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Hilbert matrix, Volterra and weighted composition operators on Banach spaces of analytic functions

Åbo Akademi University

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Preface

This thesis was carried out at the Department of Mathematics at Åbo Akademi University. I would like to express my gratitude to my supervisor, professor Mikael Lindström, and to my assistant supervisor, doctor Santeri Miihkinen, for their unwavering support. My gratitude lies also with my co-authors, Ted Eklund, Maryam M. Pirasteh and Amir H. Sanatpour. I also wish to thank doctor Karl-Mikael Perfekt for acting as a pre-examiner for my thesis and docent Hans-Olav Tylli for agreeing to act as my opponent as well as pre-examiner for my thesis. Lastly I thank my family and friends for their valuable support.

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Niklas Wikman

Svensk sammanfattning

Avsikten med denna avhandling är att studera två klassiska linjära integraloperatorer, Hilbertmatrisoperatorn $\mathcal{H}$ och den generaliserade Volterraoperatorn T_g^{φ} mellan Banachrum av analytiska funktioner på den öppna enhetsdisken i det komplexa talplanet.

Den exakta normen av Hilbertmatrisoperatorn undersöks i viktade Bergmanrum A^p_α för olika värden på parametrarna α och p. Božin och Karapetrović fann den exakta normen av Hilbertmatris operatorn på oviktade Bergmanrum A^p då $2 < p < 4$. I denna avhandling förenklas deras bevis och med hjälp av den nya bevismetoden generaliseras detta resultat partiellt till viktade Bergmanrum. Normen av Hilbertmatrisoperatorn undersöks också på Korenblumrum $H^\infty_{v_\beta}$. Ett resultat gällande viktade kompositionsoperatorer används för att få ett partiellt resultat gällande normen av Hilbertmatrisoperatorn på $H^\infty_{v_\beta}$.

För den generaliserade Volterra operatorn undersöks operatorteoretiska egenskaper, såsom begränsning, kompakthet och svag kompakthet såväl på rummet av begränsade analytiska funktioner som på rummet av begränsade analytiska funktioner med vikt samt på Bloch-liknande rum. Avsikten är att relatera dessa egenskaper till egenskaper hos de inducerande symbolerna g och φ.

Contents

Chapter 1

Introduction

The set of all analytic functions $f : \mathbb{D} \to \mathbb{C}$ on the open unit disc

$$\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$$

is denoted by $H(\mathbb{D})$. One of the reasons for confining ourselves to analytic functions on the open unit disc is that, by the Riemann mapping theorem, any open non-empty simply connected set $\Omega \subset \mathbb{C}$ can be bijectively mapped onto $\mathbb{D}$ by some analytic function, the inverse of which is also analytic.

The purpose of this thesis was to study properties of two classical linear operators on Banach spaces of analytic functions on the open unit disc, namely, the Hilbert matrix operator $\mathcal{H}$ and the generalized Volterra operator T_g^{φ}. The properties studied were norm, essential norm, boundedness, weak compactness and compactness. These notions will be discussed in detail in the following chapters.

The thesis is outlined as follows. In chapter 2 I will recall the classical Banach spaces of analytic functions on the unit disc considered in this thesis and also the properties of the linear operators mentioned above. Chapter 3 will lay the framework for the Hilbert matrix operator as well as serve as a thorough introduction of the Hilbert matrix operator on weighted Bergman spaces and Korenblum spaces. In the last chapter I will discuss the generalized Volterra operator.

1.1 List of publications

This thesis is based on the following publications. The mathematical ideas in papers I-III were developed jointly with the co-authors.

Paper I [23] T. Eklund, M. Lindström, M.M. Pirasteh, A.H. Sanatpour and N. Wikman, *Generalized Volterra operators mapping between Banach spaces of analytic functions*, Monatshefte für Mathematik, pp. 1-19, 2018.
https://doi.org/10.1007/s00605-018-1216-5

Paper II [30] M. Lindström, S. Miihkinen and N. Wikman, *Norm estimates of weighted composition operators pertaining to the Hilbert Matrix*, Proceedings of the American Mathematical Society, Volume 147, Number 6, June 2019, Pages 2425–2435.
https://doi.org/10.1090/proc/14437

Paper III [31] M. Lindström, S. Miihkinen and N. Wikman, *On the exact value of the norm of the Hilbert matrix operator on weighted Bergman spaces*, To appear in Annales Academiæ Scientiarum Fennicæ Mathematica.

Chapter 2

Banach spaces of analytic functions

2.1 Classical Banach spaces

In this section properties of the Banach spaces of analytic functions considered in this thesis is discussed. We begin with Hardy spaces H^p. Let $1 \leq p \leq \infty$. The functions $f \in H(\mathbb{D})$ satisfying

$$\|f\|_{H^p} = \sup_{0<r<1} \left(\frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p d\theta \right)^{1/p} < \infty$$

and

$$\|f\|_\infty = \sup_{z \in \mathbb{D}} |f(z)|$$

when $p = \infty$, are called Hardy functions. The space H^∞ is the space of bounded analytic functions on the unit disc. For $f \in H^p$ there exists a sharp pointwise estimate

$$|f(z)| \leq \frac{\|f\|_{H^p}}{(1-|z|^2)^{1/p}}, \quad z \in \mathbb{D}. \tag{2.1.1}$$

Let $-1 < \alpha < \infty$ and $1 \leq p < \infty$, the weighted Bergman space A^p_α is the set of functions in $H(\mathbb{D})$ satisfying

$$\|f\|_{A^p_\alpha} = \left(\int_{\mathbb{D}} |f(z)|^p dA_\alpha(z) \right)^{1/p} < \infty,$$

where

$$dA_\alpha(z) = (\alpha+1)(1-|z|^2)^\alpha dA(z)$$

and

$$dA(z) = \frac{1}{\pi} dx dy = \frac{1}{\pi} r dr d\theta, \quad z = x + iy = re^{i\theta}$$

is the normalised area measure so that $A(\mathbb{D}) = 1$ on $\mathbb{D}$. For $f \in A^p_\alpha$ there exists a sharp pointwise estimate

$$|f(z)| \leq \frac{\|f\|_{A^p_\alpha}}{(1-|z|^2)^{(2+\alpha)/p}}, \quad z \in \mathbb{D}. \tag{2.1.2}$$

We write $A^p_\alpha = A^p$ if $\alpha = 0$. The polynomials are dense in the weighted Bergman space, meaning that for any $f \in A^p_\alpha$ there exists a sequence of polynomials $\{f_n\}_{n=1}^\infty$ with the property $\|f - f_n\|_{A^p_\alpha} \to 0$ when $n \to \infty$, see [21]. For more information about Hardy spaces the reader is referred to the books by Duren [20], Garnett [24] and Koosis [28]. For information about the weighted Bergman spaces we refer the reader to the books by Hedenmalm, Korenblum and Zhu [26], Duren and Schuster [21] and Zhu [43].

2.2 Weight functions

Many of the classical Banach spaces of analytic functions on the unit disc can be generalized by adding a weight function satisfying some properties. More precisely, a weight is a continuous, strictly positive function that satisfies $\lim_{|z|\to 1} v(z) = 0$. To be able to prove more interesting results one sometimes needs to put additional regularity conditions on the weight v. The most important regularity condition is that of normality. An almost decreasing weight v, see below, is called normal if it is radial, meaning that it satisfies $v(z) = v(|z|)$ for all $z \in \mathbb{D}$, and if it has the additional properties

$$\inf_{n\in\mathbb{N}} \frac{v(1-2^{-n-1})}{v(1-2^{-n})} > 0 \text{ and } \inf_{k\in\mathbb{N}} \limsup_{n\to\infty} \frac{v(1-2^{-n-k})}{v(1-2^{-n})} < 1. \tag{2.2.1}$$

The concept of almost increasing and almost decreasing functions was introduced by Bernstein [8], $f : [a,b] \to \mathbb{R}$ is called almost increasing if there exists a constant $C > 0$ such that if $x < y$ then $f(x) < Cf(y)$. An almost decreasing function $f : [a,b] \to \mathbb{R}$ is defined similarly. Shields and Williams [39] first introduced the concept of normal weights, in their definition a weight v is said to be normal if there exists $k > \epsilon > 0$ and $r_0 < 1$ such that

$$\frac{v(r)}{(1-r)^\epsilon} \searrow 0 \text{ and } \frac{v(r)}{(1-r)^k} \nearrow \infty \quad (r_0 \leq r, r \to 1^-). \tag{2.2.2}$$

In [18] Domański and Lindström showed that conditions (2.2.1) and (2.2.2) are equivalent. The most important weights considered in this thesis are the standard weights defined by

$$v_\beta(z) = (1-|z|^2)^\beta, \quad \beta > 0.$$

It is easy to verify that the standard weights are normal, on the other hand the weight defined by

$$v_{\log,\beta}(z) = (1-\log(1-|z|^2))^{-\beta}, \quad \beta > 0$$

fails to satisfy the second condition in (2.2.2) and hence is not a normal weight.

2.3 Bloch-type and Korenblum spaces

Let $v : \mathbb{D} \to \mathbb{R}_+$ be a weight. The Bloch-type spaces considered in this thesis are defined by

$$\mathcal{B}^\infty_v = \{f \in H(\mathbb{D}) : \|f\|_{\mathcal{B}^\infty_v} = |f(0)| + \sup_{z\in\mathbb{D}} v(z)|f'(z)| < \infty\};$$

$\mathcal{B}_v^0 = \{f \in \mathcal{B}_v^\infty : \lim_{|z|\to 1^-} v(z)|f'(z)| < \infty\}$.

