 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & 0 \\ \varepsilon & \varepsilon & 5 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & 0 \\ 7 & 10 & \varepsilon & 4 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & 0 \\ 15 & 18 & 12 & 13 & 2 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & 0 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & 0 & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & 0 & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & 0 & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & 0 & \varepsilon \end{pmatrix} \Rightarrow \lambda(A_3) = 5$$

and $A_4 = \begin{pmatrix} 3 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & 0 \\ \varepsilon & 3 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & 0 \\ \varepsilon & \varepsilon & 6 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & 0 \\ 7 & 10 & \varepsilon & 8 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & 0 \\ 15 & 18 & 12 & 13 & 2 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & 0 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & 0 & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & 0 & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & 0 & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & 0 & \varepsilon \end{pmatrix} \Rightarrow \lambda(A_4) = 8$

$$A_3 \otimes A_4 = \begin{pmatrix} 6 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & 0 & 3 \\ \varepsilon & 7 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & 0 & 4 \\ \varepsilon & \varepsilon & 11 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & 0 & 5 \\ 11 & 14 & \varepsilon & 12 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & 0 & 10 \\ 20 & 23 & 18 & 21 & 4 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & 18 \\ 15 & 18 & 12 & 13 & 2 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & 0 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & 0 & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & 0 & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & 0 & \varepsilon & \varepsilon \end{pmatrix} \Rightarrow \lambda(A_3 \otimes A_4) = 12.$$

The eigenvector is $v(A_3 \otimes A_4) = (0 \quad 0 \quad 0 \quad 27 \quad 36 \quad 28 \quad 24 \quad 16 \quad 12 \quad 4)^T$

$\Longrightarrow (A_3 \otimes A_4) \otimes v = (12 \quad 12 \quad 12 \quad 39 \quad 48 \quad 40 \quad 36 \quad 28 \quad 24 \quad 16)^T$

$(\lambda(A_3) \otimes \lambda(A_4)) \otimes v = (5 \otimes 8) \otimes (0 \quad 0 \quad 0 \quad 27 \quad 36 \quad 28 \quad 24 \quad 16 \quad 12 \quad 4)^T$

$= (13 \quad 13 \quad 13 \quad 40 \quad 49 \quad 41 \quad 37 \quad 29 \quad 25 \quad 17)^T$

$\therefore (A_3 \otimes A_4) \otimes v \neq (\lambda(A_3) \otimes \lambda(A_4)) \otimes v$

Nevertheless, for the replacement and changes that we have made into the matrices A_3 and A_4 does change the maximum cycle mean of the graph in Figure 5.2 which is not equal to the eigenvalue of the matrix A.

5.1.1 Petri Nets [1]

Definition 5.1 Petri nets are directed bipartite graphs. The set of vertices V is partitioned into two disjoint subsets P and Q. The elements of P are called places and those of Q are called transitions. Places will be denoted with $p_i, i = 1, \ldots, |P|$, and transitions, $q_j, j = 1, \ldots, |Q|$ [1].

The directed arcs go from a place to a transition or vice versa. Since a Petri net is bipartite, there are no arcs from place to place or from transition to transition. In the graphical representation of Petri nets, places are drawn as circles and transitions as bars (the orientation of these bars can be anything). An example of a Petri net is given in Figure 5.3.

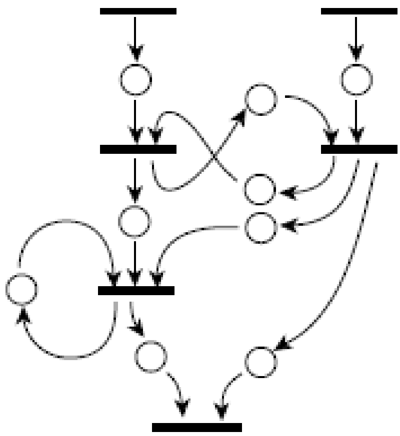

Figure 5.3: A Petri net with sources and sinks

In order to complete the formal definition of a Petri net, an initial marking must be introduced. The initial marking assigns a nonnegative integer μ_i to each place p_i. It is said that p_i is marked with μ_j initial tokens. Pictorially, μ_i dots (the tokens) are placed in the circle representing place p_i. The components μ_i form the vector μ, called the initial marking of the Petri net [1].

Definition 5.2 A Petri net is a pair (G, μ), where $G = (V, E)$ *is* a bipartite graph with a finite number of vertices (the set *V)* which are partitioned into the disjoint sets P and Q; E consists of pairs of the form (p_i, q_j) and (q_j, p_i), with $p_i \in P$ and $q_j \in Q$; the initial marking μ is a $|P|$-vector of nonnegative integers [1].

Example 5.2 Let us consider a circular track with three stations along which two trains run in one direction. The trains run from station S_1 to S_2, from S_2 to S_3, from S_3 to S_1 and so on. For safety reasons, it is assumed that a train cannot leave station S_i before the preceding train has left S_{i+1}, $i \in \underline{3}$ with $S_4 = S_1$ or, in other words, a train cannot leave a station before the platform at the next station is free. This model is symbolized in the Petri net in Figure 5.4, in which the transitions are denoted by S_i, where $i \in \underline{3}$. The trains move counterclockwise, whereas the tokens in the places in the clockwise cycle represent the conditions of the next station being free [2].

Note: $\underline{n} = \{1, \dots\dots, n\}$ for $n \in N \backslash \{0\}$.

Thus, $\underline{3} = \{1,2,3\}$

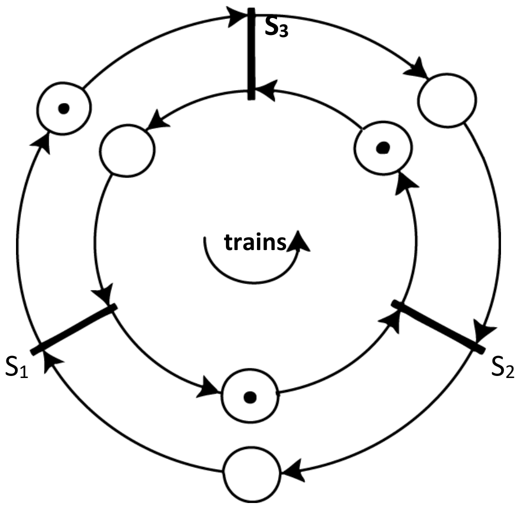

Figure 5.4: A Petri net of a circular track with two trains

If $x_i(k)$ represents the kth departure from station S_i and $a(i+1, i)$, $i \in \underline{3}$ with $a_{4,3} = a_{1,3}$ is travel time between S_i and S_{i+1}, then

$x_1(k+1) = \max\{x_3(k+1) + a_{13}, x_2(k+1)\}$,
$x_2(k+1) = \max\{x_1(k) + a_{21}, x_3(k+1)\}$,
$x_3(k+1) = \max\{x_2(k) + a_{32}, x_1(k)\}$.

This can be written as:

$$x(k+1) = \begin{pmatrix} \varepsilon & 0 & a_{13} \\ \varepsilon & \varepsilon & 0 \\ \varepsilon & \varepsilon & \varepsilon \end{pmatrix} \otimes x(k+1) \oplus \begin{pmatrix} \varepsilon & \varepsilon & \varepsilon \\ a_{21} & \varepsilon & \varepsilon \\ 0 & a_{32} & \varepsilon \end{pmatrix} \otimes x(k)$$

$$x(k+1) = \begin{pmatrix} a_{21} \oplus a_{13} & a_{13} \otimes a_{32} & \varepsilon \\ a_{21} & a_{32} & \varepsilon \\ 0 & a_{32} & \varepsilon \end{pmatrix} \otimes x(k) \tag{5.2}$$

In the derivation of the latter equality in (5.1), it has been tacitly assumed that all a_{ij} are nonnegative. Since the evolution of x_1 and x_2 is not influenced by the evolution of x_3 the third column of the latter system only has ε's, the reduced state $x_p \overset{\text{def}}{=} (x_1, x_2)^T$ can be given as

$$x_p(k+1) = \begin{pmatrix} a_{21} \oplus a_{13} & a_{13} \otimes a_{32} \\ a_{21} & a_{32} \end{pmatrix} \otimes x_p(k).$$

Now consider the railway network described in this example and assume that the travel times are random. More specifically, denote the kth travel time from station $S_i \; to \; S_{i+1}$ by $a_{i+1,k}(k)$, for $i \in \underline{2}$ and the k^{th} travel time from station $S_3 \; to \; S_1$ by $a_{1,3}(k)$. It is assumed that the travel times are stochastically independent and that the travel times for a certain track have the same distribution. This system can be modeled through $x(k) = (x_1(k), x_2(k))^T$, which satisfies

$$x(k+1) = \begin{pmatrix} a_{21}(k) \oplus a_{13}(k) & a_{13}(k) \otimes a_{32}(k) \\ a_{21}(k) & a_{32}(k) \end{pmatrix} \otimes x(k), \tag{5.3}$$

where $x_1(k)$ denotes the k^{th} departure time from station S_1 and $x_2(k)$denotes the k^{th} departure time from station S_2. Notice that the matrix on the right-hand side of that equation (5.3) is irreducible.

Example 5.3 Consider a simple railway network consisting of two stations with deterministic travel times between the stations. Specifically, the travel time from Station 2 to Station 1 equals σ', and the dwell time at Station 1 equals d, whereas the travel time from Station 1 to Station 2 equals σ and the dwell time at Station 2 equals d'. At Station 1 there is one platform at which trains can stop, whereas at Station 2 there are two platforms. Three trains circulate in the network [2].

Initially, one train is present at Station 1, one train at Station 2, and the third train is just about to enter Station 2. The time evolution of this network is described by a max-plus linear sequence of vectors $x(k) = [x_1(k), x_2(k), x_3(k), x_4(k)]^T$, where $x_1(k)$ is the kth arrival time of a train at Station 1 and $x_2(k)$is the kth departure time of a train from the Station 1, $x_3(k)$ is the kth arrival time of a train at Station 2, and $x_4(k)$ is the kth departure time of a train from Station 2. Figure (5.5) shows the Petri net model of this system [2].

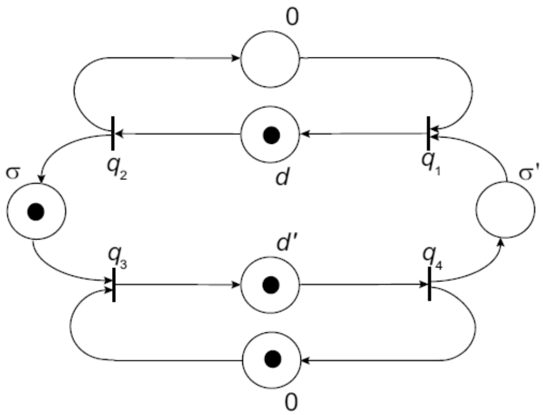

Figure 5.5: The initial state of the railway system with two platforms at Station 2

The sample-path dynamics of the network with two platforms at Station 2 is given by:

$$x_1(k + 1) = x_2(k+1) \oplus (x_4(k+1) \otimes \acute{\sigma})$$

$$x_2(k + 1) = x_1(k) \otimes d$$

$$x_3(k + 1) = (x_2(k) \otimes \sigma) \oplus x_4(k)$$

$$x_4(k+1) = x_3(k) \otimes d'$$

For $k \geq 0$. If we replace $x_2(k + 1)$ and $x_4(k + 1)$ in the first equation by the expression on the right-hand side of the second and fourth equations above, respectively, we obtain

$$x_1(k+1) = (x_1(k) \otimes d) \oplus (x_3(k) \otimes d' \otimes \sigma').$$

Hence, for $k \geq 0$

$$x_1(k+1) = (x_1(k) \otimes d) \oplus (x_3(k) \otimes d' \otimes \sigma')$$

$$x_2(k + 1) = x_1(k) \otimes d$$

$$x_3(k + 1) = (x_2(k) \otimes \sigma) \oplus x_4(k)$$

$$x_4(k+1) = x_3(k) \otimes d'.$$

This reads in vector-matrix notation

$x(k+1) = D_2 \otimes x(k)$, where

$$D_2 = \begin{pmatrix} d & \varepsilon & d' \otimes \sigma' & \varepsilon \\ d & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \sigma & \varepsilon & e \\ \varepsilon & \varepsilon & d' & \varepsilon \end{pmatrix}$$

Notice that D_2 is an irreducible matrix and that the graph of D_2 has cyclicity one.

