Neea Palojärvi

Explicit Results in Theory of Zeros and Prime Numbers

2   2   3   5   7   5   7   11

$$L(s,\chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}$$

7   11

17   17   19   23   23

13

19

31   37   37   43   43   31

29    $$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$ 47

47   61   61

47   41   47   59

67   67   53   79

71

59   67

73   83   89   97   97

71   79

83    $$\lambda(n,\tau) = \lim_{t\to\infty} \sum_{|\mathfrak{I}\rho|\leq t} \left(1 - \left(\frac{\rho}{\rho-\tau}\right)^n\right)$$

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Explicit results in theory of zeros and prime numbers

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Abstract

This thesis is comprised of three articles in which we prove explicit estimates for different number theoretical problems.

In the first article we derive an explicit Riemann-von Mangoldt formula for the Selberg class functions. Even though this kind results were already known for some functions in the Selberg class, this is the first explicit estimate which covers the whole Selberg class.

Then, in the second article, we study the τ-Li coefficients for a quite general set of functions and their connections to zero-free regions. We prove that non-negativity of certain τ-Li coefficients lead to certain zero-free regions. Moreover, we also show that negativity of certain τ-Li coefficients leads to existence of certain zeros. This is a continuation of F. C. Brown's and A. D. Droll's work with similar type of problems.

Lastly, in the third article, we prove an explicit estimate for the number of primes in arithmetic progressions assuming the generalized Riemann hypothesis. This sharpens and generalizes earlier results proved without assuming the generalized Riemann hypothesis or which apply only to some arithmetic progressions.

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Sammanfattning

Den här avhandlingen består av tre artiklar i vilka vi bevisar explicita uppskattningar för olika talteoretiska problem.

I den första artikeln härleder vi en explicit Riemann–von Mangoldt-formel för funktioner i Selbergklassen. Även om sådana resultat varit kända för några funktioner i Selbergklassen, är vårt resultat det första som behandlar hela Selbergklassen.

I den andra artikeln undersöker vi τ-Li koefficienterna för en allmän mängd och koefficienternas relationer till områden där det inte finns nollställen. Vi bevisar att om vissa τ-Li koefficienter är icke-negativa, så då existerar vissa områden utan nollställen. Vi bevisar också att om vissa τ-Li koefficienter är negativa, så då finns det minst ett nollställe som ligger i ett visst område. Den här undersökningen är en fortsättning på F. C. Browns och A. D. Drolls forskning på liknande problem.

Slutligen, i den tredje artikeln, bevisar vi en explicit uppskattning för antalet primtal i aritmetiska talföljder om vi antar den generaliserade Riemannhypotesen. Det här resultatet skärper och generaliserar tidigare resultat som har bevisats utan att anta den generaliserade Riemannhypotesen eller bara för några aritmetiska talföljder.

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Åbo, July 2020
Neea Palojärvi

List of original publications

This thesis consists of an introductory part and the following three articles:

[A] Neea Palojärvi, *On the explicit upper and lower bounds for the number of zeros of the Selberg class*, Journal of Number Theory, Vol. 194 (2019), 218-250.

[B] Neea Palojärvi, *Explicit zero-free regions and a τ-Li-type criterion*, Albanian Journal of Mathematics, Vol. 14, No. 1 (2020), 47-77.

[C]*Anne-Maria Ernvall-Hytönen and Neea Palojärvi, *Explicit bound for the number of primes in arithmetic progressions assuming the Generalized Riemann Hypothesis*. Preprint available at arXiv:2003.01925.

Articles [A] and [B] are reprinted with the permission of their respective copyright holders.

* Both authors contributed for deriving the technical details and writing the article. The second author's contribution was about 60%.

Contents

Notation

Here we collect the common notation used in Chapters 1–4. Articles [A], [B] and [C] have their own notations.

Letters

n, q	positive integers
p	a prime number
T, t, x	positive real numbers
ϵ	a (small) positive real number
s, z	complex numbers
ρ	a zero of a function
χ	a Dircihlet character modulo q
χ_0	a principal character modulo q

Some functions

$\Gamma(s)$	the gamma function
$\Lambda(n)$	the von Mangoldt function
$\zeta(s)$	the Riemann zeta function
$L(s, \chi)$	a Dirichlet L-function associated with a character χ