The Korenblum spaces, or the weighted Banach spaces of analytic functions H_v^∞ and H_v^0 are defined by

$H_v^\infty = \{f \in H(\mathbb{D}) : \|f\|_{H_v^\infty} = \sup_{z\in\mathbb{D}} v(z)|f(z)| < \infty\}$;

$H_v^0 = \{f \in H_v^\infty : \lim_{|z|\to 1^-} v(z)|f(z)| = 0\}$.

By a result of Lusky [33] it holds that $H_v^\infty \approx l^\infty$ and $H_v^0 \approx c_0$ when the weights are normal. Lusky [32] also proved that one can identify $H_v^\infty = B_w^\infty$ and $H_v^0 = B_w^0$ by using the weight $w(z) = (1-|z|)v(z)$. In this thesis we will predominantly deal with Korenblum spaces with standard weights. In some cases we have considered Korenblum spaces with normal weights, these weights satisfy the property $\lim_{|z|\to 1^-} v(z) = 0$. This is because if $\lim_{|z|\to 1^-} v(z) > 0$ then $H_v^\infty = H^\infty$. For the Korenblum spaces with standard weights it follows immediately that

$$|f(z)| \leq \frac{\|f\|_{H_{v_\beta}^\infty}}{(1-|z|^2)^\beta}, \quad z \in \mathbb{D}. \tag{2.3.1}$$

2.4 General Banach spaces satisfying some axioms

A property common for all the spaces H^p, A_α^p and $H_{v_\beta}^\infty$ is that their norm topologies are finer than the compact-open topology, this follows from the pointwise estimates that hold in these spaces. The compact-open topology defined on $H(\mathbb{D})$ is the topology generated by subsets of the form

$$B(K,U) = \{f \in H(\mathbb{D}) : f(K) \subseteq U\},$$

where $K \subset \mathbb{D}$ is compact and $U \subset \mathbb{C}$ is open. On the unit disc we have that uniform convergence on compact subsets is equivalent to convergence in the compact-open topology. We will denote $H(\mathbb{D})$ endowed with the compact-open topology by $(H(\mathbb{D}), co)$. There are also other properties that these spaces share, which will be the topic for this section.

When studying bounded linear operators on Banach spaces of analytic functions one only needs to use properties that the spaces possess. We can therefore assemble a list of conditions that we require our general space $\mathcal{X}$ to have and then give examples of spaces that satisfy the given conditions. Let $\mathcal{X}$ be a Banach space of analytic functions containing the constant functions, and let $\|\cdot\|_{\mathcal{X}}$ denote its norm. For any $z \in \mathbb{D}$, the point evaluation functional $\delta_z : \mathcal{X} \to \mathbb{C}$ is defined by $\delta_z(f) = f(z)$ for $f \in \mathcal{X}$. The following conditions will be considered on the space $\mathcal{X}$ (see [13] or [22]).

(I) The closed unit ball $B_{\mathcal{X}}$ of $\mathcal{X}$ is compact with respect to the compact-open topology.

(II) The point evaluation functionals $\delta_z : \mathcal{X} \to \mathbb{C}$ satisfy $\lim_{|z|\to 1} \|\delta_z\|_{\mathcal{X}\to\mathbb{C}} = \infty$.

(III) The linear operator $T_r : \mathcal{X} \to \mathcal{X}$ mapping $f \mapsto f_r$, where $f_r(z) := f(rz)$ is compact for every $0 < r < 1$.

(IV) The operators T_r in (III) satisfy $\sup_{0<r<1} \|T_r\|_{\mathcal{X}\to\mathcal{X}} < \infty$.

(V) The pointwise multiplication operator $M_u : \mathcal{X} \to \mathcal{X}$ satisfies $\|M_u\|_{\mathcal{X}\to\mathcal{X}} \lesssim \|u\|_\infty$ for every $u \in H^\infty$.

The notation $A \lesssim B$ indicates that there is a positive constant c, not depending on properties of A and B, such that $A \leq cB$. If both $A \lesssim B$ and $B \lesssim A$ we will write $A \asymp B$.

Condition (I) is true for all spaces considered in this thesis, except for B_v^0 and H_v^0. Furthermore, the evaluation map $\mathcal{X} \to \mathcal{X}^{**}$, $f \mapsto \tilde{f}$, where $\tilde{f}(l) = l(f)$ for $f \in \mathcal{X}$ and $l \in \mathcal{X}^*$ is a natural embedding of $\mathcal{X}$ into its second dual $\mathcal{X}^{**}$ so we can think of $\mathcal{X}$ as a subset of $\mathcal{X}^{**}$. The evaluation map is always injective, but not necessarily surjective when $\mathcal{X}$ is a Banach space. Instead we consider a subset of the dual space $\mathcal{X}^*$

$$^*\mathcal{X} = \{l \in \mathcal{X}^* : l\big|_{B_\mathcal{X}} \text{ is } co\text{-continuous}\}.$$

It turns out that $^*\mathcal{X}$ is itself a Banach space if one assumes condition (I), this follows from the Diximier-Ng Theorem [35]

Theorem 2.4.1. *Let $(\mathcal{X}, \|\cdot\|_\mathcal{X})$ be a normed space with closed unit ball $B_\mathcal{X}$. Suppose there exists a (Hausdorff) locally convex topology τ for $\mathcal{X}$ such that $B_\mathcal{X}$ is τ-compact. Then $\mathcal{X}$ itself is a Banach dual space, that is, there exists a Banach space V such that $\mathcal{X}$ is isometrically isomorphic to the dual space V^* of V (in particular, $\mathcal{X}$ is complete).*

The theorem above can be applied to Banach spaces $\mathcal{X} \subset H(\mathbb{D})$ satisfying condition (I), this is because the compact-open topology defined on $H(\mathbb{D})$ is a locally convex topology. Finally, in the proof of theorem 2.4.1 the space V is of the form

$$V = \{l \in \mathcal{X}^* : l\big|_{B_\mathcal{X}} \text{ is } \tau\text{-continuous}\}$$

and so $^*\mathcal{X}$ is a Banach space such that $\mathcal{X}$ is isometrically isomorphic to $(^*\mathcal{X})^*$. By the Hahn-Banach theorem the linear span of the set $\{\delta_z : z \in \mathbb{D}\}$ is contained and norm dense in $^*\mathcal{X}$, see [11] for further details. By using (2.1.1), (2.1.2) and (2.3.1) we can calculate the operator norm of the point evaluation functionals on A_α^p, H^p and $H_{v_\beta}^\infty$. They are

$$\|\delta_z\|_{H^p\to\mathbb{C}} = \frac{1}{(1-|z|^2)^{\frac{1}{p}}};$$
$$\|\delta_z\|_{A_\alpha^p\to\mathbb{C}} = \frac{1}{(1-|z|^2)^{\frac{2+\alpha}{p}}}, \quad \alpha > -1;$$
$$\|\delta_z\|_{H_{v_\beta}^\infty\to\mathbb{C}} = \frac{1}{(1-|z|^2)^\beta}, \quad \beta > 0.$$

From the above it is easily seen that condition (II) is satisfied in the spaces H^p, A_α^p and $H_{v_\beta}^\infty$. Condition (II) is not, however, satisfied in H^∞ since $\|\delta_z\|_{H^\infty} = 1$. The spaces H^p and

A^p_α satisfy all conditions (III)-(V) when $1 \leq p < \infty$ and $\alpha > -1$, and the same is true for H^∞_v if the weight v is normal and equivalent to its associated weight defined by

$$\tilde{v}(z) = \|\delta_z\|^{-1}_{H^\infty_v \to \mathbb{C}},$$

for proofs see [10].

2.5 Operators between Banach spaces

Let $\mathcal{X}$ and $\mathcal{Y}$ be Banach spaces and let $\mathcal{L}(\mathcal{X},\mathcal{Y})$ denote the set of all continuous linear maps $T : \mathcal{X} \to \mathcal{Y}$. A linear operator T is bounded if there exists a constant $M > 0$ such that

$$\|T(x)\|_{\mathcal{Y}} \leq M\|x\|_{\mathcal{X}}.$$

It is a well-known fact that a linear operator T is bounded if and only if T is continuous and furthermore if the operator norm

$$\|T\|_{\mathcal{X}\to\mathcal{Y}} = \sup_{\|x\|_{\mathcal{X}}\leq 1} \|T(x)\|_{\mathcal{Y}}$$

is finite. The space $\mathcal{L}(\mathcal{X},\mathcal{Y})$ equipped with the operator norm is itself a Banach space. The notion of a compact operator will be useful to us, we begin with the definition.

Definition 2.5.1. Let $\mathcal{X}$ and $\mathcal{Y}$ be Banach spaces and assume that $T \in \mathcal{L}(\mathcal{X},\mathcal{Y})$, then $T : \mathcal{X} \to \mathcal{Y}$ is a compact operator if for every bounded sequence $\{x_n\}_{n=1}^\infty$ in $\mathcal{X}$ the sequence $\{T(x_n)\}_{n=1}^\infty$ has a convergent subsequence in $\mathcal{Y}$.

The essential norm of a bounded linear operator $T : \mathcal{X} \to \mathcal{Y}$ is defined to be the distance to the compact operators, that is

$$\|T\|_{e,\mathcal{X}\to\mathcal{Y}} = \inf\{\|T-K\|_{\mathcal{X}\to\mathcal{Y}} : K : X \to Y \text{ is compact}\}.$$

Notice that $T : \mathcal{X} \to \mathcal{Y}$ is compact if and only if $\|T\|_{e,\mathcal{X}\to\mathcal{Y}} = 0$, we will denote the set of compact operators $T : \mathcal{X} \to \mathcal{Y}$ by $\mathcal{K}(\mathcal{X},\mathcal{Y})$. The definition of weak compactness of an operator between Banach spaces is given by

Definition 2.5.2. Let $\mathcal{X}$ and $\mathcal{Y}$ be Banach spaces and assume that $T \in \mathcal{L}(\mathcal{X},\mathcal{Y})$. Then $T : \mathcal{X} \to \mathcal{Y}$ is weakly compact if for every bounded sequence $\{x_n\}_{n=1}^\infty$ in $\mathcal{X}$ the sequence $\{T(x_n)\}_{n=1}^\infty$ contains a weakly convergent subsequence in $\mathcal{Y}$.