Consider the railway network, but one of the platforms at Station 2 is not available. The initial condition is the same as in the previous example. Figure 5.6 shows the Petri net of the system with one blocked platform at Station 2, yielding that

$$x_3(k + 1) = (x_2(k) \otimes \sigma) \oplus x_4(k+1)$$

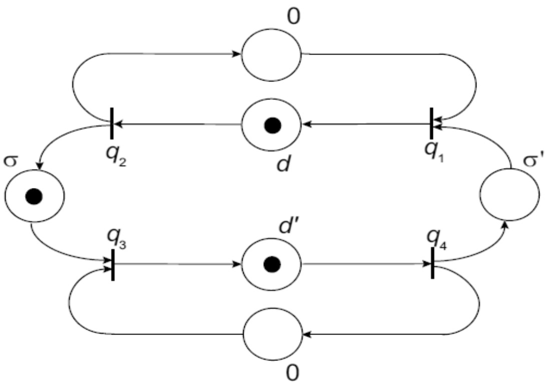

Figure 5.6: The initial state of the railway system with one blocked platform

Following the line of argument put forward for the network with two platforms at Station 2, one arrives at

$$x(k+1) = D_1 \otimes x(k), \text{ where } D_1 = \begin{pmatrix} d & \varepsilon & d' \otimes \sigma' & \varepsilon \\ d & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \sigma & d' & e \\ \varepsilon & \varepsilon & d' & \varepsilon \end{pmatrix}$$

Notice that D_1 is not irreducible, however.

Suppose that whenever the train arrives at Station 2, one platform is blocked with probability p, with $0 < p < 1$. This can be modeled by introducing $A(k)$ with distribution

$P(A(k) = D_1) = p$ and $P(A(k) = D_2) = 1 - p$.

Then, $x(k+1) = A(k) \otimes x(k)$ shows the time evolution of the system with resource restrictions.

Example 5.4 Consider the system $x(k+1) = A(k) \otimes x(k)$, with $A(k) = D_1$ and probability 0.5 , $A(k) = D_2$with also 0.5 probability D_1, D_2 are matrices from previous example (example 5.3) with numerical values $\sigma = \sigma' = d = 1$ and $d' = 2$ [2].

The matrix D_1 will be

$$D_1 = \begin{pmatrix} 1 & \varepsilon & 3 & \varepsilon \\ 1 & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & 1 & 2 & \varepsilon \\ \varepsilon & \varepsilon & 2 & \varepsilon \end{pmatrix}$$

and the precedence graph of the matrix D_1 is shown in Figure 5.7.

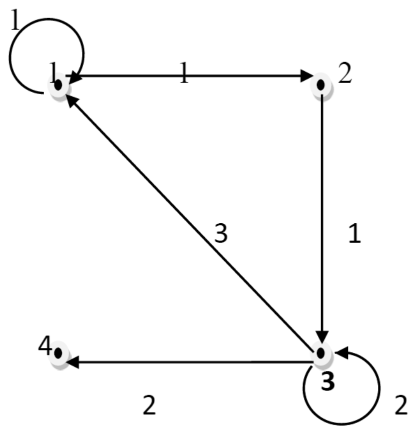

Figure 5.7: The precedence graph of D_1

If the initial times $x(0)$ are given,

i.e. $x_1(0) = x_2(0) = x_3(0) = x_4(0) = 0$ and $x(k+1) = A(k) \otimes x(k)$,

then $x(1) = \begin{pmatrix} 3 \\ 1 \\ 2 \\ 2 \end{pmatrix}$, $x(2) = \begin{pmatrix} 5 \\ 4 \\ 4 \\ 4 \end{pmatrix}$, $x(3) = \begin{pmatrix} 7 \\ 6 \\ 6 \\ 6 \end{pmatrix}$

and $x(3) - x(2) = (2 \ \ 2 \ \ 2 \ \ 2)^T$.

Hence for each $i = 1,2,3,4$ we have, $\lambda(D_1) = \frac{x_i(3) - x_i(2)}{3-2} = \frac{2}{1} = 2$.

The eigenvector of D_1 can be found by using this form:

$v\ (D_1) = x(3) \oplus \lambda \otimes x(2) \oplus \lambda^{\otimes 2} \otimes x(1)$

$$= \begin{pmatrix} 7 \\ 6 \\ 6 \\ 6 \end{pmatrix} \oplus 2 \otimes \begin{pmatrix} 5 \\ 4 \\ 4 \\ 4 \end{pmatrix} \oplus 4 \otimes \begin{pmatrix} 3 \\ 1 \\ 2 \\ 2 \end{pmatrix} = \begin{pmatrix} 7 \\ 6 \\ 6 \\ 6 \end{pmatrix}$$

and $D_1 \otimes v = \lambda \otimes v$

"Thus, the eigenvector of" $D_1 = [7 \quad 6 \quad 6 \quad 6]^T$.

According to these numerical values, $\sigma = \sigma' = d = 1$ and $d' = 2$ the matrix D_2 is:

$$D_2 = \begin{pmatrix} 1 & \varepsilon & 3 & \varepsilon \\ 1 & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & 1 & \varepsilon & 0 \\ \varepsilon & \varepsilon & 2 & \varepsilon \end{pmatrix}$$

and the precedence graph of the matrix D_2 is shown in Figure 4.8.

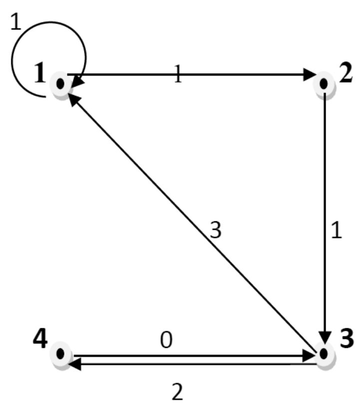

Figure 5.8: The precedence graph of D_2

If the initial times $x(0)$ are given,

i.e. $x_1(0) = x_2(0) = x_3(0) = x_4(0) = 0$ and $x(k+1) = A(k) \otimes x(k)$,

then $x(1) = \begin{pmatrix} 3 \\ 1 \\ 1 \\ 2 \end{pmatrix}$, $x(2) = \begin{pmatrix} 4 \\ 4 \\ 2 \\ 3 \end{pmatrix}$, $x(3) = \begin{pmatrix} 5 \\ 5 \\ 5 \\ 4 \end{pmatrix}$ and $x(4) = \begin{pmatrix} 8 \\ 6 \\ 6 \\ 7 \end{pmatrix}$

and $x(4) - x(1) = [5 \quad 5 \quad 5 \quad 5]^T$

Hence, $\lambda(D_2) = \frac{x_i(4)-x_i(1)}{4-1} = \frac{5}{3}$.

The eigenvector of D_2 can be found by using this form:

$$v\,(D_2) = x(4) \oplus \lambda \otimes x(3) \oplus \lambda^{\otimes 2} \otimes x(2) \oplus \lambda^{\otimes 3} \otimes x(1)$$

$$= \begin{pmatrix} 8 \\ 6 \\ 6 \\ 7 \end{pmatrix} \oplus \frac{5}{3} \otimes \begin{pmatrix} 5 \\ 5 \\ 5 \\ 4 \end{pmatrix} \oplus \left(\frac{5}{3}\right)^{\otimes 2} \otimes \begin{pmatrix} 4 \\ 4 \\ 2 \\ 3 \end{pmatrix} \oplus \left(\frac{5}{3}\right)^{\otimes 3} \otimes \begin{pmatrix} 3 \\ 1 \\ 1 \\ 2 \end{pmatrix}$$

$$= \begin{pmatrix} 8 \\ 6 \\ 6 \\ 7 \end{pmatrix} \oplus \begin{pmatrix} \frac{20}{3} \\ \frac{20}{3} \\ \frac{20}{3} \\ \frac{17}{3} \end{pmatrix} \oplus \begin{pmatrix} \frac{22}{3} \\ \frac{22}{3} \\ \frac{16}{3} \\ \frac{14}{3} \end{pmatrix} \oplus \begin{pmatrix} 8 \\ 6 \\ 6 \\ 7 \end{pmatrix} = \begin{pmatrix} 8 \\ \frac{22}{3} \\ \frac{20}{3} \\ 7 \end{pmatrix}$$

and $D_2 \otimes v = \lambda \otimes v$ (see chapter 2.2 existence of eigenvalues and eigenvectors)

Thus, the eigenvector of $D_2 = \begin{pmatrix} 8 \\ 22/3 \\ 20/3 \\ 7 \end{pmatrix}$

The Lyapunov exponent can be found by using Markov Chain theory, the derivation of the result is as follows: -

$$D_1 \otimes \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ 2 \\ 2 \end{pmatrix} \sim \begin{pmatrix} 0 \\ -2 \\ -1 \\ -1 \end{pmatrix}$$

$$D_1 \otimes \begin{pmatrix} 0 \\ -2 \\ -1 \\ -1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 1 \\ 1 \end{pmatrix} \sim \begin{pmatrix} 0 \\ -1 \\ -1 \\ -1 \end{pmatrix}$$

$$D_1 \otimes \begin{pmatrix} 0 \\ -1 \\ -1 \\ -1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 1 \\ 1 \end{pmatrix} \sim \begin{pmatrix} 0 \\ -1 \\ -1 \\ -1 \end{pmatrix}$$

$$D_2 \otimes \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ 1 \\ 2 \end{pmatrix} \sim \begin{pmatrix} 0 \\ -2 \\ -2 \\ -1 \end{pmatrix}$$

$$D_2 \otimes \begin{pmatrix} 0 \\ -2 \\ -2 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ -1 \\ 0 \end{pmatrix} \sim \begin{pmatrix} 0 \\ 0 \\ -2 \\ -1 \end{pmatrix}$$

$$D_2 \otimes \begin{pmatrix} 0 \\ 0 \\ -2 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix} \sim \begin{pmatrix} 0 \\ 0 \\ 0 \\ -1 \end{pmatrix}$$

$$D_2 \otimes \begin{pmatrix} 0 \\ 0 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ 1 \\ 2 \end{pmatrix} \sim \begin{pmatrix} 0 \\ -2 \\ -2 \\ -1 \end{pmatrix}$$

$$D_2 \otimes \begin{pmatrix} 0 \\ -2 \\ -1 \\ -1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ -1 \\ 1 \end{pmatrix} \sim \begin{pmatrix} 0 \\ -1 \\ -3 \\ -1 \end{pmatrix}$$

$$D_2 \otimes \begin{pmatrix} 0 \\ -1 \\ -3 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \\ -1 \end{pmatrix} \sim \begin{pmatrix} 0 \\ 0 \\ -1 \\ -2 \end{pmatrix}$$

$$D_2 \otimes \begin{pmatrix} 0 \\ 0 \\ -1 \\ -2 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 1 \\ 1 \end{pmatrix} \sim \begin{pmatrix} 0 \\ -1 \\ -1 \\ -1 \end{pmatrix}$$

$$D_1 \otimes \begin{pmatrix} 0 \\ 0 \\ -2 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix} \sim \begin{pmatrix} 0 \\ 0 \\ 0 \\ -1 \end{pmatrix}$$

$$D_1 \otimes \begin{pmatrix} 0 \\ 0 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ 2 \\ 2 \end{pmatrix} \sim \begin{pmatrix} 0 \\ -2 \\ -1 \\ -1 \end{pmatrix}$$

$$D_1 \otimes \begin{pmatrix} 0 \\ -2 \\ -2 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} \sim \begin{pmatrix} 0 \\ 0 \\ -1 \\ -1 \end{pmatrix}$$

$$D_1 \otimes \begin{pmatrix} 0 \\ 0 \\ -1 \\ -1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 1 \\ 1 \end{pmatrix} \sim \begin{pmatrix} 0 \\ -1 \\ -1 \\ -1 \end{pmatrix}$$

$$D_2 \otimes \begin{pmatrix} 0 \\ 0 \\ -1 \\ -1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 1 \\ 1 \end{pmatrix} \sim \begin{pmatrix} 0 \\ -1 \\ -1 \\ -1 \end{pmatrix}$$

$$D_2 \otimes \begin{pmatrix} 0 \\ -1 \\ -1 \\ -1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 0 \\ 1 \end{pmatrix} \sim \begin{pmatrix} 0 \\ -1 \\ -2 \\ -1 \end{pmatrix}$$

$$D_2 \otimes \begin{pmatrix} 0 \\ -1 \\ -2 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} \sim \begin{pmatrix} 0 \\ 0 \\ -1 \\ -1 \end{pmatrix}$$

$$D_1 \otimes \begin{pmatrix} 0 \\ -1 \\ -2 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} \sim \begin{pmatrix} 0 \\ 0 \\ -1 \\ -1 \end{pmatrix}$$

A Markov chain can be constructed with these three states, as indicated in Figure5.9

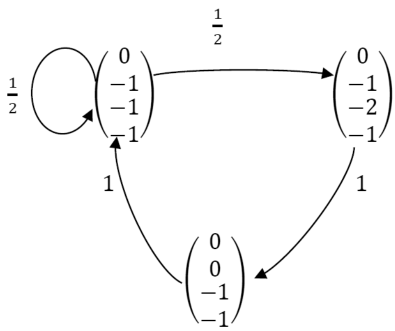

Figure 5.9: Markov chain with transition probabilities

Thus, the stationary distribution for this Markov chain is $\pi_1 = \pi_3 = \frac{1}{4}$ and $\pi_2 = \frac{1}{2}$.