Functions closely related to the number of primes

$\pi(x)$ the number of primes up to x

$$\pi(x; q, a) := \sum_{\substack{p \le x \\ p \equiv a \bmod q}} 1$$

$$\psi(x) := \sum_{n \le x} \Lambda(n)$$

$$\psi(x, \chi) := \sum_{n \le x} \chi(n)\Lambda(n)$$

$$\psi_0(x, \chi) := \begin{cases} \psi(x, \chi) - \frac{1}{2}\Lambda(x)\chi(x) & \text{if } x \text{ is a prime power} \\ \psi(x, \chi) & \text{otherwise} \end{cases}$$

$$\psi(x; q, a) := \sum_{\substack{n \le x \\ n \equiv a \bmod q}} \Lambda(n)$$

$$\theta(x; q, a) := \sum_{\substack{p \le x \\ p \equiv a \bmod q}} \log p$$

$$\mathrm{Li}(x) := \int_2^x \frac{dt}{\log t}$$

$$\mathrm{li}(x) := \lim_{\epsilon \to 0+} \int_0^{1-\epsilon} \frac{dt}{\log t} + \int_{1+\epsilon}^x \frac{dt}{\log t}$$

Asymptotics

$$
\begin{array}{ll}
f(x) \ll g(x) & |f(x)| \leq C|g(x)| \text{ for some } C \text{ and all } x \text{ large enough} \\
f(x) \ll_\epsilon g(x) & f(x) \ll g(x) \text{ where the constant } C \text{ depends on } \epsilon \\
f(x) = \mathscr{O}(g(x)) & f(x) \ll g(x) \\
f(x) = o(g(x)) & |f(x)| \leq \epsilon|g(x)| \text{ for every } \epsilon > 0 \text{ and } x \text{ large enough} \\
f(x) = \Omega(g(x)) & f(x) \geq c|g(x)| \text{ for some } c \text{ and all } x \text{ large enough}
\end{array}
$$

Note. In some cases we have more than one variable when we are using asymptotic notation. In these cases the variables depend on each other and their dependence is clarified.

Number of the zeros

$$
\begin{array}{ll}
N_F^+(t) & := |\{\rho : F(\rho) = 0, 0 \leq \Re(\rho) \leq 1, 0 \leq \Im(\rho) \leq t\}| \\
N_F^-(t) & := |\{\rho : F(\rho) = 0, 0 \leq \Re(\rho) \leq 1, -t \leq \Im(\rho) \leq 0\}| \\
N_F(t) & := |\{\rho : F(\rho) = 0, 0 \leq \Re(\rho) \leq \tau, |\Im(\rho)| \leq t\}| \\
N_F^+(t_1, t_2) & := |\{\rho : F(\rho) = 0, 0 \leq \Re(\rho) \leq 1, t_1 < \Im(\rho) \leq t_2\}| \\
N_F^-(t_1, t_2) & := |\{\rho : F(\rho) = 0, 0 \leq \Re(\rho) \leq 1, -t_2 \leq \Im(\rho) < -t_1\}| \\
N_F(t_1, t_2) & := |\{\rho : F(\rho) = 0, 0 \leq \Re(\rho) \leq \tau, t_1 < |\Im(\rho)| \leq t_2\}|
\end{array}
$$

Abbreviations

RH   the Riemann hypothesis
GRH   the generalized Riemann hypothesis
PNT   the prime number theorem

1 Introduction

We call an estimate *explicit* if there are no unknown constants in the error term. In the contrast to non-explicit estimates, the explicit ones give us actual numerical values which can be used in computations. Thus they also reveal much more information than the non-explicit ones. For example it is much more effective to say that the number of primes up to 10000 is between 1087 and 1407 [76, Theorem 2] than that it is some thousands.

In this thesis, we discuss explicit results for three different number theoretical problems. In this section, we outline the basic ideas and connections of the topics. More specific introductions to the topics can be seen in Chapters 2–4.

The first problem (see Chapter 2, article [A]) is to estimate the number of the zeros in certain rectangular regions on the complex plane for the functions in the so called *Selberg class* [69]. The class consists of functions which have somehow similar properties with the very well-known *Riemann zeta function* $\zeta(s)$. It is a function which is defined as

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \qquad \text{for} \quad \Re(s) > 1$$

and can be extended to the whole complex plane.

The Riemann zeta function was already studied in the 18th century by L. Euler (see e.g. English translations [26, 27]) and in the 19th century G. F. B. Riemann [63] continued the work by extending Euler's definition to complex variables, establishing its relation to prime numbers and studying many of its technical details. He also proposed so called the *Riemann hypothesis* (RH). The hypothesis still remains unsolved and states the following:

Conjecture 1.1 (Riemann Hypothesis) *All zeros of the Riemann zeta function with $0 < \Re(s) < 1$ lie on the line $\Re(s) = \frac{1}{2}$.*