We will denote the set of all weakly compact operators by $\mathcal{W}(\mathcal{X},\mathcal{Y})$. The following lemma is very useful to check if a linear operator, satisfying some conditions, is compact or weakly compact, see [15].

Lemma 2.5.3. *Let $\mathcal{X} \subset H(\mathbb{D})$ be a Banach space such that the closed unit ball $B_\mathcal{X}$ is compact with respect to the compact-open topology co, and let $\mathcal{Y} \subset H(\mathbb{D})$ be a Banach space such that the point evaluation functionals on $\mathcal{Y}$ are bounded. Assume that $T : \mathcal{X} \to \mathcal{Y}$ is a co-co continuous linear operator. Then $T : \mathcal{X} \to \mathcal{Y}$ is compact(respectively weakly compact) if and only if $\{T(f_n)\}_{n=1}^\infty$ converges to zero in the norm(respectively in the weak topology) of $\mathcal{Y}$ for each bounded sequence $\{f_n\}_{n=1}^\infty$ in $\mathcal{X}$ such that $f_n \to 0$ uniformly on compact subsets of $\mathbb{D}$.*

As a final note it can be seen that $\mathcal{K}(\mathcal{X},\mathcal{Y}) \subset \mathcal{W}(\mathcal{X},\mathcal{Y})$, or in other words all compact operators are weakly compact.

Chapter 3

The Hilbert matrix operator

The Hilbert matrix is an infinite matrix with entries $a_{i,j} = \frac{1}{i+j+1}$ or in matrix form

$$\mathcal{H} = \begin{pmatrix} 1 & 1/2 & 1/3 & \cdots \\ 1/2 & 1/3 & 1/4 & \cdots \\ \vdots & \vdots & \ddots & \vdots \end{pmatrix}.$$

This matrix was introduced by Hilbert in connection to his double series theorem, if $\sum_{k=0}^{\infty} a_k^2 < \infty$, then

$$0 \leq \left| \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{a_i a_j}{i+j+1} \right| \leq C \sum_{i=0}^{\infty} a_i^2.$$

The proof of this theorem was published by Weyl in his thesis [42] and the optimal constant $C = \pi$ was found by Schur [38]. Hardy and Riesz later generalized this result for $1 < p, q < \infty$, $1/p + 1/q = 1$. If $(a_k) \in \ell^p$ and $(b_k) \in \ell^q$, then

$$\sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \frac{|a_j b_k|}{j+k+1} \leq \frac{\pi}{\sin(\frac{\pi}{p})} \left(\sum_{k=0}^{\infty} |a_k|^p \right)^{1/p} \left(\sum_{k=0}^{\infty} |b_k|^q \right)^{1/q},$$

see [25]. From the above it follows that

$$\left(\sum_{k=0}^{\infty} \left| \sum_{j=0}^{\infty} \frac{a_j}{j+k+1} \right|^p \right)^{1/p} \leq \frac{\pi}{\sin(\frac{\pi}{p})} \left(\sum_{k=0}^{\infty} |a_k|^p \right)^{1/p},$$

where the constant $\frac{\pi}{\sin(\frac{\pi}{p})}$ is optimal. In other words $\mathcal{H} : l^p \to l^p, a_k \mapsto A_k$, where $A_k = \sum_{j=0}^{\infty} \frac{a_j}{j+k+1}$ is bounded and $\|\mathcal{H}\|_{l^p \to l^p} = \frac{\pi}{\sin(\frac{\pi}{p})}$. The Hilbert matrix was first considered as an operator $\mathcal{H} : \ell^2 \to \ell^2$ by Magnus [34]. He showed that the spectrum of the Hilbert matrix operator on l^2 is $[0, \pi]$.

The Hilbert matrix operator can also be defined on Banach spaces of analytic functions by its action on the Taylor coefficients. Namely let $f(z) = \sum_{k=0}^{\infty} a_k z^k$ then the Hilbert matrix operator $\mathcal{H}$ is defined as the double sum

$$\mathcal{H}(f)(z) = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{\infty} \frac{a_k}{n+k+1} \right) z^n.$$

The Hilbert matrix operator defines an analytic function on the open unit disc when the coefficients $A_n = \sum_{k=0}^{\infty} \frac{a_k}{n+k+1}$ are bounded for every $n \in \mathbb{N}$. Note that this is not always the case, let $f(z) = \frac{1}{1-z} = \sum_{k=0}^{\infty} z^k$ then

$$\sum_{k=0}^{\infty} \frac{1}{n+k+1} = \infty$$

for all $n \in \mathbb{N}$. Therefore we need to restrict $H(\mathbb{D})$ to linear subspaces of $H(\mathbb{D})$ where $\mathcal{H}$ is defined. A few examples of such spaces are H^p, A^p_α and $H^\infty_{v_\beta}$. In the Hardy case we apply Hardy's inequality to obtain

$$\sum_{k=0}^{\infty} \frac{|a_k|}{k+1} \leq \pi \|f\|_{H^1}$$

showing that the Hilbert matrix operator defines an analytic function when $f \in H^p$ with $p \geq 1$. For the spaces A^p_α and $H^\infty_{v_\beta}$ we present similar results to the above inequality, which is the topic of the next lemma.

Lemma 3.0.1. *(a) If $p \geq 2 + 2\alpha$ and $f \in A^p_\alpha$, then*

$$\sum_{k=0}^{\infty} \frac{|a_k|}{k+1} < \infty.$$

(b) if $0 < \beta < 1/2$ and $f \in H^\infty_{v_\beta}$, then

$$\sum_{k=0}^{\infty} \frac{|a_k|}{k+1} < \infty.$$

Proof. Let $f(z) = \sum_{k=0}^{\infty} a_k z^k$.

(a) Let

$$m_p(r,f)^p = \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p d\theta,$$

where $0 \leq r < 1$. Since $m_p(r,f)$ is a non-decreasing function of r it follows for $t \in (r,1)$ that

$$t(1-t^2)^\alpha m_p(r,f)^p \leq m_p(t,f)^p t(1-t^2)^\alpha$$

integrating with respect to t we get

$$\frac{1}{2(1+\alpha)}(1-r^2)^{1+\alpha} m_p(r,f)^p \leq \int_r^1 t(1-t^2)^\alpha m_p(t,f)^p dt \leq \frac{1}{2}\int_0^1 2t(1-t^2)^\alpha m_p(t,f)^p dt.$$

By rearranging terms we arrive at

$$m_p(r,f) \leq \frac{\|f\|_{A^p_\alpha}}{(1-r^2)^{(1+\alpha)/p}} \leq \frac{\|f\|_{A^p_\alpha}}{(1-r)^{(1+\alpha)/p}}.$$

Since $\alpha \geq 0$ and $p > 2+\alpha \geq 2$ we get that $m_2(r,f) \leq m_p(r,f)$ from which we get

$$\sum_{k=0}^{\infty} |a_k|^2 r^{2k} = m_2(r,f)^2 \leq \frac{\|f\|^2_{A^p_\alpha}}{(1-r)^{2(1+\alpha)/p}}.$$

By setting $r = 1-2^{-(k-1)}$, $k \geq 2$ it follows that

$$\sum_{j=2^{k-1}}^{2^k-1} |a_j|^2 \leq C 2^{(k-1)\frac{2(1+\alpha)}{p}} \|f\|^2_{A^p_\alpha}$$

for some constant $C > 0$.

(b) For the Korenblum case we use the pointwise estimate (2.3.1) to get

$$(1-r)^{2\beta} m_2(r,f)^2 \leq \|f\|^2_{H^\infty_{v_\beta}}.$$

Now by using the equality $\sum_{k=0}^{\infty} |a_k|^2 r^{2k} = m_2(r,f)^2$ we obtain

$$\sum_{k=0}^{\infty} |a_k|^2 r^{2k} \leq \|f\|^2_{H^\infty_{v_\beta}} (1-r)^{-2\beta}.$$

By putting $r = 1-2^{-(k-1)}$, $k \geq 2$ into the above inequality we arrive at the following estimate

$$\sum_{j=2^{k-1}}^{2^k-1} |a_j|^2 \leq D 2^{(k-1)2\beta} \|f\|^2_{H^\infty_{v_\beta}}$$

for some constant $D > 0$.