The Lyapunov exponent can be calculated as

as $\lambda = \pi_1 \cdot t_{21} + \left(\frac{1}{2} t_{22} + \frac{1}{2} t_{32}\right) + \pi_3 \cdot t_{13} = \frac{7}{4}$, where the t_{ij} 's are the time durations [1], [2] and [23].

Note that: $\frac{1}{2}[\lambda(D_1) + \lambda(D_2)] \neq \frac{7}{4}$, where $\lambda(D_1) = 2$ and $\lambda(D_2) = \frac{5}{3}$.

Because of $x(k + 1) = A(k) \otimes x(k)$, with $A(k) = D_1$ with probability 0.5 and $A(k) = D_2$ also with probability 0.5, the matrices D_1, D_2 are taken to be the stochastic perturbation of the original matrix D, i.e. $D_{pert} = D_1$ or D_2 .

$A =$ maxmult (D_{pert}, I) for the first iteration where I is the identity matrix.

Thus, $A = D_{(stop\ iter)}\ D_{(stop\ iter-1)} \dots \dots \dots \dots \dots \dots \dots \dots D(5)D(4)D(3)D(2)D(1)$,

where $x_k = A \otimes x_0$.

If we have taken, $k = 100$, then

$A = D(100)D(99) \cdots \cdots \cdots D(3)D(2)D(1)$

$$\Rightarrow A = \begin{pmatrix} 174 & 175 & 176 & \varepsilon \\ 173 & 174 & 175 & \varepsilon \\ 172 & 173 & 174 & \varepsilon \\ 173 & 174 & 175 & \varepsilon \end{pmatrix}$$

$$x_{100} = A \otimes x_0 = \begin{pmatrix} 176 \\ 175 \\ 174 \\ 175 \end{pmatrix}$$

$\therefore \frac{x_k}{k} = \frac{x_{100}}{100} = \begin{pmatrix} 1.76 \\ 1.75 \\ 1.74 \\ 1.75 \end{pmatrix}$ which is close to the maximal Lyapunov exponent which is in this example equal to $\frac{7}{4} = 1.75$ [1], [2] and [23].

Other simulations of length 100 give $\begin{pmatrix} 1.75 \\ 1.75 \\ 1.74 \\ 1.74 \end{pmatrix}$, $\begin{pmatrix} 1.82 \\ 1.81 \\ 1.81 \\ 1.81 \end{pmatrix}$, $\begin{pmatrix} 1.78 \\ 1.77 \\ 1.77 \\ 1.77 \end{pmatrix}$, which is also closely approximate the maximal Lyapunov exponent.

However, $\frac{1}{2}[\lambda(D_1) + \lambda(D_2)] \neq \frac{7}{4}$, where $\lambda(D_1) = 2$ and $\lambda(D_2) = \frac{5}{3}$ are the eigenvalues of D_1, D_2 respectively and we used power method to find them which is used for deterministic cases.

Example 5.5 Consider the system $x(k+1) = A(k) \otimes x(k)$, with $A(k) = D_1$ and probability 0.5 , $A(k) = D_2$ with also 0.5 probability, where

$$D_1 = \begin{pmatrix} 1 & \varepsilon & 3 & \varepsilon \\ 1 & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & 1 & 2 & 0 \\ \varepsilon & \varepsilon & 2 & \varepsilon \end{pmatrix} \text{ and } D_2 = \begin{pmatrix} 1 & \varepsilon & 2 & \varepsilon \\ 1 & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & 1 & \varepsilon & 0 \\ \varepsilon & \varepsilon & 1 & \varepsilon \end{pmatrix}.$$

Then the eigenvalues D_1 and D_2 are 2 and $4/3$, respectively.

The asymptotic growth of stochastic system where D_1 and D_2 are drawn independently with probability $\frac{1}{2}$ each can be analyzed using the same method as above. It turns out that the system generates a Markov chain on six directions as follows: -

$$D_1 \otimes \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ 2 \\ 2 \end{pmatrix} \sim \begin{pmatrix} 0 \\ -2 \\ -1 \\ -1 \end{pmatrix} \qquad D_2 \otimes \begin{pmatrix} 0 \\ -2 \\ -1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ -1 \\ 0 \end{pmatrix} \sim \begin{pmatrix} 0 \\ 0 \\ -2 \\ -1 \end{pmatrix}$$

$$D_1 \otimes \begin{pmatrix} 0 \\ -2 \\ -1 \\ -1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 1 \\ 1 \end{pmatrix} \sim \begin{pmatrix} 0 \\ -1 \\ -1 \\ -1 \end{pmatrix} \qquad D_2 \otimes \begin{pmatrix} 0 \\ 0 \\ -2 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ -1 \end{pmatrix} \sim \begin{pmatrix} 0 \\ 0 \\ 0 \\ -2 \end{pmatrix}$$

$$D_1 \otimes \begin{pmatrix} 0 \\ -1 \\ -1 \\ -1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 1 \\ 1 \end{pmatrix} \sim \begin{pmatrix} 0 \\ -1 \\ -1 \\ -1 \end{pmatrix} \qquad D_2 \otimes \begin{pmatrix} 0 \\ 0 \\ 0 \\ -2 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 1 \\ 1 \end{pmatrix} \sim \begin{pmatrix} 0 \\ -1 \\ -1 \\ -1 \end{pmatrix}$$

$$D_2 \otimes \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 1 \\ 1 \end{pmatrix} \sim \begin{pmatrix} 0 \\ -1 \\ -1 \\ -1 \end{pmatrix} \qquad D_1 \otimes \begin{pmatrix} 0 \\ 0 \\ -2 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix} \sim \begin{pmatrix} 0 \\ 0 \\ 0 \\ -1 \end{pmatrix}$$

$$D_2 \otimes \begin{pmatrix} 0 \\ -1 \\ -1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} \sim \begin{pmatrix} 0 \\ 0 \\ -1 \\ -1 \end{pmatrix} \qquad D_1 \otimes \begin{pmatrix} 0 \\ 0 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ 2 \\ 2 \end{pmatrix} \sim \begin{pmatrix} 0 \\ -2 \\ -1 \\ -1 \end{pmatrix}$$

$$D_2 \otimes \begin{pmatrix} 0 \\ 0 \\ -1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix} \sim \begin{pmatrix} 0 \\ 0 \\ 0 \\ -1 \end{pmatrix} \qquad D_1 \otimes \begin{pmatrix} 0 \\ 0 \\ 0 \\ -2 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ 2 \\ 2 \end{pmatrix} \sim \begin{pmatrix} 0 \\ -2 \\ -1 \\ -1 \end{pmatrix}$$

$$D_2 \otimes \begin{pmatrix} 0 \\ 0 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 1 \\ 1 \end{pmatrix} \sim \begin{pmatrix} 0 \\ -1 \\ -1 \\ -1 \end{pmatrix} \qquad D_1 \otimes \begin{pmatrix} 0 \\ 0 \\ -1 \\ -1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 1 \\ 1 \end{pmatrix} \sim \begin{pmatrix} 0 \\ -1 \\ -1 \\ -1 \end{pmatrix}$$

Thus, the system generates a Markov chain on six directions: $\begin{pmatrix}0\\-1\\-1\\-1\end{pmatrix}, \begin{pmatrix}0\\0\\-1\\-1\end{pmatrix}, \begin{pmatrix}0\\0\\0\\-1\end{pmatrix}, \begin{pmatrix}0\\-2\\-1\\-1\end{pmatrix}, \begin{pmatrix}0\\0\\-2\\-1\end{pmatrix}$ and $\begin{pmatrix}0\\0\\0\\-2\end{pmatrix}$.

A Markov chain can be constructed with these six states, as indicated in Figure5.10

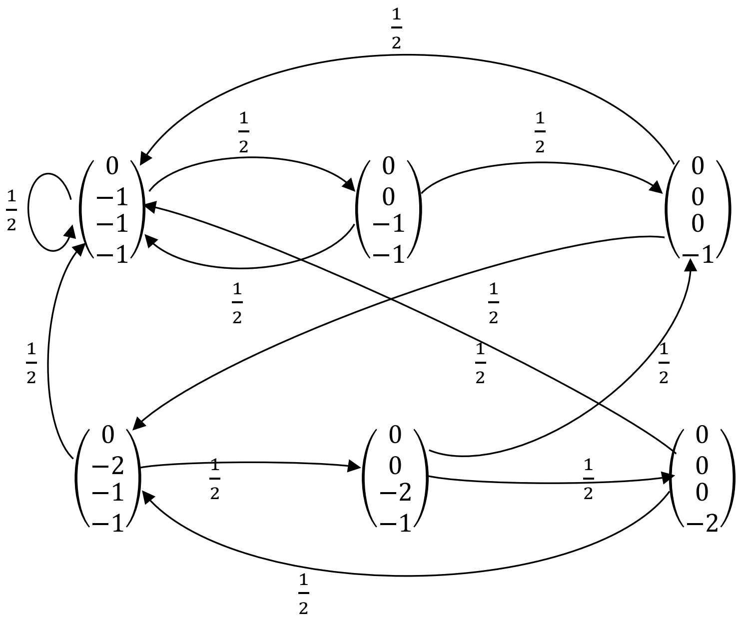

Figure 5.10: Markov chain with transition probabilities

Thus, the invariant distribution is $(24\ \ 12\ \ 7\ \ 4\ \ 2\ \ 1)/50$ and the asymptotic growth is $\frac{(24*3+\ 12*3+7*5+4*3+2*2+1*5)}{100} = 1.64$. This value is obtained by simulation as well.

Note that: $\frac{1}{2}[\lambda(D_1) + \lambda(D_2)] = \frac{1}{2}\left[2 + \frac{4}{3}\right] = \frac{5}{3} = 1.67,$ where $\lambda(D_1) = 2$ and $\lambda(D_2) = \frac{4}{3}$.

5.2 The Subadditive Ergodic Theorem

Subadditive ergodic theory is based on Kingman's subadditive ergodic theorem and its application to generalized products of random matrices. Kingman's result is formulated in terms of subadditive processes [2]. These are double-indexed processes $X = \{X_{mn}: m, n \in N\}$ satisfying the following conditions:

(S1) For $i, j, k \in N$, such that $i < j < k$, the inequality $X_{ki} \leq X_{ij} + X_{jk}$ holds with probability one.

(S2) Because the sequence $\{A(k): k \in N\}$ is an independent identically distributed one, it follows that all joint distributions of the process $X = \{X_{m+n,n+1}: m, n \in N, n > m\}$ are the same as those of $X = \{X_{mn}: m, n \in N, n > m\}$.

(S3) Because the finite entries of the $\{A(k): k \in N\}$ are all bounded so the expected value $g_n = E[X_{on}]$ exists and satisfies $g_n \geq -c \times n$ for a finite constant $c > 0$ and all $n \in N$.

A consequence of **(S1), (S3)** is that $\lambda = \lim_{n \to \infty} \frac{g_n}{n}$ exists and is finite. We can now state Kingman's subadditive ergodic theorem as follows:

Theorem 5.1 (Kingman's subadditive ergodic theorem) [1, 2]

If $X = \{X_{mn}: m, n \in N\}$ is a subadditive process, then a finite number ξ exists such that

$$\xi = \lim_{k \to \infty} \frac{X_{0k}}{k}$$

with probability one and

$$\xi = \lim_{k \to \infty} \frac{E[X_{0k}]}{k}.$$

Note that the random variables $\frac{X_{0k}}{k}$ converge, with probability one, to the same finite value, which is the limit of $\frac{E[X_{0k}]}{k}$.