Similar claims are also conjectured to hold for a more general set of functions, called as *Dirichlet L-functions* (see [5, Section 6.2]). For $\Re(s) > 1$ they are defined as series

$$L(s, \chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} \qquad \text{for} \quad \Re(s) > 1,$$

where $\chi(n)$ is a Dirichlet character, and the function can be extended to the whole complex plane. Please notice that the Riemann zeta function

is also a Dirichlet L-function with $\chi(n) = 1$ for all n. Now the *generalized Riemann hypothesis* (GRH) states the following:

Conjecture 1.2 (Generalized Riemann Hypothesis) *Assume that s is not a negative real number. Then, if $L(s,\chi) = 0$, we have $\Re(s) = \frac{1}{2}$.*

Since the Riemann zeta function is also a Dirichlet L-function, if the generalized Riemann hypothesis holds, then also the Riemann hypothesis holds.

As we have noticed, we do not know whether the Riemann hypothesis or the generalized Riemann hypothesis holds or not. But, to make the question a little bit simpler, could we instead make some conclusions about zero-free regions for some functions? Or, could we at least prove some criteria for them?

In the second problem (see Chapter 3, article [B]) we study one this kind of criterion. We consider the functions whose number of zeros satisfies similar formulas as we derived for the Selberg class to answer the first problem. The idea is to apply these formulas and show that certain values of so called τ-*Li coefficients* (see e.g. [28]) lead to certain zero-free regions or zeros in certain regions. The τ-Li coefficients, generalized from so called *Li coefficients* [41], consist of an infinite sequence of certain numbers. It is known [28] that if all of them are non-negative, then the Riemann zeta function has no zeros with $\Re(s) > \tau/2$. These kind of results have also been proved for other functions.

The zero-free regions also lead us to the third problem (see Chapter 4, article [C]). It is about the number of primes in arithmetic progressions assuming the generalized Riemann hypothesis. Knowledge about the zero-free regions for the Dirichlet L-functions or assuming the generalized Riemann hypothesis give sharper estimates for the number of primes in arithmetic progressions [14, Chapter 20].

Furthermore, this kind of property is also known for the Selberg class —the class we considered in the first problem. Let us first underlay the topic a little bit by saying a couple of words about the prime counting function. The *prime number theorem* (PNT) tells that the number of primes up to x is essentially about $x/\log x$ [15, 30]. The result can be derived by showing that the function $\psi(x) = \sum_{n \leq x} \Lambda(n)$, where $\Lambda(n)$ is the von Mangoldt function, is essentially about a size x for large values x. We know the following equivalence between this theorem and the Riemann zeta function (see e.g. [49, Chapter 8]):

Theorem 1.3 *PNT is equivalent to that $\zeta(1 + it) \neq 0$ for all real numbers t.*

Now we are ready to move on to the Selberg class connection. Let us

define the *generalized von Mangoldt function* $\Lambda_F(n)$ by

$$-\frac{F'(s)}{F(s)} = \sum_{n=1}^{\infty} \frac{\Lambda_F(n)}{n^s}$$

and $\psi_F(x) = \sum_{n \le x} \Lambda_F(n)$, where the function F is a function in the Selberg class. Please now notice that we can write

$$-\frac{\zeta'(s)}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s}$$

and thus the definition of the function $\psi_F(x)$ coincides with the definition of the function $\psi(x)$ when $F = \zeta$. Further, it is conjectured, called as the *prime number theorem for the Selberg class*, that $\psi_F(x) \sim kx$, where k is the order of the function F at $s = 1$. Keeping in mind the definitions of the functions $\psi_F(x), \psi(x)$ and the connection between the function $\psi(x)$ and the PNT, this is a very natural generalization of the PNT. Now, by [36], we know the following connection between the PNT and the zeros of a function F in the Selberg class:

Theorem 1.4 *Let a function F be in the Selberg class. The PNT defined for the Selberg class function F is true if and only if $F(1 + it) \ne 0$ for all real numbers t.*

Theorem 1.4 is a generalization of Theorem 1.3. Thus we have seen some very surprising connections between the number of primes and the zeros of some functions.

   As we see, the three problems under the consideration are beautifully connected to each other. Using the formulas for the number of the zeros, we can determine zero-free regions for the functions and estimate the number of (certain) primes. Furthermore, the knowledge about the zero-free regions of the Riemann zeta function or the Dirichlet L-functions also leads to better estimates for the number of primes and the number of primes in arithmetic progressions.

   For further reading, we suggest to take a look at the book [5] which gives a comprehensive introduction to the world of the Riemann zeta function and related problems. A reader may also enjoy A. W. Dudek's PhD thesis [18] in which he proves explicit estimates for problems connected to prime numbers. Other references mentioned in this chapter may be interesting but most of them require much stronger mathematical background than this introduction.