To complete the proof we observe that in both case (a) and case (b) it holds that

$$\begin{aligned}
\sum_{k=2}^{\infty} \frac{|a_k|}{k+1} = \sum_{k=2}^{\infty} \sum_{j=2^{k-1}}^{2^k-1} \frac{|a_j|}{j+1} &\leq \sum_{k=2}^{\infty} \sum_{j=2^{k-1}}^{2^k-1} \frac{|a_j|}{2^{k-1}} \\
&\leq \sum_{k=2}^{\infty} 2^{1-k} \left(\sum_{j=2^{k-1}}^{2^k-1} 1^2 \right)^{\frac{1}{2}} \left(\sum_{j=2^{k-1}}^{2^k-1} |a_j|^2 \right)^{\frac{1}{2}} \\
&= \sum_{k=2}^{\infty} 2^{1-k} 2^{\frac{k-1}{2}} \left(\sum_{j=2^{k-1}}^{2^k-1} |a_j|^2 \right)^{\frac{1}{2}}.
\end{aligned}$$

Now for the (a) case we get

$$\sum_{k=2}^{\infty} \frac{|a_k|}{k+1} \leq C\|f\|_{A^p_\alpha} \sum_{k=2}^{\infty} 2^{-\frac{k-1}{2}(1-\frac{2(1+\alpha)}{p})} < \infty$$

if $p > 2 + 2\alpha$. For the (b) case we get

$$\sum_{k=2}^{\infty} \frac{|a_k|}{k+1} \leq D\|f\|_{H^\infty_{v_\beta}} \sum_{k=2}^{\infty} 2^{-\frac{k-1}{2}(1-2\beta)} < \infty$$

if $0 < \beta < \frac{1}{2}$, completing the proof. □

The objective was to show that the Hilbert matrix operator defines an analytic function in the weighted Bergman case when $p > 2 + \alpha$, $\alpha \geq 0$ and in the Korenblum space when $0 < \beta < 1$. Therefore Lemma 3.0.1 is not enough for boundedness of $\mathcal{H}$ and we need another approach. The key is using the pointwise estimate for weighted Bergman spaces to show that $\mathcal{H}(f)$ defines an analytic function on $\mathbb{D}$ when $f \in A^p_\alpha$, $\alpha \geq 0$ and $p > 2 + \alpha$ and then using the fact that $H^\infty_{v_\beta} \subseteq A^p_\alpha$ for p large enough.

Theorem 3.0.2. *Let $\alpha \geq 0$ and $p > 2 + \alpha$. If $f \in A^p_\alpha$ then $\mathcal{H}(f) \in H(\mathbb{D})$.*

Proof. Let $f \in A^p_\alpha$. By using (2.1.2) we deduce that

$$\left| \int_0^1 \frac{f(t)}{1-tz} dt \right| \leq \int_0^1 \frac{|f(t)|}{|1-tz|} dt \leq \frac{\|f\|_{A^p_\alpha}}{1-|z|} \int_0^1 \frac{1}{(1-t^2)^{(2+\alpha)/p}} dt < \infty,$$

since $p > 2 + \alpha$. From the above it follows that

$$\int_0^1 |f(t)| dt \leq C(\alpha, p)\|f\|_{A^p_\alpha} \tag{3.0.1}$$

and

$$\sup_n \left| \int_0^1 t^n f(t) dt \right| < \infty.$$

Now let $f = \sum_{k=0}^{\infty} a_k z^k$ and define $S_N f(z) = \sum_{k=0}^{N} a_k z^k$. By [21, Lemma 1], it holds that $\|f - S_N f\|_{A^p_\alpha} \to 0$, $N \to \infty$ if and only if $\sup_{N\geq 1} \|S_N\| < \infty$. By the boundedness of the Riesz projection on H^p, $1 < p < \infty$ there exists a constant $C > 0$ independent of N and f such that

$$\int_0^{2\pi} |S_N f(e^{i\theta})|^p d\theta \leq C \int_0^{2\pi} |f(e^{i\theta})|^p d\theta.$$

Applying this result to the functions $f_r \in H^p$, $0 \leq r < 1$ we get

$$\begin{aligned}
\|S_N f\|^p_{A^p_\alpha} &= \frac{\alpha+1}{\pi} \int_0^1 \int_0^{2\pi} |S_N f(re^{i\theta})|^p d\theta r(1-r^2)^\alpha dr \\
&\leq C \frac{\alpha+1}{\pi} \int_0^1 \int_0^{2\pi} |f(re^{i\theta})|^p d\theta r(1-r^2)^\alpha dr \\
&= C\|f\|^p_{A^p_\alpha},
\end{aligned}$$

showing that $\sup_{N\geq 1}\|S_N\| < \infty$. Now using estimate (3.0.1) we have

$$\begin{aligned}\left|\int_0^1 t^n f(t)dt - \sum_{k=0}^{N}\frac{a_k}{n+k+1}\right| &= \left|\int_0^1 t^n f(t)dt - \int_0^1\left(\sum_{k=0}^{N} a_k t^k\right)t^n dt\right|\\ &= \left|\int_0^1 t^n(f(t) - S_N f(t))dt\right|\\ &\leq C(\alpha,p)\|f - S_N f\|_{A^p_\alpha},\end{aligned}$$

which converges to 0 when $N\to\infty$. This shows that

$$\int_0^1 t^n f(t)dt = \sum_{k=0}^{\infty}\frac{a_k}{n+k+1}$$

for each $n\in\mathbb{N}$ and further that $\mathcal{H}(f)(z) = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{\infty}\frac{a_k}{n+k+1}\right)z^n$ is an analytic function on the open unit disc. □

To show that the Hilbert matrix operator defines an analytic function in the Korenblum spaces with $0<\beta<1$ we use the fact that $H^\infty_{v_\beta}\subseteq A^p_{p\beta}$ for $p>\frac{2}{1-\beta}$. Indeed, if $f\in H^\infty_{v_\beta}$ then

$$\begin{aligned}\|f\|^p_{A^p_{p\beta}} &= \int_{\mathbb{D}}|f(z)|^p(1-|z|^2)^{p\beta}(p\beta+1)dA(z)\\ &\leq \|f\|^p_{H^\infty_{v_\beta}}\int_{\mathbb{D}}\frac{(1-|z|^2)^{p\beta}}{(1-|z|^2)^{p\beta}}(p\beta+1)dA(z)\\ &= \|f\|^p_{H^\infty_{v_\beta}}(p\beta+1)<\infty.\end{aligned}$$

Hence, $\mathcal{H}(f)\in H(\mathbb{D})$ when $H^\infty_{v_\beta}$ with $0<\beta<1$.

3.1 Integral representation

For our purposes it is necessary to write $\mathcal{H}$ in a different way, namely as an integral mean of weighted composition operators, see [17]. Through this representation it is possible to calculate the norm precisely in the A^p_α case when $\alpha\geq 0$ and $2+\alpha<p<\infty$, given some further constraints. In this thesis we will give a thorough account of why this integral representation holds in the A^p_α and $H^\infty_{v_\beta}$ case. First we give another representation of the Hilbert matrix operator, namely

$$\mathcal{H}(f)(z) = \int_0^1\frac{f(t)}{1-tz}dt.$$

This representation is valid when $f\in A^p_\alpha$ and $f\in H^\infty_{v_\beta}$. Indeed, let $f(z)=\sum_{k=0}^{\infty}a_k z^k\in A^p_\alpha$ and define

$$\mathcal{S}(f)(z) = \int_0^1\frac{f(t)}{1-tz}dt,\quad z\in\mathbb{D}.$$

By the proof of Theorem 3.0.2 we have that

$$\sup_n \left| \int_0^1 t^n f(t) dt \right| < \infty$$

and for all $n \in \mathbb{N}$ that

$$\int_0^1 t^n f(t) dt = \sum_{k=0}^{\infty} \frac{a_k}{n+k+1}.$$

Thus by Fubini's theorem we get

$$S(f)(z) = \int_0^1 \left(\sum_{n=0}^{\infty} f(t)(tz)^n \right) dt = \sum_{n=0}^{\infty} \left(\int_0^1 t^n f(t) \right) z^n = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{\infty} \frac{a_k}{n+k+1} \right) z^n = \mathcal{H}(f)(z)$$

for all $z \in \mathbb{D}$. Since $H^\infty_{v_\beta} \subset A^p_{p\beta}$, we conclude from the above that for each $f \in H^\infty_{v_\beta}$ with $0 < \beta < 1$ it holds that $S(f) = \mathcal{H}(f) \in H(\mathbb{D})$, if we choose $p > \frac{2}{1-\beta}$. The integral representation for $\mathcal{H}$ also holds in H^p spaces, but we will not prove it here.

3.2 Weighted composition representation

In this section we will rewrite $\mathcal{H}$ as an integral of weighted composition operators. Recall that

$$\mathcal{H}(f)(z) = \int_0^1 \frac{f(s)}{1-sz} ds.$$

By changing the path of integration,

$$s = s(t) = \frac{t}{(t-1)z+1}, \qquad 0 \le t \le 1,$$

we arrive at an alternate representation for $\mathcal{H}(f)$

$$\mathcal{H}(f)(z) = \int_0^1 \omega_t(z) f(\phi_t(z)) dt,$$

where

$\omega_t(z) = \frac{1}{(t-1)z+1}$, $0 \le t \le 1$

$\phi_t(z) = \frac{t}{(t-1)z+1}$, $0 \le t \le 1$.