We will apply Kingman's subadditive ergodic theorem to the maximal (resp., minimal) finite element of $x(k)$, with $x(k)$ defined in equation (5.1):

$$x(k+1) = A(k) \otimes x(k), \qquad k \geq 0$$

The basic concepts are defined in the following. For $A \in \mathbb{R}_{\max}^{n \times n}$, the minimal finite entry of A, denoted by $||A||_{\min}$, is given by

$$||A||_{\min} = \min\{a_{ij} \mid (i, j) \in D(A)\},$$

where $||A||_{\min} = \varepsilon' (= +\infty)$ if $D(A) = \emptyset$. (Recall that $D(A)$ denotes the set of arcs in the communication graph of A). In the same vein, we denote the maximal finite entry of $A \in \mathbb{R}_{\max}^{n\times n}$ by $||A||_{\max}$, which implies

$$||A||_{\max} = \max\{a_{ij} \mid (i,j) \in D(A)\}$$

where $||A||_{\max} = \varepsilon$ if $D(A) = \emptyset$. A direct consequence of the above definitions is that for any regular $A \in \mathbb{R}_{\max}^{n\times n}$

$$||A||_{\min} \le ||A||_{\max}.$$

Notice that $||A||_{\min}$ and $||A||_{\max}$ can have negative values. For regular $A \in \mathbb{R}_{\max}^{n\times m}$ and regular $B \in \mathbb{R}_{\max}^{m\times l}$

$$||A \otimes B||_{\max} \le ||A||_{\max} \otimes ||B||_{\max} \tag{5.4}$$

Proof: Note that $A_{ij} \le ||A||_{\max}$ so

$$\|A \otimes B\|_{\max} = \max(a_{ik} + b_{kj}) \le \max(\max_{i,k} a_{ik} + \max_{k,j} b_{kj})$$
$$= \max(a_{ik}) \otimes \max(b_{kj})$$
$$= \|A\|_{\max} \otimes \|B\|_{\max}$$

■

and

$$||A \otimes B||_{\min} \ge ||A||_{\min} \otimes ||B||_{\min} \tag{5.5}$$

Proof: $\|\boldsymbol{A} \otimes \boldsymbol{B}\|_{\min} = \min(a_{ik} + b_{kj}) \ge \min(\min_{i,k} a_{ik} + \min_{k,j} b_{kj})$

$$= \min(a_{ik}) \otimes \min(b_{kj})$$
$$= \|A\|_{\min} \otimes \|B\|_{\min}$$

■

The basic max-plus recurrence relation is

$$x(k+1) = A(k) \otimes x(k), \text{ for } k \ge 0, \text{with } x(0) = x_0.$$

To indicate the initial value of the sequence, we sometimes use the notation

$$x(k; x_0) = \bigotimes_{n=0}^{k-1} A(n) \otimes x_0 \quad , k \in N \tag{5.6}$$

To abbreviate the notation, we set for m ≥ n ≥ 0

$$A[m,n] \stackrel{\text{def}}{=} \otimes_{n=0}^{k-1} A(k)$$

With this, equation (5.6) can be written as

$$x(k; x_0) = A[k,0] \otimes x_0 \quad \text{for } k \geq 0.$$

Notice that for $0 \leq n \leq p \leq m$

$$A\,[m,n] = A\,[m,p] \otimes A\,[p,n] \tag{5.7}$$

Lemma 5.1 Let $\{A(k) : k \in N\}$ be an independent identically distributed sequence of integrable matrices such that $A(k)$ is regular with probability one [2].

Then: $\left\{-\left\|A[m,n] \otimes u\right\|_{\min} : m > n \geq 0\right\}$ and $\{||A[m,n] \otimes u||_{\max} : m > n \geq 0\}$ are subadditive ergodic processes.

Proof: Note that for $2 \leq m$ and $0 \leq n < p < m$

$$A[m,n] \otimes u \leq ||A[p,n] \otimes u||_{\max} \otimes u \tag{5.8}$$

and

$$A[m,n] \otimes u \geq ||A[p,n] \otimes u||_{\min} \otimes u \tag{5.9}$$

For $2 \leq m$ and $0 \leq n < p < m$ we obtain

$$\begin{aligned} ||A[m,n] \otimes u||_{\max} &= ||A[m,p] \otimes A[p,n] \otimes u||_{\max} \\ &\leq ||A[m,p] \otimes (||A[p,n] \otimes u||_{\max} \otimes u)||_{\max} \\ &\leq ||A[m,p] \otimes u||_{\max} + ||A[p,n] \otimes u||_{\max} \end{aligned} \tag{5.10}$$

Following a similar line of argument, where (5.7) and (5.3) are used for establishing the inequalities, it follows that

$$||A[m,n] \otimes u||_{\min} \geq ||A[m,p] \otimes u||_{\min} + ||A[p,n] \otimes u||_{\min} \tag{5.11}$$

for $2 \leq m$ and $0 \leq n < p < m$. Repeated application of (5.8) implies

$||A[m,0] \otimes u||_{\max} \leq ||A(m-1) \otimes u||_{\max} + \cdot \cdot \cdot + ||A(0) \otimes u||_{max}$, and, using the fact that

$\{A(k) : k \in N\}$ is an independent identically distributed sequence. This yield

$$E||A[m,0] \otimes u||_{\max} \leq m \times E\,||A(0) \otimes u||_{\max} \qquad (5.12)$$

Following a similar line of argument, it follows that

$$E||A[m,0] \otimes u||_{\min} \geq m \times E\,||A(0) \otimes u||_{\min} \qquad (5.13)$$

We now turn to conditions (S_1) to (S_3). For $E||A[m,0] \otimes u||_{\max}$, ($S_1$) follows from (5.10), and (S_1) follows for $-||A\,[m,n] \otimes u||_{\min}$ from (5.11).

The stationary condition (S_2) follows immediately from the independent identically distributed hypothesis for $\{A(k) : k \in N\}$ [2].

We now turn to condition (S_3) for $E||A[m,n] \otimes u||_{\max}$. The fact that $\{A(k): k \in N\}$ is an independent identically distributed sequence implies that

$$\begin{aligned} E[||A[k,0] \otimes u||_{\max}] &\geq E[\,||A(0) \otimes u||_{\min}] \\ &\geq k \times E[\,||A(0) \otimes u||_{\min}] \\ &\geq k \times E[\,||A(0)\,||_{\min}] \\ &\geq k \times (-|E[\,||A(0)\,||_{\min}]|) \end{aligned}$$

where we have used for the one but last inequality the fact that $||u||_{\min} = 0$ in combination with (5.5).

Integrability of $A(0)$ together with regularity implies that $E[\,||A(0)||_{\min}]$ is finite (for a proof use the fact that $\min(X,Y) \leq |X| + |Y|$).

This establishes condition (S_3) for $||A[m,n] \otimes u||_{\max}$.

For the proof that $-||A[m,n] \otimes u||_{\min}$ and $||A\,[m,n] \otimes u||_{\min}$ satisfies (S_3), follows from multiplying (5.11) by -1 [2].

The above lemma shows that Kingman's subadditive ergodic theorem can be applied to

$||A[k,0] \otimes u||_{\min}$ and $||A[k,0] \otimes u||_{\max}$. The precise statement is given in the following theorem.

Theorem 5.2 Let $\{A(k) : k \in N\}$ be an independent identically distributed sequence of integrable matrices such that $A(k)$ is regular with probability one. Then, finite constants λ^{top} and λ^{bot} exist such that with probability one [1], [2] and [23].

$$\lambda^{bot} \stackrel{\text{def}}{=} \lim_{k\to\infty} \frac{1}{k} \|A[k,0] \otimes u\|_{\min} \leq \lambda^{top} \stackrel{\text{def}}{=} \lim_{k\to\infty} \frac{1}{k} \|A[k,0] \otimes u\|_{\max}$$

and

$$\lambda^{bot} = \lim_{k\to\infty} \frac{1}{k} E[\|A[k,0] \otimes u\|_{\min}], \qquad \lambda^{top} = \lim_{k\to\infty} \frac{1}{k} E[\|A[k,0] \otimes u\|_{\max}]$$

The constant λ^{top} is called the top or maximal Lyapunov exponent of $\{A(k): k \in N\}$, and λ^{bot} is called the bottom or minimal Lyapunov exponent of $\{A(k) : k \in N\}$.

The top and bottom Lyapunov exponents of $A(k)$ are related to the asymptotic growth rate of $x(k)$ defined in 5.1 as follows. The top Lyapunov exponent equals the asymptotic growth rate of the maximal entry of $x(k)$, and the bottom Lyapunov exponent equals the asymptotic growth rate of the minimal entry of $x(k)$. The precise statement is given in the following corollary.

Corollary 5.1 [2]. Let $\{A(k): k \in N\}$ be an independent identically distributed sequence of integrable matrices such that $A(k)$ is regular with probability one. Then, for any finite and integrable initial condition x_0, it holds with probability one that

$$\lambda^{bot} = \lim_{k\to\infty} \frac{\|x(k;\, x_0)\|_{\min}}{k} \leq \lambda^{top} = \lim_{k\to\infty} \frac{\|x(k;\, x_0)\|_{\max}}{k}$$

and

$$\lambda^{bot} = \lim_{k\to\infty} \frac{1}{k} E[\|x(k;\, x_0)\|_{\min}], \qquad \lambda^{top} = \lim_{k\to\infty} \frac{1}{k} E[\|x(k;\, x_0)\|_{\max}]$$

Proof*:* Note that $x(k;\, x_0) = A[k,0] \otimes x_0$ for any $k \in N$. Provided that x_0 is finite, it follows by monotonicity arguments that

$$A[k,0] \otimes (||x_0||_{\min} \otimes u) \leq x(k;\, x_0) \leq A[k,0] \otimes (||x_0||_{\max} \otimes u).$$

It is easily checked that this implies

$$||A[k,0] \otimes u||_{\min} \otimes ||x_0||_{\min} \leq ||x(k;\, x_0)||_{min} \leq ||A[k,0] \otimes u||_{\min} \otimes ||x_0||_{\max}$$

Dividing the above row of inequalities by k and letting $k \to \infty$ yields

$$\lim_{k\to\infty} \frac{1}{k} \|x(k;\, x_0)\|_{\min} = \lambda^{bot}$$

with probability one. The proof for the other limit follows from the same line of argument.

Arguments used for the proof of the first part of the corollary are still valid when expected values are applied.

5.3 Matrices with a Fixed Structure

5.3.1 Irreducible Matrices

In this section, we consider stationary sequences $\{A(k): k \in N\}$ of integrable and irreducible matrices in $\mathbb{R}_{\max}^{n\times n}$ [2]. The additional property is that all finite elements are non-negative and that all diagonal elements are non-negative such that with probability one

(i) Finite entries are bounded from below by a finite constant and
(ii) The communication graph realizes a subgraph that is strongly connected, has cyclicity one and is independent of k.

As we will show in the following theorem, the setting of this section implies that $\lambda^{\text{top}} = \lambda^{\text{bot}}$, which in particular implies convergence of $\frac{x_i(k)}{k}$ as $k \to \infty$, for $i \in \underline{n}$. The main technical result is submitted in the following lemma [2].

Lemma 5.2 [2]. Let $D \in \mathbb{R}_{\max}^{n\times n}$, be a non-random irreducible matrix such that its communication graph has cyclicity one. If $A(k) \geq D$ with probability one, for any k, then there exist integers L and N exist such that for any $k \geq N$

$$\|x(k)\|_{\min} \geq \|x(k-L)\|_{\max} + (\|D\|_{\min})^{\otimes L}$$

Proof: Denote the communication graph of D by $G = (N, D)$, and note that G is of cyclicity one. Denote the number of elementary cycles in G by q, and let β_i denote the length of cycle ζ_i, for $i \in q$. Then the greatest common divisor of $\{\beta_1, \dots, \beta_q\}$ is equal to one. A natural number N exists such that for all $k \geq N$ there are integers $n_1, \dots, n_q \geq 0$ such that

$$k = n_1\beta_1 + n_2\beta_{2+} \dots \dots + n_q\beta_q .$$

Let l_{ij} denote the minimal length of a path from j to i containing all vertices of G. Such paths exist because D is irreducible (and, hence, G is strongly connected).

Let the maximal length of all these paths be denoted by, i.e. $l = \max_{i,j\in\underline{n}} l_{ij}$.

Next, choose an L with $L \geq N + l$. Then for any $i, j \in n$, there is a path from j to i of length L. Indeed, take any $i, j \in \underline{n}$ and choose a path, as mentioned above, from j to i containing all vertices of G and having minimal length l_{ij}. Clearly, the path has at least one node in common with each of the q cycles in G. As $L - l_{ij} \geq N$, there are integers $n_1, \dots, n_q \geq 0$ such that

$$L - l_{ij} = n_1\beta_1 + n_2\beta_{2+} \dots\dots + n_q\beta_q.$$

Hence, by adding n_1 copies of cycle ζ_1 , and so on, up to n_q copies of cycle ζ_q to the chosen path from j to i of length l_{ij} , a new path from j to i is created of length L.

In graph theoretical terms, the element $[A(K, K - L)]_{ij}$ denotes the maximal weight of a path of length L from node j to node i on the "interval" $[K - L, k)$.