   This thesis is organized as follows: Chapters 2, 3 and 4 correspond to articles [A], [B] and [C] respectively. At the beginning of the each chapter, there is an introduction to the topic. After that the main results of the article are described followed by the sketches of the proofs. At the end of the each chapter, we have a short discussion of the topic.

2 On the Selberg class and the number of the zeros

In article [A], we study an explicit Riemann-von Mangoldt formula for the functions in the Selberg class. It estimates the number of the zeros of the functions in the Selberg class. Thus we start with an introduction to the Selberg class and the Riemann-von Mangoldt formula.

2.1 The Selberg class

The *Selberg class S* was introduced by A. Selberg [69] in 1989. It consists of functions $F(s) = \sum_{n=1}^{\infty} \frac{a(n)}{n^s}$, where the terms $a(n)$ are complex numbers and the series expansion is absolutely convergent for $\Re(s) > 1$, and which satisfies the following four conditions:

(a) **Ramanujan hypothesis:** For any $\epsilon > 0$ we have $a(n) \ll_{\epsilon} n^{\epsilon}$.

(b) **Analytic continuation:** There is an integer $k \geq 0$ such that

$$(s-1)^k F(s)$$

is an entire function of a finite order.

(c) **Functional equation:** We have

$$\phi_F(s) = \omega \overline{\phi_F(1-\bar{s})}$$

where

$$\phi_F(s) = F(s) Q^s \prod_{j=1}^{f} \Gamma(\lambda_j s + \mu_j)$$

and

$$f, j \in \mathbb{Z}_+, \quad Q \in \mathbb{R}, \quad \lambda_j \in \mathbb{R}_+, \quad \omega, \mu_j, d_F, \lambda \in \mathbb{C}$$

such that

$$j \in [1, f], \quad d_F = 2 \sum_{j=1}^{f} \lambda_j, \quad \lambda = \prod_{j=1}^{f} \lambda_j^{2\lambda_j}, \quad |\omega| = 1, \quad \Re(\mu_j) \geq 0.$$

(d) **Euler product:** We have

$$F(s) = \prod_p F_p(s),$$

where

$$F_p(s) = \exp\left(\sum_{l=1}^{\infty} \frac{b(p^l)}{p^{ls}} \right)$$

with coefficients $b(p^l)$ satisfying $b(p^l) \ll p^{l\theta}$ for some real number $\theta < \frac{1}{2}$.

One obvious example of the functions in the Selberg class is the Riemann zeta function. The Dirichlet L-functions associated with primitive characters are also contained in the Selberg class. It is expected, but not known, that all functions $F \in S$ are automorphic L-functions.

Furthermore, it is worth noticing that the data coming from the functional equation is not unique. Instead, it can be proved that the term d_F, called *degree*, is unique for the function F (see e.g. [50, p. 119]). This also means that the error terms described in Theorems 2.3 and 2.5 and Corollary 2.4 are not unique even though the main terms are and the results are explicit.

Let us now move on from the axioms to the problems related to the Selberg class. It turned out that there are many interesting problems in relation the Selberg class. What degrees d_F can the functions have? If we select a certain degree d_F, which functions have this degree? Even though the parameters λ_j are not unique, is it always possible, no matter which function is under the consideration, to find some certain kind of terms λ_j e.g. $\lambda_j = \frac{1}{2}$ for some j? What other invariants are there? Can we prove that there exists an integer n_F associated to F such that

$$\sum_{p \leq x} \frac{|a(p)|^2}{p} = n_F \log \log x + \mathcal{O}(1)?$$

Furthermore, knowing that the Riemann zeta function is in the Selberg class, one may ask whether the functions satisfy the Riemann hypothesis or not. Already Selberg [69] himself conjectured this:

Conjecture 2.1 *Let $F(s) \in S$. Then $F(s) \neq 0$ if $\Re(s) > \frac{1}{2}$.*

It is not known whether the functions $F \in S$ satisfy Conjecture 2.1 or not. Instead, there are several examples about functions violating some of the axioms (a)–(d) and not satisfying Conjecture 2.1. For example, without assuming the Euler product, we can consider the function

$$\left(1 - 2^{1-s}\right) \zeta(s) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s}.$$

It clearly violates Conjecture 2.1 since it has a zero at $s = 1 + 2\pi i / \log 2$. For more examples, see [12].