Here ω_t is a bounded analytic function on the open unit disc for all $0 \le t \le 1$, and since

$$|\phi_t(z)| = \frac{t}{|(t-1)z+1|} \le \frac{t}{1-(1-t)|z|} \le 1,$$

it holds that $\phi_t : \mathbb{D} \to \mathbb{D}$ or in other words ϕ_t is an analytic self-map of the unit disc. Furthermore it holds that $\phi_t(\mathbb{D})$ is a disc with center $\frac{1}{2-t}$ and radius $\frac{1-t}{2-t}$. We denote $T_t(f)(z) = \omega_t(z) f(\phi_t(z))$. Now T_t is a weighted composition operator and we have that

$$\mathcal{H}(f)(z) = \int_0^1 T_t(f)(z) dt.$$

3.3 The norm of the Hilbert matrix operator on H^p

In [17] Diamantopoulos and Siskakis studied the boundedness of $\mathcal{H} : H^p \to H^p$. They proved that $\mathcal{H} : H^p \to H^p$ is bounded for $1 < p < \infty$ and they also found an upper bound on the norm of $\mathcal{H}$ when $2 \le p < \infty$, namely

$$\|\mathcal{H}\|_{H^p \to H^p} \le \frac{\pi}{\sin\left(\frac{\pi}{p}\right)}. \tag{3.3.1}$$

$\mathcal{H}$ is not bounded on H^1 or H^∞, for the H^∞ case consider

$$\mathcal{H}(1)(z) = \int_0^1 \frac{1}{1-tz} dt = \int_0^1 \left(\sum_{k=0}^{\infty} t^k z^k \right) dt = \sum_{k=0}^{\infty} \frac{z^k}{k+1} = \frac{1}{z} \log\left(\frac{1}{1-z}\right),$$

which is clearly unbounded on $\mathbb{D}$. As such $\mathcal{H}$ is unbounded on H^∞. In the H^1 case we can use the function

$$f(z) = \frac{z^2}{(1-z)(\log\left(\frac{1}{1-z}\right))^2} \in H^1$$

and it can be shown that $\mathcal{H}(f) \notin H^1$. The lower bound of the norm of $\mathcal{H}$ was proven to be the same as the right side of (3.3.1) for all $1 < p < \infty$ by Dostanić, Jevtić and Vukotić in [19]. They used test functions of the form

$$f_\gamma(z) = (1-z)^{\frac{-\gamma}{p}},$$

where $\gamma \in (\epsilon, 1)$ for $\epsilon \in (0,1)$. For the upper bound of the norm of $\mathcal{H}$, Dostanić et al. [19] used the exact norm of the Riesz projection for Hardy spaces to determine the above upper bound for $1 < p < \infty$. This method does not work in the Bergman case when $2 < p < 4$ and therefore we need another approach.

3.4 An upper estimate of the norm of $\mathcal{H}$ on weighted Bergman spaces

Recently the norm of the Hilbert matrix operator on Bergman spaces has been under active study. In [16] Diamantopoulos showed that $\mathcal{H} : A^p \to A^p$ is bounded when $2 < p < \infty$ and determined an upper bound for the norm of $\mathcal{H}$

$$\|\mathcal{H}\|_{A^p \to A^p} \le \frac{\pi}{\sin \frac{2\pi}{p}},$$

when $4 \le p < \infty$. He also managed to find an upper bound of the norm of $\mathcal{H}$ when $2 < p < 4$, although less precise. In their work [19], Dostanić et al. improved the estimate when $2 < p < 4$ and they also determined the lower bound for $\mathcal{H} : A^p \to A^p$ thus proving the exact value of the norm of $\mathcal{H}$ on $4 \le p < \infty$. In the same article the authors also conjectured that the exact value of the norm of $\mathcal{H}$ on A^p when $2 < p < 4$ is the same as in the $4 \le p < \infty$ case. Later on Božin and Karapetrović [12] proved the conjecture in the affirmative by reducing the problem of determining the norm of $\mathcal{H}$ to estimates concerning Beta functions. In paper II we simplified the proof in [12] by removing the dependency on Sturm's theorem [37] and giving a partial new proof of one of the key lemmas. By this new proof the lemma could then partly be generalized to the weighted Bergman case, as was done in paper III. In [27] Karapetrović studied $\mathcal{H}$ on A^p_α when $\alpha \geq 0$ and $2+\alpha < p < \infty$. He found that the norm of $\mathcal{H}$ is

$$\|\mathcal{H}\|_{A^p_\alpha \to A^p_\alpha} = \frac{\pi}{\sin\frac{(2+\alpha)\pi}{p}},$$

when $4 \leq 2(2+\alpha) \leq p < \infty$ and found an estimate of the norm of $\mathcal{H}$ when $2 \leq 2+\alpha < p < 2(2+\alpha)$. In paper III we improved the result by showing that $\mathcal{H} : A^p_\alpha \to A^p_\alpha$ has the same norm for a subinterval of $2 \leq 2+2\alpha < p < 2(2+\alpha)$ and a condition for the remaining p in the mentioned interval, more on this later. We now return to calculating an upper bound for the operator norm of $\mathcal{H}$. We get

$$\begin{aligned}
\|\mathcal{H}(f)\|_{A^p_\alpha} &= \left((\alpha+1)\int_{\mathbb{D}} |\mathcal{H}(f)(z)|^p(1-|z|^2)^\alpha dA(z)\right)^{1/p} \\
&= (\alpha+1)^{1/p}\left(\int_{\mathbb{D}}\left|\int_0^1 T_t(f)(z)dt\right|^p (1-|z|^2)^\alpha dA(z)\right)^{1/p} \\
&\leq (\alpha+1)^{1/p}\int_0^1\left(\int_{\mathbb{D}} |T_t(f)(z)|^p(1-|z|^2)^\alpha dA(z)\right)^{1/p} dt \\
&= \int_0^1 \|T_t(f)\|_{A^p_\alpha} dt,
\end{aligned}$$

where the second last inequality is motivated by the continuous version of Minkowski's inequality. We have arrived at the estimate

$$\|\mathcal{H}(f)\|_{A^p_\alpha} \leq \int_0^1 \|T_t(f)\|_{A^p_\alpha} dt, \tag{3.4.1}$$

which motivates the study of the norm of the weighted composition operator T_t. Estimating the norm $\|T_t(f)\|_{A^p_\alpha}$ is done via the change of variables $w = \phi_t(z)$, $dA(w) = |\phi_t'(z)|^2 dA(z)$. We have

$$\begin{aligned}
\|T_t(f)\|_{A^p_\alpha} &= (\alpha+1)^{1/p}\left(\int_{\mathbb{D}} |T_t(f)(z)|^p(1-|z|^2)^\alpha dA(z)\right)^{1/p} \\
&= (\alpha+1)^{1/p}\left(\int_{\mathbb{D}} |\omega_t(z)f(\phi_t(z))|^p(1-|z|^2)^\alpha dA(z)\right)^{1/p} \\
&= (\alpha+1)^{1/p}\left(\int_{D_t} |f(w)|^p|\omega_t(\phi_t^{-1}(w))|^p(1-|\phi_t^{-1}(w)|^2)^\alpha|\phi_t'(\phi_t^{-1}(w))|^{-2} dA(w)\right)^{1/p}.
\end{aligned}$$

By a few calculations we get the following

$$|\omega_t(\phi_t^{-1}(w))| = \tfrac{|w|}{t};$$

$|\phi_t'(\phi_t^{-1}(w))|^{-2} = \frac{t^2}{(1-t)^2|w|^4};$

$1-|\phi_t^{-1}(w)|^2 = \frac{t}{1-t}\frac{2\mathfrak{Re}(w)-t-(2-t)|w|^2}{(1-t)|w|^2}.$

So we arrive at the following expression for the norm of T_t

$$\|T_t(f)\|_{A^p_\alpha}^p = (\alpha+1)\frac{t^{2+\alpha-p}}{(1-t)^{2+2\alpha}}\int_{D_t}|w|^{p-2\alpha-4}|f(w)|^p g_t(w)^\alpha\, dA(w),$$

where $g_t(w) = 2\mathfrak{Re}(w)-t-(2-t)|w|^2$. Now if $\alpha \geq 0$ we can estimate g_t^α.

$$g_t(w) = 2\mathfrak{Re}(w)-t-(2-t)|w|^2 \leq 1+|w|^2-t-(2-t)|w|^2 = (1-|w|^2)(1-t).$$

This gives us now

$$g_t(w)^\alpha \leq (1-|w|^2)^\alpha(1-t)^\alpha.$$

Using the above inequality we get the following upper bound for the norm of T_t.

$$\|T_t(f)\|_{A^p_\alpha} \leq \frac{t^{\frac{2+\alpha}{p}-1}}{(1-t)^{\frac{2+\alpha}{p}}}\left((\alpha+1)\int_{D_t}|w|^{p-2\alpha-4}|f(w)|^p(1-|w|^2)^\alpha\, dA(w)\right)^{1/p}. \tag{3.4.2}$$

We will now show how the norm of the Hilbert matrix operator can be calculated immediately from the above expression in the case when $p \geq 2(2+\alpha)$. Indeed, since $|w|^{p-2\alpha-4} \leq 1$ and $D_t \subset \mathbb{D}$,

$$\begin{aligned}\|\mathcal{H}(f)\|_{A^p_\alpha} &\leq \int_0^1 \|T_t(f)\|_{A^p_\alpha}\, dt\\ &\leq (\alpha+1)^{1/p}\int_0^1 \frac{t^{\frac{2+\alpha}{p}-1}}{(1-t)^{\frac{2+\alpha}{p}}}\left(\int_{D_t}|w|^{p-2\alpha-4}|f(w)|^p(1-|w|^2)^\alpha\, dA(w)\right)^{1/p} dt\\ &\leq B\left(\frac{2+\alpha}{p}, 1-\frac{2+\alpha}{p}\right)\|f\|_{A^p_\alpha},\end{aligned}$$

where

$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}\, dt,$$

for $x,y \in \mathbb{C}$ satisfying $\mathfrak{Re}(x) > 0, \mathfrak{Re}(y) > 0$, is the Beta function. The Beta function can also be defined via the Gamma function

$$\Gamma(z) = \int_0^\infty x^{z-1}e^{-x}\, dx, \mathfrak{Re}(z) > 0.$$

The Beta function can then be related to the Gamma function by the relation

$$B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}.$$

We will also use the following equality concerning the Beta function

$$B(x,1-x) = \frac{\pi}{\sin(\pi x)}. \tag{3.4.3}$$

These and more properties concerning the Beta function can be found in the book by Abramowitz and Stegun [1]. By using property (3.4.3) we deduce

$$\|\mathcal{H}(f)\|_{A^p_\alpha} \leq \frac{\pi}{\sin\frac{(2+\alpha)\pi}{p}}\|f\|_{A^p_\alpha}$$

in the case when $p \geq 2(2+\alpha)$.