Since $A[K, K - L] \geq D^{\otimes L}$ by assumption, it follows that for all $k \geq N$ and all $i \in \underline{n}$

$$x_i(k) = \oplus_{j=1}^{n}[A(k, k - L)]_{ij} \otimes x_j(k - L)$$

$$\geq \oplus_{j=1}^{n}\left[D^{\otimes L}\right]_{ij} \otimes x_j(k - L)$$

$$\geq \oplus_{j=1}^{n}(\|D\|_{\min})^{\otimes L} \otimes x_j(k - L)$$

$$\geq (\|D\|_{\min})^{\otimes L} \oplus_{j=1}^{n} x_j(k - L)$$

implying that

$$\|x(k)\|_{\min} \geq \|x(k - L)\|_{\max} + (\|D\|_{\min})^{\otimes L}$$

The condition that $A(k) \geq D$ with probability one for any $k \in N$ and with D being irreducible will be referred to as condition (H_1).

(H_1)*:* There is a non-random irreducible matrix D whose communication graph is of cyclicity one such that $A(k) \geq D$ for any $k \in N$, with probability one.

Matrix D in (H_1) is called the minimal support matrix of $A(k)$.

Theorem 5.3 [2]. Let $\{A(k) : k \in N\}$ be a random sequence of integrable matrices satisfying (H_1). For $x(k)$ defined in (5.1) it holds, with probability one, that

$$\lim_{k\to\infty}\frac{1}{k}\|x(k; x_0)\|_{\min} = \lim_{k\to\infty}\frac{1}{k}\, x_i(k; x_0) = \lim_{k\to\infty}\frac{1}{k}\|x(k; x_0)\|_{\max}$$

For any $i \in n$ and any finite initial state x_0.

Proof: Let D be given as in (H_1); then D satisfies the condition put forward in Lemma 5.2, and finite positive numbers L and N exist such that for $k \geq N$

$$\|x(k; x_0)\|_{\min} \geq \|x(k - L; x_0)\|_{\max} + (\|D\|_{\min})^{\otimes L}$$

Dividing both sides of the above inequality by k and letting k tend to ∞ yields

$$\lim_{k\to\infty} \frac{1}{k} \|x(k;x_0)\|_{\min} \geq \lim_{k\to\infty} \frac{1}{k} \|x(k;x_0)\|_{\max} \tag{5.14}$$

for any finite initial vector x_0. The existence of the above limits is guaranteed by Corollary 5.1, where we use the fact that (H_1) implies that $A(k)$ is regular with probability one [2].

Following the line of argument in the proof of Corollary 5.1, the limits in (5.12) are independent of the initial state. Combining (5.12) with the obvious fact that

$\|x(k;x_0)\|_{\max} \geq x_j(k;x_0) \geq \|x(k;x_0)\|_{\min}$ for $j \in \underline{n}$

proves the claim.

Example 5.6 Consider the matrix $A = \begin{pmatrix} 2 & 6 & \varepsilon & -3 \\ 1 & \varepsilon & -1 & \varepsilon \\ \varepsilon & 5 & -2 & \varepsilon \\ \varepsilon & 4 & -2 & \varepsilon \end{pmatrix}$ and the precedence graph of A is shown in Figure 1.4-Example1.10 [1].

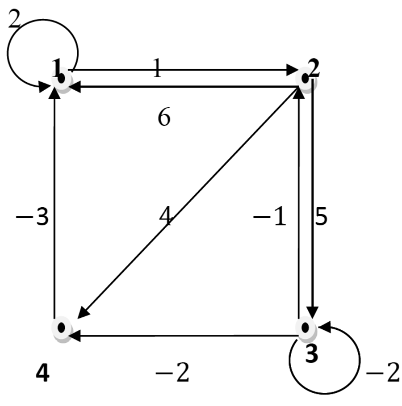

Figure 1.4: The precedence graph of A

If the initial times $x(0)$ are given,

i.e. $x_1(0) = x_2(0) = x_3(0) = x_4(0) = 0$ and $x(k+1) = A(k) \otimes x(k)$,

$x(1) = A \otimes x(0)$

$x(2) = A \otimes x(1) = A^{\otimes 2} \otimes x(0)$

$x(3) = A \otimes x(2) = A^{\otimes 2} \otimes x(1) = A^{\otimes 3} \otimes x(0)$

$x(4) = A \otimes x(3) = A^{\otimes 2} \otimes x(2) = A^{\otimes 3} \otimes x(1) = A^{\otimes 4} \otimes x(0)$

$x(5) = A \otimes x(4) = A^{\otimes 2} \otimes x(3) = A^{\otimes 3} \otimes x(2) = A^{\otimes 4} \otimes x(1) = A^{\otimes 5} \otimes x(0)$

then $x(1) = \begin{pmatrix} 6 \\ 1 \\ 5 \\ 4 \end{pmatrix}$, $x(2) = \begin{pmatrix} 8 \\ 7 \\ 6 \\ 5 \end{pmatrix}$, $x(3) = \begin{pmatrix} 13 \\ 9 \\ 12 \\ 11 \end{pmatrix}$, $x(4) = \begin{pmatrix} 15 \\ 14 \\ 14 \\ 13 \end{pmatrix}$, $x(5) = \begin{pmatrix} 20 \\ 16 \\ 19 \\ 18 \end{pmatrix}$

and $x(5) - x(3) = [7 \quad 7 \quad 7 \quad 7]^T$

$\therefore \lambda(A) = \frac{x_i(5) - x_i(3)}{5-3} = \frac{7}{2}$

The eigenvector of A can be found by using this form:

$v(A) = x(5) \oplus \lambda \otimes x(4) \oplus \lambda^{\otimes 2} \otimes x(3) \oplus \lambda^{\otimes 3} \otimes x(2) \oplus \lambda^{\otimes 4} \otimes x(1)$

$$v(A) = \begin{pmatrix} 20 \\ 16 \\ 19 \\ 18 \end{pmatrix} \oplus \frac{7}{2} \otimes \begin{pmatrix} 15 \\ 14 \\ 14 \\ 13 \end{pmatrix} \oplus \left(\frac{7}{2}\right)^{\otimes 2} \otimes \begin{pmatrix} 13 \\ 9 \\ 12 \\ 11 \end{pmatrix} \oplus \left(\frac{7}{2}\right)^{\otimes 3} \otimes \begin{pmatrix} 8 \\ 7 \\ 6 \\ 5 \end{pmatrix} \oplus \left(\frac{7}{2}\right)^{\otimes 4} \otimes \begin{pmatrix} 6 \\ 1 \\ 5 \\ 4 \end{pmatrix}$$

$$= \begin{pmatrix} 20 \\ 16 \\ 19 \\ 18 \end{pmatrix} \oplus \begin{pmatrix} 37/2 \\ 35/2 \\ 35/2 \\ 33/2 \end{pmatrix} \oplus \begin{pmatrix} 20 \\ 16 \\ 19 \\ 18 \end{pmatrix} \oplus \begin{pmatrix} 37/2 \\ 35/2 \\ 33/2 \\ 31/2 \end{pmatrix} \oplus \begin{pmatrix} 20 \\ 15 \\ 19 \\ 18 \end{pmatrix}$$

$\therefore v(A) = \begin{pmatrix} 20 \\ 35/2 \\ 19 \\ 18 \end{pmatrix}$ is the eigenvector of A

And $A \otimes v = \lambda \otimes v = \begin{pmatrix} 47/2 \\ 21 \\ 45/2 \\ 43/2 \end{pmatrix}$

$v(A) = \begin{pmatrix} 0 \\ -5/2 \\ -1 \\ -2 \end{pmatrix}$ is also an eigenvector of A

Now, for the matrix $A = \begin{pmatrix} 2 & 6 & \varepsilon & -3 \\ 1 & \varepsilon & -1 & \varepsilon \\ \varepsilon & 5 & -2 & \varepsilon \\ \varepsilon & 4 & -2 & \varepsilon \end{pmatrix}$

we know that $A^{\otimes n} \otimes v = \lambda^{\otimes n} \otimes v$

If we replace $a_{42} = 4$ by 5 and replace $a_{33} = -2$ by -1, the eigenvalue $\lambda(A) = \frac{7}{2}$.

If we replace $a_{14} = -3$ by -1 and replace $a_{43} = -2$ by -1, the eigenvalue $\lambda(A) = \frac{7}{2}$.

Thus, let $A_1 = \begin{pmatrix} 2 & 6 & \varepsilon & -3 \\ 1 & \varepsilon & -1 & \varepsilon \\ \varepsilon & 5 & -2 & \varepsilon \\ \varepsilon & 5 & -2 & \varepsilon \end{pmatrix} \Longrightarrow \lambda(A_1) = \frac{7}{2}$

$A_2 = \begin{pmatrix} 2 & 6 & \varepsilon & -3 \\ 1 & \varepsilon & -1 & \varepsilon \\ \varepsilon & 5 & -1 & \varepsilon \\ \varepsilon & 4 & -2 & \varepsilon \end{pmatrix} \Longrightarrow \lambda(A_2) = \frac{7}{2}$, $A_3 = \begin{pmatrix} 2 & 6 & \varepsilon & -1 \\ 1 & \varepsilon & -1 & \varepsilon \\ \varepsilon & 5 & -2 & \varepsilon \\ \varepsilon & 4 & -2 & \varepsilon \end{pmatrix} \Longrightarrow \lambda(A_3) = \frac{7}{2}$

and $A_4 = \begin{pmatrix} 2 & 6 & \varepsilon & -3 \\ 1 & \varepsilon & -1 & \varepsilon \\ \varepsilon & 5 & -2 & \varepsilon \\ \varepsilon & 5 & -1 & \varepsilon \end{pmatrix} \Longrightarrow \lambda(A_4) = \frac{7}{2}, \dots \dots$

All of the replacement and changes that we have made into the matrices $A_1, A_2, A_3,$ and A_4 does not change the maximum cycle mean of the graph in Figure1.4 which is equal to the eigenvalue of the matrix A.

$A_1 \otimes A_2 \otimes A_3 \otimes A_4 = \begin{pmatrix} 14 & 15 & 12 & 6 \\ 10 & 14 & 8 & 5 \\ 13 & 14 & 11 & 5 \\ 13 & 14 & 11 & 5 \end{pmatrix} \Longrightarrow$ The eigenvalue $\lambda(A_1 \otimes A_2 \otimes A_3 \otimes A_4) = 14$

and the eigenvector is $v(A_1 \otimes A_2 \otimes A_3 \otimes A_4) = \begin{pmatrix} 0 \\ -1 \\ -1 \\ -1 \end{pmatrix}$

$$(A_1 \otimes A_2 \otimes A_3 \otimes A_4) \otimes v = \begin{pmatrix} 14 & 15 & 12 & 6 \\ 10 & 14 & 8 & 5 \\ 13 & 14 & 11 & 5 \\ 13 & 14 & 11 & 5 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ -1 \\ -1 \\ -1 \end{pmatrix} = \lambda^{\otimes 4} \otimes v = \left(\frac{7}{2}\right)^{\otimes 4} \otimes \begin{pmatrix} 0 \\ -1 \\ -1 \\ -1 \end{pmatrix}$$

$$= \begin{pmatrix} 14 \\ 13 \\ 13 \\ 13 \end{pmatrix}$$

$\therefore (A_1 \otimes A_2 \otimes A_3 \otimes A_4) \otimes v = \lambda^{\otimes 4} \otimes v$

but $A^{\otimes 4} = \begin{pmatrix} 14 & 15 & 12 & 6 \\ 10 & 14 & 8 & 5 \\ 13 & 14 & 11 & 5 \\ 12 & 13 & 11 & 4 \end{pmatrix} \Longrightarrow$ The eigenvalue $\lambda(A^{\otimes 4}) = 14$

and the eigenvector is $v(A^{\otimes 4}) = \begin{pmatrix} 0 \\ -1 \\ -1 \\ -2 \end{pmatrix}$

$$\Longrightarrow A^{\otimes 4} \otimes v = \lambda^{\otimes 4} \otimes v = \begin{pmatrix} 14 \\ 13 \\ 13 \\ 12 \end{pmatrix}$$