Let us now take a closer look at the zeros of the function $F \in S$. According to the Euler product the function $F \in S$ does not have zeros ρ with $\Re(\rho) > 1$. Thus and by the functional equation, the zeros $\Re(\rho) \leq 0$ are the poles of the factors $\Gamma(\lambda_j s + \mu_j)$. Since the gamma function has poles at non-positive integers, the zeros with non-positive real parts are

$\rho = -\frac{m+\mu_j}{\lambda_j}$ where $m = 0, 1, 2, \ldots$. These zeros are called *trivial zeros*. They are well-known and hence we are interested in *non-trivial zeros* meaning the zeros which do not come from the poles of the gamma function and thus their real parts are in the interval $[0, 1]$. Notice that according to the definitions, the function may have a trivial and a non-trivial zero at the same point.

In the next section, we continue with the non-trivial zeros of the Selberg class functions. There we take a closer look at their number up to some height T on the complex plane.

This introduction to the Selberg class is based on the previous references and literature [34, 35, 59, 60]. Especially the latest references also provide an interesting survey of the Selberg class.

2.2 The Riemann-von Mangoldt formula

The *Riemann-von Mangoldt formula* estimates the number of the non-trivial zeros of the Riemann zeta function with imaginary parts at least zero and at most T:

Theorem 2.2 (Riemann-von Mangoldt formula) *Let T be a positive real number and $N(T) := |\{\rho : 0 < \Re(\rho) < 1, 0 < \Im(\rho) \leq T\}|$. Then we have*

$$N(T) = \frac{T}{2\pi} \log \frac{T}{2\pi} - \frac{T}{2\pi} + \mathcal{O}(\log T).$$

Theorem 2.2 was conjectured by G. F. B. Riemann [63] and proved by H. von Mangoldt who also proved an explicit version of the formula [78, 79]. The explicit formula has been improved several times (e.g. [1, 65, 74]). For the latest improvement, see [74].

Since the non-trivial zeros of the Riemann zeta function lie symmetrically with respect to the real axis [5, Theorem 2.6], the Riemann-von Mangoldt formula also gives an estimate for the number of all non-trivial zeros of the Riemann zeta function. Thus we know a quite sharp estimate for (at least) one function in the Selberg class. Let us now take a look at the whole Selberg class.

By J. Steuding [71] the number of the non-trivial zeros of the function $F \in S$ up to height T is

$$\frac{d_F}{2\pi} T \log \frac{T}{e} + \frac{T}{2\pi} \log(\lambda Q^2) + \mathcal{O}(\log T). \tag{2.1}$$

A little bit more precise version of this formula is given in [70]. In addition to the Riemann zeta function, there are also explicit results for the number of the non-trivial zeros of the Dirichlet *L*-functions and the Dedekind zeta function [46, 75]. Explicit results concerning the whole Selberg class are described in the next section.

2.3   Results

In article [A], we prove explicit formulas of type (2.1) for the Selberg class. Due to the length of the results, we do not write the explicit constants here in full length. Instead, we refer to article [A].

Theorem 2.3 (Article [A]) *Let $F \in S$ and T_0, T be positive real numbers which are large enough and explicitly given in Lemma 5.1 in article [A]. Then*

$$\left| N_F^{\pm}(T_0, T) - \frac{d_F}{2\pi} T \log \frac{T}{e} - \frac{T}{2\pi} \log(\lambda Q^2) \right|$$
$$< c_{F,1} \log T + c_{F,2}(T_0) + \frac{c_{F,3}(T_0)}{T}$$

where the terms $c_{F,1}$, $c_{F,j}(T_0)$ ($j = 2, 3$) are explicitly given in Theorem 5.2 in article [A]. They depend on the function F and the terms $c_{F,j}(T_0)$ also on the number T_0.

As we already mentioned in Section 2.1, the error terms in Theorem 2.3 are not unique. Furthermore, Theorem 2.3 gives an estimate for a large set of different functions and is not very sharp.

By formula (2.1) we can conclude that there exists only finitely many zeros up to height T_0. Thus we can also conclude the following:

Corollary 2.4 (Article [A]) *Let $F \in S$ and T_0, T be positive real numbers which are large enough and explicitly given in Lemma 5.1 in article [A]. Then*

$$\left| N_F^{\pm}(T) - \frac{d_F}{2\pi} T \log \frac{T}{e} - \frac{T}{2\pi} \log(\lambda Q^2) \right|$$
$$< c_{F,1} \log T + C_{F,2}(T_0) + \frac{c_{F,3}(T_0)}{T}$$

where the terms $c_{F,1}$, $c_{F,3}(T_0)$ are explicitly given in Theorem 5.2 and $C_{F,2}(T_0)$ in Corollary 5.3 in article [A].