As was seen the case $p \geq 2(2+\alpha)$ follows immediately from (3.4.2). The case $p < 2(2+\alpha)$ has proven to be much more cumbersome to solve, because in the above expression the term $|w|^{p-2\alpha-4}$ grows large. The following theorem, which is the main result in paper III, gives a partial solution to the problem of determining the norm of $\mathcal{H}$ when $2+2\alpha < p < 2(2+\alpha)$.

Theorem 3.4.1. *Let $\alpha \geq 0$. Suppose that either of the following conditions holds*

(a) $2+\alpha+\sqrt{\alpha^2+\frac{7}{2}\alpha+3} \leq p < 2(2+\alpha)$;

(b) $2+2\alpha < p < 2+\alpha+\sqrt{\alpha^2+\frac{7}{2}\alpha+3}$ *and*

$$\int_0^1 I_t\left(\frac{2+\alpha}{p}, 1-\frac{2+\alpha}{p}\right)t^{2p-4\alpha-5}(1-t^4)^\alpha\,dt - \frac{1}{4(\alpha+1)} \leq 0.$$

Then

$$\|\mathcal{H}\|_{A^p_\alpha\to A^p_\alpha} \leq \frac{\pi}{\sin\frac{(2+\alpha)\pi}{p}}.$$

Here I_t refers to the regularized incomplete Beta function and is defined by

$$I_t = \frac{B_t(x,y)}{B(x,y)}, \quad B_t(x,y) = \int_0^t s^{x-1}(1-s)^{y-1}\,ds.$$

The proof rests upon two useful lemmas, the first being due to Bhayo and Sándor [9]. This handles the (a) case in 3.4.1.

Lemma 3.4.2. *Let $x > 1, 0 < y < 1$. Then*

(a) $B(x,y) < \frac{1}{xy}(x+y-xy)$;

(b) $B(x,y) \geq \frac{1}{xy}\frac{x+y}{1+xy}$.

The inequalities reverse when $x,y \in (0,1]$.

For the following lemma we need to introduce two new functions. Let $\alpha \geq 0$, $2+2\alpha < p < 2(2+\alpha)$. Define for $s \in [0,1]$, $0 < t < 1$

$H_{\alpha,p}(s) = \sum_{k=0}^\infty \binom{\alpha}{k}(-1)^k\frac{1}{p-2\alpha-2+2k} - \frac{1}{\alpha(\alpha+1)}(1-s^4)^{\alpha+1}$;

$K_{\alpha,p}(s,t) = \sum_{k=0}^\infty \binom{\alpha}{k}(-1)^k\frac{1}{p-2\alpha-2+2k}\max(s^2,t^2)^{p-2\alpha-2+2k}$, where

$$\binom{\alpha}{k} = \frac{\alpha(\alpha-1)(\alpha-2)\cdots(\alpha-k+1)}{k!}, \quad \binom{\alpha}{0} = 1$$

are the generalized binomial coefficients. It is easily seen that the two sums above are convergent for every $\alpha \geq 0$ and $2+2\alpha < p < 2(2+\alpha)$. With the help of these two functions we get the second important lemma.

Lemma 3.4.3. *Let $\alpha \geq 0$, $2+2\alpha < p < 2(2+\alpha)$, and define $\psi_{\alpha,p}(t) = t^{\frac{2+\alpha}{p}-1}(1-t)^{-\frac{2+\alpha}{p}}$. The following two conditions are equivalent:*

(a) $B\left(\frac{2+\alpha}{p}, 1-\frac{2+\alpha}{p}\right)H_{\alpha,p}(0) - \int_0^1 \psi_{\alpha,p}(t)K_{\alpha,p}(0,t)dt \leq 0$

(b) $\int_0^1 I_t\left(\frac{2+\alpha}{p}, 1-\frac{2+\alpha}{p}\right)t^{2p-4\alpha-5}(1-t^4)^\alpha dt - \frac{1}{4(\alpha+1)} \leq 0.$

Moreover, if $\alpha \in [0,1]$ or $\alpha \in [2,3]$, then

$$\begin{aligned}
&\frac{1}{2p-4\alpha-4} - \frac{1}{(2p-4\alpha-4)^2}\frac{1}{B\left(\frac{2+\alpha}{p}, 2p-4\alpha-4\right)} \\
&-\alpha\left(\frac{1}{2p-4\alpha} - \frac{1}{(2p-4\alpha)^2}\frac{1}{B\left(\frac{2+\alpha}{p}, 2p-4\alpha\right)}\right) \\
&+\frac{\alpha(\alpha-1)}{2}\left(\frac{1}{2p-4\alpha+4} - \frac{1}{(2p-4\alpha+4)^2}\frac{1}{B\left(\frac{2+\alpha}{p}, 2p-4\alpha+4\right)}\right) - \frac{1}{4(\alpha+1)} \leq 0
\end{aligned}$$

implies that (a) and (b) hold.

The last inequality in the above lemma can be shown to hold when $\alpha = 0$, we get

$$\frac{1}{2p-4} - \frac{1}{(2p-4)^2}\frac{1}{B\left(\frac{2}{p}, 2p-4\right)} - \frac{1}{4} \leq 0.$$

The above inequality is equivalent to

$$B\left(\frac{2}{p}, 2p-4\right) \leq \frac{1}{(p-2)(4-p)}.$$

Lemma 3.4.4. *Let $2 < p < 4$. Then*

$$B\left(\frac{2}{p}, 2p-4\right) \leq \frac{1}{(p-2)(4-p)}.$$

Proof. (1) Case $2 < p < \frac{5}{2}$. By using $B(x,y) \leq \frac{1}{xy}, x,y \in (0,1]$ and observing that both parameters in the Beta function belong to the interval $(0,1]$, we have

$$B\left(\frac{2}{p}, 2(p-2)\right) \leq \frac{1}{(p-2)(4-p)}.$$

(2) Case $\frac{5}{2} < p < 4$. Now $2(p-2) > 1$ and $\frac{2}{p} < 1$. Hence by Lemma 3.4.2, we have

$$B\left(\frac{2}{p}, 2(p-2)\right) < \frac{1}{\frac{2}{p}\cdot 2(p-2)}\left(\frac{2}{p} + 2(p-2) - \frac{2}{p}\cdot 2(p-2)\right)$$
$$= \frac{1}{2(p-2)}(p^2 - 4p + 5).$$

The claim follows from the inequality

$$\frac{1}{2}(p^2 - 4p + 5) \leq \frac{1}{4-p},$$

which is equivalent to

$$(p-3)^2(p-2) \geq 0.$$

□

Note that inequality (b) in lemma 3.4.3 does not hold for all $\alpha \geq 0$ and $2 + 2\alpha < p < 2 + \alpha + \sqrt{\alpha^2 + \frac{7}{2}\alpha + 3}$. In the case when $\alpha = 1$ we have the following example.

Example 3.4.5. Let $\alpha = 1$. Then condition (b) in 3.4.3 does not hold when $4 < p \leq 5.1$ but it holds when $5.5 \leq p < 5.74$. This is shown in paper III.

3.5 Lower bound of the Hilbert matrix operator

To get the exact norm of the Hilbert matrix operator on the weighted Bergman spaces we still need to find the lower bound for this operator. To find the lower bound for the operator norm it is sufficient to find suitable test functions. Karapetrović [27] proved for $1 < 2 + \alpha < p < \infty$ that the following holds

$$\|\mathcal{H}\|_{A^p_\alpha \to A^p_\alpha} \geq \frac{\pi}{\sin\frac{(2+\alpha)\pi}{p}}.$$

This was done through the use of test functions of the form

$$f_\gamma(z) = (1-z)^{-\frac{\gamma}{p}}, \quad z \in \mathbb{D}$$

for $1 < \gamma < \alpha + 2 < p$. For $\alpha \geq 0$ it is easy to show that $f_\gamma \in A^p_\alpha$ for the aforementioned values on γ. Indeed,

$$\begin{aligned}
\|f_\gamma\|^p_{A^p_\alpha} &= \int_{\mathbb{D}} |f_\gamma(z)|^p (1-|z|^2)^\alpha (\alpha+1)\,dA(z) \\
&= \int_{\mathbb{D}} |(1-z)|^{-\gamma} (1-|z|^2)^\alpha (\alpha+1)\,dA(z) \\
&\leq (\alpha+1)\int_{\mathbb{D}} |(1-z)|^{-\gamma}\,dA(z) \\
&= 2(\alpha+1)\int_0^{\frac{\pi}{2}}\int_0^{2\cos(\theta)} r^{1-\gamma}\,dr\,d\theta \\
&\leq \frac{(\alpha+1)2^{3-\gamma}}{2-\gamma}\int_0^{\frac{\pi}{2}} \cos^{2-\gamma}(\theta)\,d\theta < \infty,
\end{aligned}$$

where the final inequality holds because $\gamma < 2$. In the above we changed to polar coordinates centered around $z = 1$. The rest of the proof uses Stirling's formula and some properties of hypergeometric functions, see [27] for further details.

3.6 The Hilbert matrix operator on Korenblum spaces

Recall that the Korenblum spaces are defined as

$$H_v^\infty = \{f \in H(\mathbb{D}) : \|f\|_{H_v^\infty} = \sup_{z\in\mathbb{D}} v(z)|f(z)| < \infty\}.$$

For our purposes we are only interested in the Korenblum spaces with the standard weights $v(z) = (1-|z|^2)^\beta$. The boundedness of $\mathcal{H}$ on the Korenblum spaces with standard weight was noted by Aleman, Montes-Rodríguez and Sarafoleanu in [3]. Since $\mathcal{H}$ is not bounded on $H_{v_\beta}^\infty$ for $\beta = 1$ and $\beta = 0$ we will focus on the scale $0 < \beta < 1$. In the same way as in the weighted Bergman case, take a $f \in H_{v_\beta}^\infty$ then

$$\begin{aligned}
\|\mathcal{H}(f)\|_{H_{v_\beta}^\infty} &= \sup_{z\in\mathbb{D}} \left| \int_0^1 T_t(f)(z)dt(1-|z|^2)^\beta \right| \\
&\leq \int_0^1 \sup_{z\in\mathbb{D}} |T_t(f)(z)|(1-|z|^2)^\beta dt \\
&= \int_0^1 \|T_t(f)\|_{H_{v_\beta}^\infty} dt.
\end{aligned}$$

So we are again able to reduce the problem of determining the norm of $\mathcal{H}$ to finding the norm of the weighted composition operator T_t. The following lemma is important in establishing the upper bound of the norm of $\mathcal{H}$.