However, if we replace $a_{12} = 6$ by 5 as in matrix A_5, the eigenvalue $\lambda(A_5) = 3$. If we replace $a_{12} = 6$ by 5 and $a_{21} = 1$ by 0 as in matrix A_6, the eigenvalue $\lambda(A_6) = 2.5$. If we replace $a_{12} = 6$ by 4, $a_{21} = 1$ by 2 and $a_{32} = 5$ by 4 as in matrix A_7, the eigenvalue $\lambda(A_7) = 3$. Finally, if we replace $a_{23} = -1$ by 3 as in matrix A_8, the eigenvalue $\lambda(A_8) = 4$, so in these cases the matrices are

$$A_5 = \begin{pmatrix} 2 & 5 & \varepsilon & -3 \\ 1 & \varepsilon & -1 & \varepsilon \\ \varepsilon & 5 & -2 & \varepsilon \\ \varepsilon & 4 & -1 & \varepsilon \end{pmatrix} \Longrightarrow \lambda(A_5) = 3 \text{ and } v(A_5) = \begin{pmatrix} 0 \\ -2 \\ 0 \\ -1 \end{pmatrix}$$

$$A_6 = \begin{pmatrix} 2 & 5 & \varepsilon & -3 \\ 0 & \varepsilon & -1 & \varepsilon \\ \varepsilon & 5 & -2 & \varepsilon \\ \varepsilon & 4 & -2 & \varepsilon \end{pmatrix} \Rightarrow \lambda(A_6) = 2.5 \text{ and } v(A_6) = \begin{pmatrix} 0 \\ -2.5 \\ 0 \\ -1 \end{pmatrix}$$

$$A_7 = \begin{pmatrix} 2 & 4 & \varepsilon & -3 \\ 2 & \varepsilon & -1 & \varepsilon \\ \varepsilon & 4 & -2 & \varepsilon \\ \varepsilon & 4 & -2 & \varepsilon \end{pmatrix} \Rightarrow \lambda(A_7) = 3 \text{ and } v(A_7) = \begin{pmatrix} 0 \\ -1 \\ 0 \\ 0 \end{pmatrix}$$

$$A_8 = \begin{pmatrix} 2 & 6 & \varepsilon & -3 \\ 1 & \varepsilon & 3 & \varepsilon \\ \varepsilon & 5 & -2 & \varepsilon \\ \varepsilon & 4 & -2 & \varepsilon \end{pmatrix} \Rightarrow \lambda(A_8) = 4 \text{ and } v(A_8) = \begin{pmatrix} 0 \\ -2 \\ -1 \\ -2 \end{pmatrix}$$

$$A_5 \otimes A_6 \otimes A_7 \otimes A_8 = \begin{pmatrix} 11 & 15 & 12 & 6 \\ 10 & 14 & 10 & 5 \\ 10 & 13 & 12 & 4 \\ 9 & 12 & 11 & 3 \end{pmatrix} \Rightarrow \text{The eigenvalue } \lambda(A_5 \otimes A_6 \otimes A_7 \otimes A_8) = 14 \text{ and}$$

$$\text{the eigenvector } v(A_5 \otimes A_6 \otimes A_7 \otimes A_8) = \begin{pmatrix} 0 \\ -1 \\ -2 \\ -3 \end{pmatrix}$$

$$\Rightarrow (A_5 \otimes A_6 \otimes A_7 \otimes A_8) \otimes v = \begin{pmatrix} 11 & 15 & 12 & 6 \\ 10 & 14 & 10 & 5 \\ 10 & 13 & 12 & 4 \\ 9 & 12 & 11 & 3 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ -1 \\ -2 \\ -3 \end{pmatrix} = \begin{pmatrix} 14 \\ 13 \\ 12 \\ 11 \end{pmatrix}$$

$$(\lambda(A_5) \otimes \lambda(A_6) \otimes \lambda(A_7) \otimes \lambda(A_8)) \otimes v = (3 \otimes 2.5 \otimes 3 \otimes 4) \otimes \begin{pmatrix} 0 \\ -1 \\ -2 \\ -3 \end{pmatrix} = 12.5 \otimes \begin{pmatrix} 0 \\ -1 \\ -2 \\ -3 \end{pmatrix} = \begin{pmatrix} 12.5 \\ 11.5 \\ 10.5 \\ 9.5 \end{pmatrix}$$

$\therefore (A_5 \otimes A_6 \otimes A_7 \otimes A_8) \otimes v \neq (\lambda(A_5) \otimes \lambda(A_6) \otimes \lambda(A_7) \otimes \lambda(A_8)) \otimes v$

Nevertheless, for the replacement and changes that we have made into the matrices $A_5, A_6, A_7,$ and A_8 does change the maximum cycle mean of the graph in Figure1.4 that is not equal to the eigenvalue of the matrix A.

Theorem 5.4 [1] and [2]. Let $\{A(k): k \in N\}$ be a stationary sequence of integrable and irreducible matrices in $\mathbb{R}_{max}^{n\times n}$ satisfying *(H₁)*. Then, it holds that $\lambda = \lambda^{\text{top}} = \lambda^{\text{bot}}$ and for any finite integrable initial condition x_0 it holds with probability one for all j.

The constant, defined in Theorem 5.4, is referred to as the max-plus Lyapunov exponent of the sequence of random matrices $\{A(k): k \in N\}$. There is no ambiguity in denoting the Lyapunov exponent of $\{A(k): k \in N\}$ and the eigenvalue of a matrix A by the same symbol, since the Lyapunov exponent of $\{A(k): k \in N\}$ is just the eigenvalue of A whenever $A(k) = A$ for all $k \in N$ [2].

Example 5.7 Consider the system $x(k+1) = A(k) \otimes x(k)$, with $A(k) = A_1$ and probability 0.5 , $A(k) = A_2$with also 0.5 probability, A_1, A_2 are matrices from previous example (example 5.5), so the matrices A_1, A_2 are

$$A_1 = \begin{pmatrix} 2 & 6 & \varepsilon & -3 \\ 1 & \varepsilon & 3 & \varepsilon \\ \varepsilon & 5 & -2 & \varepsilon \\ \varepsilon & 5 & -2 & \varepsilon \end{pmatrix} \Rightarrow \lambda(A_1) = \frac{7}{2}, \text{and } v(A_1) = (0 \quad -2.5 \quad -1 \quad -2)^T$$

$$A_2 = \begin{pmatrix} 2 & 6 & \varepsilon & -3 \\ 1 & \varepsilon & -1 & \varepsilon \\ \varepsilon & 5 & -1 & \varepsilon \\ \varepsilon & 4 & -2 & \varepsilon \end{pmatrix} \Rightarrow \lambda(A_2) = \frac{7}{2}, \text{and } v(A_2) = (0 \quad -2.5 \quad -1 \quad -2)^T$$

The Lyapunov exponent can be found by using Markov Chain theory as $\lambda = \frac{7}{2}$.

Note that: $\frac{1}{2}[\lambda(A_1) + \lambda(A_2)] = \frac{7}{2}$, where $\lambda(A_1) = \frac{7}{2}$ and $\lambda(A_2) = \frac{7}{2}$. See also Lemma 5.3.

Now, consider the system $x(k+1) = A(k) \otimes x(k)$, with $A(k) = A_1$ and probability 0.5, $A(k) = A_5$ with 0.5 probability, A_1, A_5 are matrices from pervious example (Example 5.5), so the matrices A_1, A_5 are

$$A_1 = \begin{pmatrix} 2 & 6 & \varepsilon & -3 \\ 1 & \varepsilon & -1 & \varepsilon \\ \varepsilon & 5 & -2 & \varepsilon \\ \varepsilon & 5 & -2 & \varepsilon \end{pmatrix} \Rightarrow \lambda(A_1) = \frac{7}{2}, \text{and } v(A_1) = (0 \quad -2.5 \quad -1 \quad -2)^T$$

$$A_5 = \begin{pmatrix} 2 & 5 & \varepsilon & -3 \\ 1 & \varepsilon & -1 & \varepsilon \\ \varepsilon & 5 & -2 & \varepsilon \\ \varepsilon & 4 & -2 & \varepsilon \end{pmatrix} \Longrightarrow \lambda(A_5) = 3, \text{and } v(A_5) = (0 \quad -2 \quad 0 \quad -1)^T$$

The Lyapunov exponent can be found by using Markov Chain theory as $\lambda = \frac{7}{2}$.

Note that: $\frac{1}{2}[\lambda(A_1) + \lambda(A_5)] \neq \frac{7}{2}$, where $\lambda(A_1) = \frac{7}{2}$ and $\lambda(A_5) = 3$.

Because of $x(k+1) = A(k) \otimes x(k)$, with $A(k) = A_1$ with probability 0.5 and $A(k) = A_5$ with probability 0.5, the matrices A_1, A_5 are taken to be the stochastic perturbation of the original matrix A,

i.e. $A_{pert} = A_1$ or A_5 .

$A =$ maxmult (A_{pert}, I) for the first iteration where I is the identity matrix.

Thus, $A = A_{(stop\ iter)}\ A_{(stop\ iter-1)} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots A(3)A(2)A(1)$

and $x_k = A \otimes x_0$.

If we have taken, $k = 100$, then

$A = A(100)A(100-1) \cdots\cdots\cdots A(3)A(2)A(1)$

$$\Rightarrow A = \begin{pmatrix} 351 & 351 & 349 & 343 \\ 348 & 348 & 346 & 340 \\ 350 & 350 & 348 & 342 \\ 350 & 350 & 348 & 342 \end{pmatrix}$$

$$x_{100} = A \otimes x_0 = \begin{pmatrix} 351 \\ 348 \\ 350 \\ 350 \end{pmatrix}$$

$\therefore \frac{x_{100}}{100} = \begin{pmatrix} 3.51 \\ 3.48 \\ 3.50 \\ 3.50 \end{pmatrix}$ which is close to the maximal Lyapunov exponent which is in this case equal to $\frac{7}{2}$ =3.5.

Other simulations of length 100 give $\begin{pmatrix} 3.52 \\ 3.50 \\ 3.51 \\ 3.51 \end{pmatrix}$, $\begin{pmatrix} 3.49 \\ 3.46 \\ 3.49 \\ 3.48 \end{pmatrix}$, $\begin{pmatrix} 3.50 \\ 3.49 \\ 3.49 \\ 3.48 \end{pmatrix}$, which is also closely approximate the maximal Lyapunov exponent.

$\Rightarrow \frac{1}{2}[\lambda(A_1) + \lambda(A_5)] \neq \frac{7}{2},$ where $\lambda(A_1) = \frac{7}{2}$ and $\lambda(A_5) = 3$ are the eigenvalues of A_1, A_5 respectively and we used power method to find them which is used for deterministic cases.

Lemma 5.3 If A_1 and A_2 have the same eigenvector then the Lyapunov exponent is $p \cdot \lambda_1 + q \cdot \lambda_2$ where, prob $(A_k = A_1) = p \in (0,1)$ and prob $(A_k = A_2) = q = 1 - p \in (0,1)$.

Proof: A_1 and A_2 have the same eigenvector and eigenvalues $\lambda_1$and λ_2 respectively.

$$A_1 \otimes v = \lambda_1 \otimes v$$
$$A_2 \otimes v = \lambda_2 \otimes v$$

Let prob $(A_k = A_1) = p \in (0,1)$

prob $(A_k = A_2) = q = 1 - p \in (0,1)$ independently for all $k = 1,2,..$

If v is the eigenvector for all products, then

$$(A_1 \otimes A_2) \otimes v = A_1 \otimes (A_2 \otimes v) = A_1 \otimes (\lambda_2 \otimes v) = \lambda_2 \otimes (A_1 \otimes v) = \lambda_2 \otimes (\lambda_1 \otimes v) = (\lambda_2 \otimes \lambda_1) \otimes v$$
$$= (\lambda_1 \otimes \lambda_2) \otimes v$$

By induction;

$$(A(n) \otimes A(n-1) \otimes \cdots \cdots \cdots \otimes A(2) \otimes A(1)) \otimes v$$
$$= [\lambda(A(n)) \otimes \lambda(A(n-1)) \otimes \cdots \cdots \cdots \otimes \lambda(A(2)) \otimes \lambda(A(1))] \otimes v$$

where $\lambda(A(i)) = \lambda_1$ if $A(i) = A_1$ and

$\lambda(A(i)) = \lambda_2$ if $A(i) = A_2$ for $1 \leq i \leq n,\ n = 1,2, \ldots$

Hence, $\lambda(A(n)) \otimes \lambda(A(n-1)) \otimes \cdots \cdots \cdots \otimes \lambda(A(2)) \otimes \lambda(A(1))$ is an eigenvalue of $A(n) \otimes A(n-1) \otimes \cdots \cdots \cdots \otimes A(2) \otimes A(1)$.

However, this number is: $\lambda_1 \cdot n$number of $A_1{}'$s + $\lambda_2 \cdot n$number of $A_2{}'$s.