In addition to Theorem 2.3 and Corollary 2.4, the following bound is also mentioned in article [A]:

Theorem 2.5 (Article [A]) *Let $F \in S$ and T be a positive real number which is large enough and explicitly given in Theorem 4.2 in article [A]. Then*

$$\left| N_F^{\pm}(T, 2T) - \frac{d_F}{2\pi} T \log \frac{T}{e} - \frac{T}{2\pi} \log(\lambda Q^2) \right|$$
$$< c_1 \log T + c_2 + \frac{c_3}{T}$$

*where the terms c_j are explicitly given in Remark 5.4 in article [A]. They depend on the function F but do **not** depend on the term T.*

Theorem 2.5 allows us to consider the number of the zeros in certain strips without the dependence of the term T_0 which is needed in Theorem 2.3 and Corollary 2.4. It is worth noticing that Theorem 2.3 does not straightly follow from Theorem 2.5 as we will next see. When we compute the number of the zeros with imaginary parts in $(T_0, T]$, we have an error term

$$\sum_{n=0}^{\log_2\left(\frac{T}{T_0}\right)} \log\left(2^n T_0\right) = \sum_{n=0}^{\log_2\left(\frac{T}{T_0}\right)} \Omega\left(n\right) = \Omega\left((\log T)^2\right).$$

This is bigger than the wanted error term $\mathscr{O}(\log T)$.

In addition to Theorems 2.3 and 2.5 and Corollary 2.4, numerical examples of the constants $c_{F,1}$, $c_{F,j}(T_0)$ $(j = 1, 2)$, $C_{F,2}(T_0)$, and c_j $(j = 1, 2, 3)$ for L-functions associated with holomorphic newforms are provided at the end of article [A] (see Table 1).

2.4 Main steps of the proofs

Theorems 2.3 and 2.5 and Corollary 2.4 can be proved using the same steps. Thus, in this section, we describe the steps only for the proof of Theorem 2.3. The proof follows the same steps as the proof for the asymptotic case described in article [71] and in book [72, Chapters 6, 7]. The main idea is to estimate the number of the zeros using certain integrals and then estimate the integrals.

First we recognize that the functions $F(\bar{s})$ and $\overline{F(\bar{s})} = \sum_{n=1}^{\infty} \frac{\overline{a(n)}}{n^s}$ have the same zeros and thus we need to only consider the cases where $\Im(s) > 0$. As a starting point, we apply Littlewood's lemma (see [72, Chapter 7.1] or Littlewood's original paper [42]) which connects zeros of a function to a contour integral:

Theorem 2.6 (Littlewood's lemma) *Let A and B be real numbers, $A < B$ and $f(s)$ be an analytic function on the rectangle*

$$\mathscr{R} := \{s \in \mathbb{C} : A \leq \sigma \leq B, |t| \leq T\}.$$

Assume that the function $f(s)$ does not vanish on the right edge $\sigma = B$. Furthermore, let $\mathscr{R}'$ be the rectangle $\mathscr{R}$ minus the union of horizontal cuts from the zeros of the function f in $\mathscr{R}$ to the left edge of $\mathscr{R}$, and choose a single-valued branch of $\log(f(s))$ in the interior of $\mathscr{R}'$. Denote by $\nu(\sigma, T)$ the number of the zeros $\rho = \beta + i\gamma$ of the function $f(s)$ inside the rectangle with $\beta > \sigma$ including the zeros with $\gamma = T$ but not those with $\gamma = -T$. Then

$$\int_{\partial\mathscr{R}} \log f(s)ds = -2\pi i \int_A^B \nu(\sigma, T)d\sigma.$$

By the assumption of the analytic continuation and the Euler product, the only possible poles of the function are at $s = 1$ and the function $F(s)$ does not vanish for $\Re(s) > 1$. Thus we apply Littlewood's lemma for the function $F(s)$ and a rectangle $\mathscr{R}$ with the vertices $B + iT_0$, $B + iT$, $A + iT$ and $A + iT_0$. Here the numbers $B > 1$, $T_0 > 1$ and $T > T_0$ are large enough and the number $A < 0$ small enough. More detailed version of the selection of the constants is given in article [A].

Subtracting the formula containing A from the formula containing $A + 1$ we get to the following formula:

$$
2\pi N_F^+(T_0, T) + 2\pi \sum_{\substack{T_0 < \Im(\rho) \leq T \\ 0 > \Re(\rho) > A+1}} 1 + 2\pi \sum_{\substack{T_0 < \Im(\rho) \leq T \\ A+1 \geq \Re(\rho) > A}} (\Re(\rho) - A)
$$

$$
= \int_{T_0}^{T} \left(\log |F(A + it)| - \log |F(A + 1 + it)| \right) dt
$$

$$
- \int_{A}^{B} \arg F(\sigma + iT_0) d\sigma + \int_{A+1}^{B} \arg F(\sigma + iT_0) d\sigma \qquad (2.2)
$$

$$
+ \int_{A}^{B} \arg F(\sigma + iT) d\sigma - \int_{A+1}^{B} \arg F(\sigma + iT) d\sigma
$$

$$
:= I_1(T_0, T, A) - I_1(T_0, T, A + 1) - I_2(T_0, A, B)
$$

$$
+ I_2(T_0, A + 1, B) + I_2(T, A, B) - I_2(T, A + 1, B).
$$

Since we know the locations of the non-trivial zeros, the last two terms on the left-hand side of formula (2.2) can be estimated easily. Thus, it is sufficient to estimate the integrals defined on the right-hand side of formula (2.2).