Lemma 3.6.1. *Let $0 < \beta < 1$. Then*

$$\|T_t\|_{H_{v_\beta}^\infty \to H_{v_\beta}^\infty} = \begin{cases} \frac{t^{\beta-1}}{(1-t)^\beta} \text{ if } 0 < \beta \leq 2/3 \text{ and } 0 < t < 1 \text{ or if } 2/3 < \beta < 1 \text{ and } \frac{3\beta-2}{4\beta-2} \leq t < 1 \\ (1-x_0)^{2\beta-1}\left(\frac{1-|\frac{x_0}{1-t}|^2}{(1-x_0)^2-t^2}\right)^\beta \text{ if } 2/3 < \beta < 1 \text{ and } 0 < t < \frac{3\beta-2}{4\beta-2}, \end{cases}$$

where

$$x_0 = \frac{\beta + 2\beta t - t - \sqrt{4\beta^2 t - 2\beta t + \beta^2 - 2\beta + 1}}{2\beta - 1}.$$

The proof of the above lemma relies on a representation of the norm of weighted composition operators, namely let $u \in H_{v_\beta}^\infty$ and let φ be an analytic self-map of the unit disc, then the norm of the weighted composition operator uC_φ can be written as

$$\|uC_\varphi\|_{H_{v_\beta}^\infty \to H_{v_\beta}^\infty} = \sup_{z\in\mathbb{D}} |u(z)| \left(\frac{1-|z|^2}{1-|\varphi(z)|^2} \right)^\beta,$$

see [14]. Before we move to the main proof concerning the norm on Korenblum spaces in paper II we will briefly discuss the lower bound of the norm of $\mathcal{H}$.

Theorem 3.6.2. *If $\mathcal{H} : H^\infty_{v_\beta} \to H^\infty_{v_\beta}$, then*

$$\|\mathcal{H}\|_{H^\infty_{v_\beta} \to H_{v^\infty_\beta}} \geq \frac{\pi}{\sin(\pi\beta)}$$

holds for $0 < \beta < 1$.

The proof of this theorem is done by estimating the operator norm of $\mathcal{H}$ with test functions of the form

$$f_\beta(z) = \frac{1}{(1-z)^\beta}, \quad 0 < \beta < 1, \quad z \in \mathbb{D}.$$

Indeed, for $0 < \beta < 1$ we have $f_\beta \in H^\infty_{v_\beta}$, since

$$\|f_\beta\|_{H^\infty_{v_\beta}} = \sup_{z\in\mathbb{D}} \frac{(1-|z|^2)^\beta}{(1-z)^\beta} \leq \sup_{z\in\mathbb{D}} (1+|z|)^\beta = 2^\beta.$$

Furthermore, if $r \in (0,1)$ it holds that

$$|f_\beta(r)|(1-r^2)^\beta = \frac{(1-r^2)^\beta}{(1-r)^\beta} = (1+r)^\beta,$$

letting $r \to 1^-$ we get $|f_\beta(r)|(1-r^2)^\beta \to 2^\beta$ and so

$$\|f_\beta\|_{H^\infty_{v_\beta}} = 2^\beta.$$

By the help of the above lemma and the lower bound of the norm of $\mathcal{H}$ we manage to find the exact norm of the Hilbert matrix operator on the $0 < \beta < 2/3$ scale, and an upper estimate on the $2/3 < \beta < 1$ scale. The result in paper II is as follows

Theorem 3.6.3. *Let $0 < \beta < 2/3$, and let $\mathcal{H} : H^\infty_{v_\beta} \to H^\infty_{v_\beta}$ be the Hilbert matrix operator. Then*

$$\|\mathcal{H}\|_{H^\infty_{v_\beta} \to H^\infty_{v_\beta}} = \frac{\pi}{\sin(\beta\pi)}.$$

For $2/3 < \beta < 1$, we have the following upper bound.

$$\|\mathcal{H}\|_{H^\infty_{v_\beta} \to H^\infty_{v_\beta}} \leq \int_0^{\frac{3\beta-2}{4\beta-2}} G(x_0)\,dt + \int_{\frac{3\beta-2}{4\beta-2}}^1 t^{\beta-1}(1-t)^{-\beta}\,dt,$$

where

$$G(x) = (1-x)^{2\beta-1}\left(\frac{1-|\frac{x}{1-t}|^2}{(1-x)^2-t^2}\right)^\beta$$

and

$$x_0 = \frac{\beta+2\beta t - t - \sqrt{4\beta^2 t - 2\beta t + \beta^2 - 2\beta + 1}}{2\beta-1}.$$

Chapter 4

The Volterra operator

The generalized Volterra operator T_g^{φ}, for a fixed function $g \in H(\mathbb{D})$ and an analytic selfmap $\varphi : \mathbb{D} \to \mathbb{D}$ is defined as

$$T_g^{\varphi}(f)(z) = \int_0^{\varphi(z)} f(\xi)g'(\xi)d\xi, \quad z \in \mathbb{D}, f \in H(\mathbb{D}).$$

Note that $T_g^{\varphi}(f)$ defines an analytic function when $f \in H(\mathbb{D})$. Li and Stević [29] introduced the operator T_g^{φ} in the form

$$T_g^{\varphi} = C_{\varphi} \circ T_g,$$

where C_{φ} is the composition operator $f \mapsto f \circ \varphi$ and T_g is defined below. The classical Volterra operator is obtained in the case when $\varphi(z) = z$,

$$T_g(f)(z) = \int_0^{z} f(\xi)g'(\xi)d\xi, \quad z \in \mathbb{D}, f \in H(\mathbb{D}).$$

The operator T_g has been extensively studied in the last decades, beginning with the paper by Pommerenke [36]. Later on Aleman and Siskakis [4], [5] and Aleman and Cima [2] continued investigating T_g on H^p and characterized boundedness and compactness. One of the remaining open problems was characterizing boundedness and compactness for $T_g : H^{\infty} \to H^{\infty}$, which is the topic of the next section.

4.1 Boundedness and compactness results of $T_g : H_{v_\beta} \to H^{\infty}$

In [6] Anderson, Jovovic and Smith studied the boundedness of $T_g : H^{\infty} \to H^{\infty}$ in terms of its symbol g and conjectured that the set

$$T[H^{\infty}] = \{g \in H(\mathbb{D}) : T_g : H^{\infty} \to H^{\infty} \text{ is bounded}\}$$

would be the same as the space of analytic functions on the unit disc with bounded radial variation

$$BRV = \{f \in H(\mathbb{D}) : \sup_{0 \le \theta < 2\pi} \int_0^1 |f'(re^{i\theta})| dr < \infty\}.$$

It is easily seen that $BRV \subseteq T[H^\infty]$ since if $g \in$ BRV then for any $f \in H^\infty$ we have

$$
\begin{aligned}
\|T_g(f)\|_{H^\infty} = \sup_{z\in\mathbb{D}}\left|\int_0^z f(\xi)g'(\xi)d\xi\right| &= \sup_{0\le\theta<2\pi}\sup_{0\le R<1}\left|\int_0^{Re^{i\theta}} f(\xi)g'(\xi)d\xi\right| \\
&= \sup_{0\le\theta<2\pi}\sup_{0\le R<1}\left|\int_0^{R} f(re^{i\theta})g'(re^{i\theta})dr\right| \\
&\le \|f\|_{H^\infty}\sup_{0\le\theta<2\pi}\int_0^1 |g'(re^{i\theta})|dr
\end{aligned}
$$

and thus $T_g : H^\infty \to H^\infty$ is bounded. In [40] Smith, Stolyarov and Volberg proved the reverse inclusion $T[H^\infty] \subseteq$ BRV when g is univalent, or in other words

$$
T[H^\infty] \cap \{g \in H(\mathbb{D}) : g \text{ is univalent}\} \subset BRV. \tag{4.1.1}
$$

The same paper also contains a counterexample to the general conjecture proving that $T[H^\infty] \not\subseteq$ BRV. The proof of (4.1.1) utilizes a result concerning uniform approximation of Bloch functions, for details see [40] and [41]. Let β and r be positive constants and let $\mathcal{B}(\Omega_\beta^r)$ denote the class of analytic functions in the open sector

$$
\Omega_\beta^r = \left\{z \in \mathbb{C} : 0 < |z| < r \text{ and } \frac{-\beta}{2} < \arg(z) < \frac{\beta}{2}\right\}
$$

with the property

$$
|F'(z)| \le \frac{C_F}{|z|} \quad \text{for } z \in \Omega_\beta^r.
$$

The constant C_F depends only on β, r and the function F. In the theorem below $\Omega_\beta = \Omega_\beta^1$ and $\widetilde{u}$ denotes the harmonic conjugate of u with $\widetilde{u}(\frac{1}{2}) = 0$.