Dividing by n, we get that

$\frac{1}{n}(A(n) \otimes A(n-1) \otimes \cdots \cdots \cdots \otimes A(2) \otimes A(1)) \otimes v$ is $\lambda_1 \cdot proportion$ of $A_1{}'$s + $\lambda_2 \cdot proportion$ of $A_2{}'$s

By the law of large numbers, the $proportion$ of $A_1's \rightarrow p$ and

$proportion$ of $A_2's \rightarrow q = 1 - p$

∴ The Lyapunov exponent is $\boldsymbol{p \cdot \lambda_1 + q \cdot \lambda_2}$

■

Example 5.8 Let $A_1 = \begin{pmatrix} 2 & 6 & \varepsilon & -3 \\ 1 & \varepsilon & -1 & \varepsilon \\ \varepsilon & 5 & -2 & \varepsilon \\ \varepsilon & 5 & -2 & \varepsilon \end{pmatrix} \Longrightarrow \lambda(A_1) = \frac{7}{2}$, and $v(A_1) = (0 \quad -2.5 \quad -1 \quad -2)^T$

Let $A_2 = \begin{pmatrix} 2.5 & 6.5 & \varepsilon & -2.5 \\ 1.5 & \varepsilon & -0.5 & \varepsilon \\ \varepsilon & 5.5 & -0.5 & \varepsilon \\ \varepsilon & 4.5 & -1.5 & \varepsilon \end{pmatrix} \Longrightarrow \lambda(A_2) = 4$, and $v(A_2) = (0 \quad -2.5 \quad -1 \quad -2)^T$

The Lyapunov exponent can be found by using the Lemma 5.3.

Note that: $\frac{1}{2}[\lambda(A_1) + \lambda(A_2)] = 3.75$, where $\lambda(A_1) = \frac{7}{2}$ and $\lambda(A_2) = 4$.

Because of $x(k + 1) = A(k) \otimes x(k)$, with $A(k) = A_1$ with probability 0.5 and $A(k) = A_2$ with probability 0.5, the matrices A_1, A_2 are taken to be the stochastic perturbation of the original matrix$A$.

i.e. $A_{pert} = A_1$ or A_2 .

$A =$ maxmult (A_{pert}, I) for the first iteration where I is the identity matrix.

Thus, $A = A_{(stop\ iter)}\, A_{(stop\ iter-1)}$...$A(3)A(2)A(1)$

and $x_k = A \otimes x_0$.

If we have taken, $k = 200$, then

$A = A(200)A(199) \cdots\cdots\cdots A(3)A(2)A(1)$

$$\Rightarrow A = \begin{pmatrix} 747.5 & 751.5 & 745.5 & 742.5 \\ 746.5 & 747.5 & 744.5 & 738.5 \\ 746.5 & 750.5 & 744.5 & 741.5 \\ 746.5 & 750.5 & 744.5 & 741.5 \end{pmatrix}$$

$$x_{200} = A \otimes x_0 = \begin{pmatrix} 751.5 \\ 747.5 \\ 750.5 \\ 750.5 \end{pmatrix}$$

$\therefore \frac{x_{200}}{200} = \begin{pmatrix} 3.7575 \\ 3.7375 \\ 3.7525 \\ 3.7525 \end{pmatrix}$ which is close to the maximal Lyapunov exponent which is in this case equal to 3.75

$\Rightarrow \frac{1}{2}[\lambda(A_1) + \lambda(A_5)] = 3.75,$ where $\lambda(A_1) = \frac{7}{2}$ and $\lambda(A_2) = 4$ are the eigenvalues of A_1, A_2 respectively and we used power method to find them which is used for deterministic cases.

5.3.2 Beyond Irreducible Matrices

We will now drop the obligation that $A(k)$ has a minimal support matrix that is irreducible.To deal with this case, we assume that the position of finite elements of $A(k)$ is fixed and independent of k, and we decompose $A(k)$ into its irreducible parts. The limit theorem, to be submitted briefly, then states that the Lyapunov exponent of the overall matrix equals the maximum of the Lyapunov exponent of its irreducible components [2].

Let $\{A(k): k \in N\}$ be a sequence of matrices in $\mathbb{R}_{max}^{n \times n}$ such that the arc set of the communication graph of $A(k)$ is independent of k and non-random.

For $i \in n$, $[i]$ denotes the set of vertices of the maximal strongly connected subgraph that contains node i, and denote by $\lambda_{[i]}$ the Lyapunov exponent associated to the matrix obtained by restricting $A(k)$ to the vertices in $[i]$. We state the theorem without proof.

Theorem 5.5 [2] Let $\{A(k): k \in N\}$ be an independent identically distributed sequence of regular and integrable matrices in $\mathbb{R}_{max}^{n \times n}$ such that the communication graph of $A(k)$ has cyclicity one and is independent of k and non-random [2]. For any finite integrable initial value x_0, it holds with probability one that

$$\lim_{k \to \infty} \frac{1}{k} \left\| \frac{x_j(k; x_0)}{k} \right\|_{\min} = \lim_{k \to \infty} \frac{1}{k} [x_j(k; x_0)] = \lambda_j$$

with

$$\lambda_j = \bigoplus_{i \in \pi^*(j)} \lambda_{[j]} \quad , \quad j \in \underline{n}.$$

5.4 State Reduction for Matrices with Dominant Maximal Cycle

In this Section we describe a method to simplify the calculations in some special cases. The amount of computational work can be reduced significantly, even in the stochastic case, if the maximal cycle of the involved matrices dominates the weights of the edges between all other nodes in the graph. State reduction for matrices in max-plus algebra with dominant maximal cycle mean for special case (stochastic case) can be described into the following definition and two propositions.

Definition 5.3 Let A and B be two matrices. Suppose A and B have maximum cycles involving $\{i_1,\ i_2,\cdots,i_k\}$ only. Further assume that all weights involving other nodes are less than the entries in the maximal cycles of both A and B. We say that A and B have a dominant maximum cycle.

In the sequel we will talk about cycles in the matrices A and B instead of the more correct expressions cycles in the precedence graphs of the matrices A and B.

Proposition 5.1 Let A and B have dominant maximum cycle, so that all elements involving other nodes are less than the entries in the maximal cycles of both A and B. Consider the stochastic case where the probabilities of A and B are p and *1-p*, respectively. Then the Lyapunov exponent of the system can be obtained by restricting the attention to $\{i_1,\ i_2,\cdots,i_k\}$ only.

Proof: Suppose the maximal cycle of the matrices A and B is $i_1,\ i_2,\cdots,i_k$. Suppose that we substitute this for another cycle with edges outside of the set $\{i_1,\ i_2,\cdots,i_k\}$. Then necessarily its weight is decreased because all edges going outside the given set have lower weights than any of the weights in the maximal cycle.

■

Proposition 5.2 Let A and B be two matrices and c a scalar. If A and B are replaced by $A+c$ and $B+c$ respectively then the Lyapunov exponent increases by c.

Proof: All edges of the matrices A and B are increased by c. Any entry in the product $A \otimes B$ is increased by *2c*: $((A+c)\otimes(B+c))_{ik} = \max_j\,((A+c)_{ij} + (B+c)_{jk}) = \max_j\,(A_{ij}+c+B_{jk}+c) = (A \otimes B)_{ik} + 2c$.

Similarly, the entries in any product of *N* factors of A's and B's are increased by Nc. The mean of all cycles is thus increased by c and so the Lyapunov exponent, too, is increased by c.

■

Example 5.9 Let A and B are two matrices with prob $(A) = \frac{1}{2}$ and prob $(B) = \frac{1}{2}$, where

$$A = \begin{pmatrix} 2 & 6 & 1 & 2 \\ 1 & \varepsilon & 2 & 0 \\ 0 & 1 & -1 & -1 \\ 0 & 2 & 2 & 0 \end{pmatrix} \text{ has eigenvalue } \lambda(A) = 3.5$$

$$A - 3 = \begin{pmatrix} -1 & 3 & -2 & -1 \\ -2 & \varepsilon & -1 & -3 \\ -3 & -2 & -4 & -4 \\ -3 & -1 & -1 & -3 \end{pmatrix}$$

$$\lambda(A-3) = 0.5 = \lambda(A) - 3$$

Example 5.10 Let $A = \begin{pmatrix} 2 & 6 & 0 & 0 \\ 1 & \varepsilon & 0 & 0 \\ 0 & 0 & -1 & -1 \\ 0 & -1 & 0 & 0 \end{pmatrix}$ and the eigenvalue $\lambda(A) = 3.5$

$A_{restr.} = \begin{pmatrix} 2 & 6 \\ 1 & \varepsilon \end{pmatrix}$ involves nodes 1 and 2 only. The eigenvalue of $A_{restr.} = \lambda(A) = 3.5$

$$A - 3 = \begin{pmatrix} -1 & 3 & -3 & -3 \\ -2 & \varepsilon & -3 & -3 \\ -3 & -3 & -4 & -4 \\ -3 & -4 & -3 & -3 \end{pmatrix} \text{ has eigenvalue } 0.5 = \lambda(A) - 3$$

$$A_{restr.} - 3 = \begin{pmatrix} -1 & 3 \\ -2 & \varepsilon \end{pmatrix}$$

Let $B = \begin{pmatrix} 2 & 5 & 0 & 0 \\ 1 & \varepsilon & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ has eigenvalue $\lambda(B) = 3$

$$B - 3 = \begin{pmatrix} -1 & 2 & -3 & -3 \\ -2 & \varepsilon & -3 & -3 \\ -3 & -3 & -3 & -3 \\ -3 & -3 & -3 & -3 \end{pmatrix} \text{ has eigenvalue } 0 = \lambda(B) - 3$$

$$B_{restr.} - 3 = \begin{pmatrix} -1 & 2 \\ -2 & \varepsilon \end{pmatrix}$$

To show that by using the graph theory:

The graph of the matrix A (see Figure 5.9) shows that the maximum cycle mean is $\frac{A_{12}+A_{21}}{2} = \frac{6+1}{2} = 3.5$

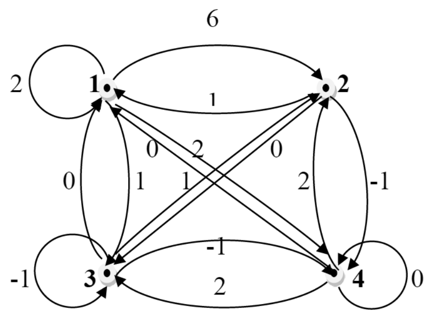

Figure 5.11: The precedence graph of the matrix A

And the graph of the $A_{restr.} = \begin{pmatrix} 2 & 6 \\ 1 & \varepsilon \end{pmatrix}$ (see Figure 5.10) which is also includes the maximum cycle mean:

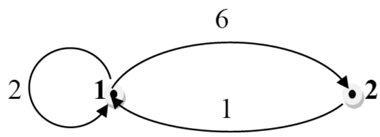

Figure 5.12: The precedence graph of the matrix $A_{restr.}$

In the next example we going to show how dimension reduction can be done by uniformly dominant circuit.

Example 5.11 Let A and B are two matrices with prob $(A) = \frac{1}{2}$ and prob $(B) = \frac{1}{2}$, where

$$A = \begin{pmatrix} \varepsilon & 0 & \varepsilon & \varepsilon \\ 2 & \varepsilon & \varepsilon & -1 \\ 0 & 0 & \varepsilon & \varepsilon \\ -1 & 0 & 0 & \varepsilon \end{pmatrix} \text{ has eigenvalue } \lambda(A) = 1$$

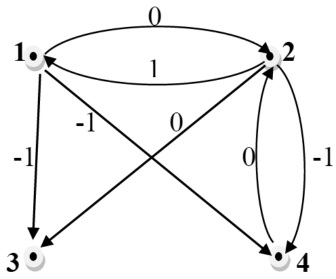

Figure 5.13: The precedence graph of the matrix A

If we reduce the dimensions by reduction uniformly dominant circuit, then we get

$A_1 = \begin{pmatrix} \varepsilon & 0 \\ 2 & \varepsilon \end{pmatrix}$ which it has the same eigenvalue

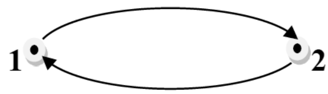

Figure 5.14: The precedence graph of the matrix A_1

And for the matrix $B = \begin{pmatrix} -1 & 2 & \varepsilon & -1 \\ 3 & -2 & \varepsilon & \varepsilon \\ \varepsilon & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ which has the eigenvalue $\lambda(B) = 2.5$

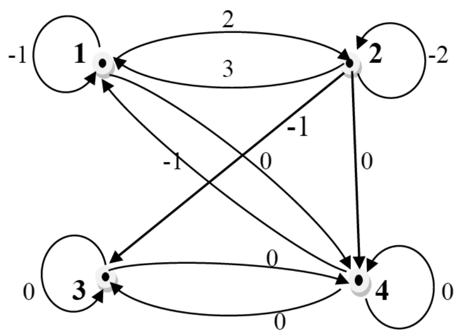

Figure 5.15: The precedence graph of the matrix B

If we reduce the dimensions by reduction uniformly dominant circuit, then we get

$B_1 = \begin{pmatrix} -1 & 2 \\ 3 & -2 \end{pmatrix}$

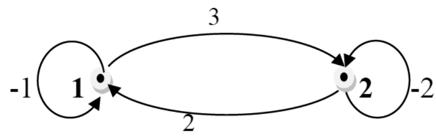

Figure 5.16: The precedence graph of the matrix B_1

Next, we will show how to use the dimension reduction to compute the Lyapunov exponent more easily. We learn from ([2] Chapter 11 pp. 175) that the stochastic matrix products give rise to a Markov chain on the set of *directions* of vectors in $\mathrm{R}^d_{\max}$. The Lyapunov exponent can be calculated by using the stationary probability distribution π of the Markov chain on directions: the exponent is the average growth (with respect to π) under multiplication by A and B (independently of each other, with the given probabilities p and $1\text{-}p$).