First we look at the difference $I_1(T_0, T, A) - I_1(T_0, T, A + 1)$**.** The function $F(s)$, and thus also the function $\log F(s)$, can be easily estimated when $\Re(s) > 1$ is large enough by using the series expansion. Applying the functional equation, we reach the case where it is sufficient to estimate the integral of the difference of the terms $\log F(s)$, where $\Re(s)$ is large enough, and the terms which depend on certain products related to the gamma function. The first difference is easily estimated using the Taylor series expansion for the logarithm. For the second difference, we apply the following estimate for the logarithm of the gamma function (see [73, paragraph 9]):

Lemma 2.7 *Let z be a complex number such that $|\arg(z)| < \pi$. Then we have*

$$
\log \Gamma(z) = z \log z - z + \frac{1}{2} \log \frac{2\pi}{z} + E(z),
$$

where $E(z)$ is a holomorphic function satisfying

$$
|E(z)| \leq \frac{1}{12\,|z|} \sec^2 \left(\frac{\arg z}{2} \right).
$$

These estimates produce the main term $\frac{d_F}{2\pi} T \log \frac{T}{e} + \frac{T}{2\pi} \log(\lambda Q^2)$ and explicit error terms of sizes $\mathscr{O}(\log T)$ and $\mathscr{O}(1)$.

Now we estimate the integrals I_2 defined on the right-hand side of formula (2.2). First, we recognize that if the term $\Re\left(F(\sigma + it)\right)$ has N zeros with $\sigma \in [a, b]$, then

$$\left| \int_a^b \arg F(\sigma + it) \right| \leq (N + 1)(b - a)\pi.$$

Thus the goal is to estimate the term N.

We would like to estimate the term N in such a way that it is connected to the function $F(s)$ and has some formula which can be (easily) estimated. Thus, let

$$g(z) = \frac{1}{2}\left(F(z + it) + \overline{F(\bar{z} + it)} \right)$$

and let $n(r)$ denote the number of the zeros of the function $g(z)$ with $|z - b| \leq r$. For real numbers σ we have $g(\sigma) = \Re\left(F(\sigma + it)\right)$. Thus $N \leq n(b - a)$. By Jensen's formula [33] we have

$$n(b - a) \log 2 \leq \int_0^{2(b-a)} \frac{n(r)}{r} dr$$
$$= \frac{1}{2\pi} \int_0^{2\pi} \log|g(b + 2(b - a)e^{i\theta})| d\theta - \log|g(b)|.$$

The term $\log|g(b)|$, where the number $b > 1$ is large enough, can be estimated using the series expansion for the term $F(s)$.

The integral of the term $\log|g(b + 2(b - a)e^{i\theta})|$ is a little bit more complicated to estimate since the real part $\Re(b + 2(b - a)e^{i\theta})$ is not necessarily big enough for making the trivial estimates. Thus we apply the Phragmén-Lindelöf principle for a strip [32, Theorem 5.53]. It tells us an estimate for a function inside a strip if we can estimate the function on the left and right edges of the strip. Since by the functional equation we can easily estimate the function $F(s)$ with small and large real parts of s, we obtain the wanted result by the Phragmén-Lindelöf principle. Thus we have estimated the term $n(a - b)$ and hence also the integrals I_2. These estimates produce explicit error terms of sizes $\mathscr{O}(\log T)$, $\mathscr{O}(1)$ and $\mathscr{O}(1/T)$. Together with the estimates for the term $I_1(T_0, T, A) - I_1(T_0, T, A + 1)$, we have obtained the results described in Section 2.3.

The main differences between the proof of Steuding's asymptotic result and the explicit results described here are that we refer to explicit results instead of the asymptotic ones and some steps are done

more precisely. For example, instead of using the asymptotic version of Stirling's formula, we apply the explicit one mentioned in Lemma 2.7. Also, for instance, obtaining an explicit estimate for the integral of the term $\log |g(b + 2(b - a)e^{i\theta})|$ needs a little bit more careful work and more case by case work than the asymptotic one.

More detailed proofs can be found in article [A].