Theorem 4.1.1. *Let $0 < \gamma < \beta < \pi$ and $\epsilon > 0$. Then there is a number $\delta(\epsilon) > 0$ such that for each $F \in \mathcal{B}(\Omega_\gamma^{1/2})$ there exists a harmonic function $u : \Omega_\beta \to \mathbb{R}$ with the properties*

(1) $|\mathfrak{Re}(F(x)) - u(x)| \le \epsilon$, for $x \in (0, \delta(\epsilon)]$;

(2) $|\tilde{u}(z)| \le C(\epsilon, \gamma, \beta, C_F) < \infty$, for $z \in \Omega_\beta$.

In [15] Contreras, Peláez, Pommerenke and Rättyä proved that the Volterra operator $T_g : H_{v_1}^\infty \to H^\infty$ is bounded if and only if g is a constant function, with this in mind and since $H_{v_\alpha}^\infty \subset H_{v_\beta}^\infty$, $\alpha \le \beta$ the only bounded Volterra operator $T_g : H_{v_\alpha}^\infty \to H^\infty$ when $\alpha \ge 1$ is the zero operator, as such the only interesting case left is when $0 \le \alpha < 1$. In paper I we showed that a similar condition to (4.1.1) characterizes boundedness of the Volterra operator $T_g : H_{v_\alpha}^\infty \to H^\infty$ for a univalent symbol $g \in H(\mathbb{D})$ and standard weights $v_\alpha(z) = (1 - |z|^2)^\alpha$.

Theorem 4.1.2. *If $g \in H(\mathbb{D})$ is univalent and $0 \le \alpha < 1$, then $T_g : H_{v_\alpha}^\infty \to H^\infty$ is bounded if and only if*

$$
\sup_{0\le\theta<2\pi}\int_0^1 \frac{|g'(re^{i\theta})|}{(1-r^2)^\alpha}dr < \infty.
$$

The proof of this theorem leans heavily on Theorem 4.1.1. In [6] the authors also studied compactness of the Volterra operator $T_g : H^\infty \to H^\infty$ and suggested the space

$$BRV_0 = \left\{ f \in H(\mathbb{D}) : \lim_{t \to 1^-} \sup_{0 \le \theta < 2\pi} \int_t^1 |f'(re^{i\theta})| dr = 0 \right\}$$

of functions analytic in the unit disc with derivative uniformly integrable on radii as a possible candidate for the set of such functions g. The main result of paper I is proving this conjecture in the affirmative when g is univalent. The result is as follows

Theorem 4.1.3. *If $g \in H(\mathbb{D})$ is univalent and $0 \le \alpha < 1$, then $T_g : H^\infty_{v_\alpha} \to H^\infty$ is compact if and only if*

$$\lim_{t \to 1^-} \sup_{0 \le \theta < 2\pi} \int_t^1 \frac{|g'(re^{i\theta})|}{(1-r^2)^\alpha} dr = 0.$$

Note that the proof of the conjecture by the authors of [6] appears as the $\alpha = 0$ case in the above theorem. The proof utilizes Theorem 4.1.1 as well as Lemma 2.5.3, where the last lemma is allowed to be used because of the following lemma.

Lemma 4.1.4. *Let $g \in H(\mathbb{D})$ and let φ be an analytic self-map of $\mathbb{D}$. Then $T_g^\varphi : H(\mathbb{D}) \to H(\mathbb{D})$ is co-co continuous.*

Proof. Begin by assuming that $f_n \to 0$ uniformly on compact subsets of $\mathbb{D}$. Choose such a compact subset $K \subset \mathbb{D}$. Since the image $\varphi(K)$ is contained in some closed disk $\overline{D(0, r_k)}$ for some $0 < r_k < 1$ we have that

$$\begin{aligned}\sup_{z \in K} |T_g^\varphi(f_n)(z)| = \sup_{z \in K} \left| \int_0^{\varphi(z)} f_n(\xi) g'(\xi) d\xi \right| &\le \sup_{z \in \varphi(K)} \int_0^z |f_n(\xi)||g'(\xi)| d|\xi| \\ &\le \sup_{\eta \in \overline{D(0,r_k)}} |f_n(\eta)| \sup_{z \in \varphi(K)} \int_0^z |g'(\xi)| d|\xi|.\end{aligned}$$

Now, since $f_n \to 0$ uniformly on compact sets, the first term in the last expression tends to zero. Therefore $T_g^\varphi(f_n) \to 0$ uniformly on compact sets. This shows that $T_g^\varphi : H(\mathbb{D}) \to H(\mathbb{D})$ is *co-co* continuous. □

We did not use this method to prove boundedness and compactness results for the Volterra operator $T_g : H^\infty_{v_\alpha} \to H^\infty_{v_\beta}$, where $\alpha \ge 1$ and $\beta > 0$ since much better results hold, see the next section and the papers [7], [15].

4.2 Boundedness, compactness and weak compactness results for T_g^φ

To see how the situation changes for the Volterra operator when the target space H^∞ changes to a weighted Banach space H^∞_v we also studied generalized Volterra operators of the type $T_g^\varphi : \mathcal{X} \to H^\infty_v$ and $T_g : \mathcal{X} \to B^\infty_v$, here $\mathcal{X}$ is a general Banach space satisfying the conditions outlined in section 2.4. In this case the differentiated Volterra operator $D \circ T_g^\varphi = (g \circ \varphi)' C_\varphi$ is a weighted composition operator, and as such we can apply results concerning weighted composition operators from [22] to get the following estimates of the norm and essential norm of T_g^{φ}, proved in paper I.

Theorem 4.2.1. *Let $\mathcal{X}$ be a Banach space of analytic functions on $\mathbb{D}$ satisfying condition (I) and assume that $\varphi(0) = 0$.*

(i) if the weight v is normal, then

$$\|T_g^{\varphi}\|_{\mathcal{X}\to H_v^{\infty}} \asymp \sup_{z\in\mathbb{D}}(1-|z|)v(z)|(g\circ\varphi)'(z)|\,\|\delta_{\varphi(z)}\|_{\mathcal{X}\to\mathbb{C}}.$$

(ii) For any weight v,

$$\|T_g^{\varphi}\|_{\mathcal{X}\to B_v^{\infty}} = \sup_{z\in\mathbb{D}} v(z)|(g\circ\varphi)'(z)|\,\|\delta_{\varphi(z)}\|_{\mathcal{X}\to\mathbb{C}}.$$

Theorem 4.2.2. *Let $\mathcal{X}$ be a Banach space of analytic functions on $\mathbb{D}$ satisfying conditions (I) - (IV) and assume that $\varphi(0) = 0$.*

(i) If the weight v is normal and $T_g^{\varphi} : \mathcal{X} \to H_v^{\infty}$ is bounded, then

$$\|T_g^{\varphi}\|_{e,\mathcal{X}\to H_v^{\infty}} \asymp \limsup_{|\varphi(z)|\to 1}(1-|z|)v(z)|(g\circ\varphi)'(z)|\,\|\delta_{\varphi(z)}\|_{\mathcal{X}\to\mathbb{C}}.$$

(ii) For any weight v, if $T_g^{\varphi} : \mathcal{X} \to B_v^{\infty}$ is bounded, then

$$\|T_g^{\varphi}\|_{e,\mathcal{X}\to B_v^{\infty}} \asymp \limsup_{|\varphi(z)|\to 1} v(z)|(g\circ\varphi)'(z)|\,\|\delta_{\varphi(z)}\|_{\mathcal{X}\to\mathbb{C}}.$$

The above norm estimates can for instance be applied to the weighted Bergman case by recalling that

$$\|\delta_z\|_{A_\alpha^p\to\mathbb{C}} = (1-|z|^2)^{\frac{-2-\alpha}{p}}$$

one then arrives at the expression

$$\|T_g^{\varphi}\|_{e,A_\alpha^p\to H_v^{\infty}} \asymp \limsup_{|\varphi(z)|\to 1} \frac{1-|z|}{(1-|\varphi(z)|)^{\frac{2+\alpha}{p}}} v(z)|(g\circ\varphi)'(z)|.$$

By using the essential norm estimates above we are also able to relate compactness of $T_g^{\varphi} : \mathcal{X} \to H_v^{\infty}$ to compactness of $T_g^{\varphi} : \mathcal{X} \to H_v^0$, and a corresponding result for Bloch-type spaces, in the following way

Theorem 4.2.3. *Let $\mathcal{X}$ be a Banach space of analytic functions and assume that $\varphi(0) = 0$.*

(i) If the space $\mathcal{X}$ satisfies conditions (I) -(IV) and the weight v is normal, then $T_g^{\varphi} : \mathcal{X} \to H_v^{\infty}$ is compact if and only if $T_g^{\varphi} : \mathcal{X} \to H_v^0$ is compact.

(ii) If the space $\mathcal{X}$ satisfies conditions (I) and (IV), then for any weight v, $T_g^{\varphi} : \mathcal{X} \to B_v^0$ is compact if and only if $T_g^{\varphi} : \mathcal{X} \to B_v^{\infty}$ is compact and $g\circ\varphi \in B_v^0$.

We also had some results showing that compactness and weak compactness coincide for operators of the type $T : H_v^\infty \to \mathcal{Y}$ and $T : B_v^\infty \to \mathcal{Y}$, where $\mathcal{Y} \subset H(\mathbb{D})$ is a Banach space. The theorem is as follows

Theorem 4.2.4. *Let v be a normal weight and assume that the Banach space $\mathcal{Y} \subset H(\mathbb{D})$ satisfies condition (I).*

(i) If the restriction $T\big|_{B_{H_v^\infty}}$ is co-co continuous then $T : H_v^\infty \to \mathcal{Y}$ is compact if and only if it is weakly compact.

(ii) If the restriction $T\big|_{B_{B_v^\infty}}$ is co-co continuous then $T : B_v^\infty \to \mathcal{Y}$ is compact if and only if it is weakly compact.

The above theorem can be applied to the generalized Volterra operator T_g^φ by Lemma 4.1.4.

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