The direction of a vector u is simply u (max plus $-$) divided by its length.
Let $u = (u_1\, u_2\, \ldots.u_k)^T$. Its length is taken to be the (max plus-sum or) maximum entry $\max_i u_i$, $1 \le i \le k$, and its direction is then the vector $u - \max_i u_i$ which obviously has length 0. In our examples below, the sign $\sim$ denotes equal direction of two vectors.
The case of norming by "$sum = \max$" is show in the following example.

Example 5.12 Let C and D are two matrices with prob $(C) = \frac{1}{2}$ and prob $(D) = \frac{1}{2}$, where
$C = \begin{pmatrix} 1 & 1 \\ 3 & 0 \end{pmatrix}$, $D = \begin{pmatrix} 1 & 3 \\ 0 & 2 \end{pmatrix}$ and $\lambda(C) = \lambda(D) = 2$. It turns out that the process generates a Markov chain on three directions:

$$C\otimes\begin{pmatrix}0\\0\end{pmatrix} = \begin{pmatrix}1\\3\end{pmatrix} \sim \begin{pmatrix}-2\\0\end{pmatrix}$$

$$C\otimes\begin{pmatrix}-2\\0\end{pmatrix} = \begin{pmatrix}1\\1\end{pmatrix} \sim \begin{pmatrix}0\\0\end{pmatrix}$$

$$D\otimes\begin{pmatrix}0\\0\end{pmatrix} = \begin{pmatrix}3\\2\end{pmatrix} \sim \begin{pmatrix}0\\-1\end{pmatrix}$$

$$D\otimes\begin{pmatrix}0\\-1\end{pmatrix} = \begin{pmatrix}2\\1\end{pmatrix} \sim \begin{pmatrix}0\\-1\end{pmatrix}$$

$$D\otimes\begin{pmatrix}-2\\0\end{pmatrix} = \begin{pmatrix}3\\2\end{pmatrix} \sim \begin{pmatrix}0\\-1\end{pmatrix}$$

$$C\otimes\begin{pmatrix}0\\-1\end{pmatrix} = \begin{pmatrix}1\\3\end{pmatrix} \sim \begin{pmatrix}-2\\0\end{pmatrix}$$

we know that; $P(C) = P(D) = \frac{1}{2}$

Thus:

	$\binom{0}{0}$	$\binom{-2}{0}$	$\binom{0}{-1}$
$\binom{0}{0}$	0	$\frac{1}{2}$	$\frac{1}{2}$
$\binom{-2}{0}$	$\frac{1}{2}$	0	$\frac{1}{2}$
$\binom{0}{-1}$	0	$\frac{1}{2}$	$\frac{1}{2}$

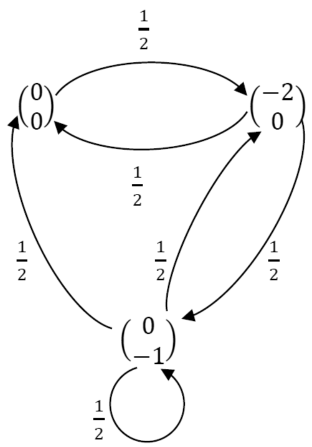

Figure 5.17: Markov chain with transition probabilities

Thus, the stationary distribution for the Markov chain is $\pi = \begin{pmatrix} \frac{1}{6} & \frac{1}{3} & \frac{1}{2} \end{pmatrix}$

The average growth of maximal Lyapunov exponent is in this case equal to $\frac{6+8+15}{12} = \frac{29}{12} = 2.41667$ where $\frac{1}{2}[\lambda(C) + \lambda(D)] = \frac{4}{2} = 2.$

In the case of Example 5.11. the average growth of maximal Lyapunov exponent is equal to $\frac{180}{102} = 1.7647$, while $\frac{1}{2}[\lambda(A) + \lambda(B)] = \frac{3.5}{2} = 1.75,$ where $\lambda(A) = 2.5$ and $\lambda(B) = 1$ are the eigenvalues of A, B respectively.

Here the Markov chain on directions is $\binom{0}{0}$, $\binom{-2}{0}$, $\binom{0}{1}$ and $\binom{-1}{0}$ and the stationary distribution $\pi = \begin{pmatrix} \frac{1}{8} & \frac{1}{4} & \frac{1}{4} & \frac{3}{8} \end{pmatrix}.$

Note. If the dominant cycle consists of one element only, then the Lyapunov exponent is simply $\frac{1}{2}[\lambda(A) + \lambda(B)].$

Example 5.13 In 4.4.2, M2. The production of D and E using only one of the units for D, during the filling stage and this has been formulated by using a max-plus model and after introduction of numerical values from Table 4.1, the A-matrix of the system becomes

$A = \begin{pmatrix} 10 & 4 \\ 14 & 8 \end{pmatrix}$, the eigenvalue $\lambda(A) = 10$

Now take

$B = \begin{pmatrix} 10 & 12 \\ 14 & 16 \end{pmatrix}$ which it has the eigenvalue 16.

It turns out that the process generates a Markov chain on one direction:

$A \otimes \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 10 \\ 14 \end{pmatrix} \sim \begin{pmatrix} -4 \\ 0 \end{pmatrix}$

$A \otimes \begin{pmatrix} -4 \\ 0 \end{pmatrix} = \begin{pmatrix} 6 \\ 10 \end{pmatrix} \sim \begin{pmatrix} -4 \\ 0 \end{pmatrix}$

$B \otimes \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 12 \\ 16 \end{pmatrix} \sim \begin{pmatrix} -4 \\ 0 \end{pmatrix}$

$B \otimes \begin{pmatrix} -4 \\ 0 \end{pmatrix} = \begin{pmatrix} 12 \\ 16 \end{pmatrix} \sim \begin{pmatrix} -4 \\ 0 \end{pmatrix}$

Suppose that; $P(A) = P(B) = \frac{1}{2}$. The Markov chain is trivial, on only one direction.

This is a case where both matrices have the same eigenvector. In other words, we can immediately use Lemma 5.3. Thus, the Lyapunov exponent is 13.

We can make the matrix A stochastic by changing, for instance, 4 to 6 and 14 to 16, in this case, the A-matrix becomes;

$A = \begin{pmatrix} 10 & 6 \\ 16 & 8 \end{pmatrix}$ which it has the eigenvalue 11.

It turns out that the process generates a Markov chain on one direction:

$A \otimes \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 10 \\ 16 \end{pmatrix} \sim \begin{pmatrix} -6 \\ 0 \end{pmatrix}$

$A \otimes \begin{pmatrix} -6 \\ 0 \end{pmatrix} = \begin{pmatrix} 6 \\ 10 \end{pmatrix} \sim \begin{pmatrix} -4 \\ 0 \end{pmatrix}$

$A \otimes \begin{pmatrix} -4 \\ 0 \end{pmatrix} = \begin{pmatrix} 6 \\ 12 \end{pmatrix} \sim \begin{pmatrix} -6 \\ 0 \end{pmatrix}$

$B \otimes \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 12 \\ 16 \end{pmatrix} \sim \begin{pmatrix} -4 \\ 0 \end{pmatrix}$

$B \otimes \begin{pmatrix} -4 \\ 0 \end{pmatrix} = \begin{pmatrix} 12 \\ 16 \end{pmatrix} \sim \begin{pmatrix} -4 \\ 0 \end{pmatrix}$

$B \otimes \begin{pmatrix} -6 \\ 0 \end{pmatrix} = \begin{pmatrix} 12 \\ 16 \end{pmatrix} \sim \begin{pmatrix} -4 \\ 0 \end{pmatrix}$

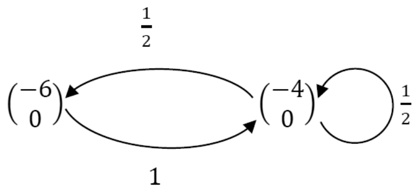

Figure 5.18: Markov chain with transition probabilities

Here the Markov chain on directions is $\begin{pmatrix} -6 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} -4 \\ 0 \end{pmatrix}$ and the stationary distribution

$\pi = \begin{pmatrix} \frac{1}{3} & \frac{2}{3} \end{pmatrix}$.

The average growth of maximal Lyapunov exponent is in this case equal to 13.67 where $\frac{1}{2}[\lambda(A) + \lambda(B)] = \frac{11+16}{2} = 13.5$

Chapter 6

General Conclusion

The aim of this thesis was to provide an introductory text on max-plus algebra and to present results on advanced topics and, in particular, how it is useful in applications. An overview of the basic notions of the max-plus algebra and max-plus linear discrete event systems (DES) was presented. The basic operations, definitions, theorems, and properties of the max-plus algebra were introduced. The main feature of max-plus algebra is that addition is replaced by max, and multiplication is replaced by addition. This is useful especially for modeling, simulating and optimizing nonlinear scheduling problems using linear algebra methods.

Chapters 1-2 aimed to be a guide through basic max-plus algebra, where Chapter 1 presented simple introduction of max-plus algebra properties, matrices and graphs. Chapter 2 implemented the solution of linear systems such as $A \otimes x = b$, linear independence and dependence, and presented the max-plus eigenproblem in detail. Efficient methods for finding all eigenvalues and describing all eigenvectors for any square matrix over $\mathbb{R}_{\max} = \mathbb{R} \cup \{-\infty\}$, with all the necessary proofs, were presented. The modeling of production systems using max-plus DES was discussed, and two examples were also presented. Analogue to characteristic equation and the Cayley–Hamilton theorem in max-plus algebra were introduced.

Chapter 3 described how a max-plus model for a train system can be constructed. Meeting conditions caused by having only a single track, and other physical constrains, have been modeled. A state update equation of the form $x(k) = A \otimes x(k-1)$ is desirable, which was made possible by using cross-substitutions and extending the state space with delayed states. Static and dynamic delay sensitivity of the network has been analyzed by modifying the A-matrix and using eigenvalue calculations. The obtained results were compared to recovery matrix-based calculations found in the literature. A recovery matrix for the chosen extended state space becomes large and contains even irrelevant information. Guidelines for finding and interpreting the relevant information from the recovery matrix have been discussed. Max-plus formalism was used throughout this chapter.

The main contribution of this thesis is found in Chapter 4, where it was described how a max-plus model for a manufacturing system can be constructed, and an optimal schedule was found without optimization. The scheduling of production systems consisting of many stages and different units was considered, where some of the units were used for multiple production stages. If a production unit is used for different stages, cleaning is needed in between, while no cleaning is needed between stages of the same type. The obtained state update equation was in this case also rewritten in the form $x(k) = A \otimes x(k-1)$ using several cross-substitutions, and extension of the state space with delayed states. Structural decisions, such as using a unit for different tasks, were found difficult to formulate in max-plus algebra. Three possible operation modes with the structure fixed was identified and modeled separately using max plus. The central driving factor for structural switches was durability constraints, which were present in the production. Thus, only a part of the schedule was obtained by solving eigenvalue problems of the max-plus model, the structural decisions were made on the basis of a few alternative schedules obtained using max-plus. This was based on the finding that structural switches should be postponed as late as possible, so the criterion used was to do the switch one step before the step that was the first that violated at least one durability constraint. Using this strategy, an optimal schedule was obtained without any optimization.

Chapter 5 provides a thorough review and explanation of the theory of stochastic max-plus linear systems, which has seen fast developments in the last two decades. Ergodic theory for stochastic max-plus linear systems has been presented where the common approaches were discussed. Examples of stochastic max-plus linear systems production system were presented, and the Lyapunov exponent was found by using simulation. Connections to Petri nets was also discussed in this chapter.

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