2.5 Discussion

In article [A], we derive an explicit Riemann-von Mangoldt type formula for the Selberg class. The main steps of the proofs are described in Section 2.4.

Since the results are quite general, the coefficients described in Theorems 2.3–2.5 are probably not the most optimal ones. This can be already seen by comparing the numerical results obtained from Theorems 2.3–2.5 to the known coefficients for the Riemann zeta function [74]. One possible way to improve the results may be to apply the so called *Backlund's trick* [1] which has been successfully applied for the Riemann zeta function [1, 74], the Dedekind zeta functions and the Dirichlet *L*-functions [75] and the Hecke-Landau zeta-function [29]. The idea of the trick is to show that if there exist zeros of $\Re\left(F(\sigma + iT)^N\right)$ with $\sigma \in [\frac{1}{2}, \sigma_1]$, then there exist zeros of $\Re\left(F(\sigma + iT)^N\right)$ with $\sigma \in [1 - \sigma_1, \frac{1}{2}]$.

In addition to the Backlund's trick, proving estimates for some specific functions in the Selberg class also leads to sharper results. In Section 2.2 we already mentioned these kind of the results.

There are plenty of applications of the formula for the number of the zeros. Already in Chapter 1 we mentioned connections between zero-free regions and the number of the zeros in addition to their connections to PNT. One example of the first connection is discussed in Chapter 3 and an example of the second one in Chapter 4.

Furthermore, the number of the zeros can also be used to determine the locations of the zeros. Let us define $S_F(T) = \frac{1}{\pi} \arg F(1/2 + iT)$. The term $S_F(T)$ is closely related to the number of the zeros since by [70, page 838] we have an asymptotic formula

$$N_F^\pm(T) = \frac{d_F}{2\pi} T \log \frac{T}{e} + \frac{T}{2\pi} \log(\lambda Q^2) + C_1 \log T + S_F(T) + \mathcal{O}(1/T).$$

Similarly as in the case of the Riemann zeta function (see e.g. [64, 65]), we can determine the locations of the zeros using the term $S_F(T)$. It also plays an important part in Turing's method [77] to determine the zeros. Thus, proving good, explicit bounds for the term $S_F(T)$ would be an interesting research question. Moreover, it is known that assuming the RH we obtain a sharper estimate for the term $S_F(T)$ coming from

the Riemann zeta function [42, 74]. Hence, it would be interesting to investigate similar results for the Selberg class.

3 On explicit $\mathcal{T}$-Li type criteria

In article [B], we study the τ-Li coefficients and their connections to explicit zero-free regions. We start with an introduction to the Li and the τ-Li coefficients.

3.1 On Li and $\mathcal{T}$-Li coefficients

In 1997 X.-J. Li [41] proved that the RH is equivalent to that each of the terms

$$\lambda_n = \frac{1}{(n-1)!} \frac{d^n}{ds^n} \left[s^{n-1} \log \left(s(s-1)\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s)\right)\right]_{s=1},$$

where n is a positive integer, is non-negative. This criterion is called *Li's criterion* and the terms λ_n are *Li coefficients*. It is worth noticing that the same kind of constants were already introduced by J. B. Keiper [37] in 1992. In addition to the Li's criterion, Li also proved that the Li coefficients can be written as

$$\lambda_n = \sum_\rho \left(1 - \left(1 - \frac{1}{\rho}\right)^n\right), \qquad n \in \mathbb{Z}_+, \tag{3.1}$$

where the sum runs over the non-trivial zeros of the Riemann zeta function.

The Li's criterion raises a couple of questions:

1. Could we also prove similar results for the Li coefficients if the terms $\Re(\rho)$ satisfy other conditions than the condition $\Re(\rho) = 1/2$? E.g. could we look generally at the zeros ρ with $\Re(\rho) > 1/2$?

2. Are there generalizations for other functions?

3. What can be deduced considering only finitely many of the Li coefficients?

The first question was answered by P. Freitas [28]. He proved that all zeros of the Riemann zeta function satisfy the condition $\Re(s) \leq \tau/2$, where $\tau \in [0.5, \infty)$, if and only if the numbers

$$\frac{1}{\tau}\sum_\rho \left(1 - \left(\frac{\rho}{\rho - \tau}\right)^n\right), \qquad n \in \mathbb{Z}_+, \tag{3.2}$$

are non-negative. Here the sum runs over the non-trivial zeros of the Riemann zeta function and terms including zeros ρ and $1 - \rho$ are paired together. In the case $\tau = 1$ formulas (3.2) and (3.1) are equal.

Regarding the second question there are several generalizations to different sets of complex numbers ρ and various functions. For example, similar conditions to Li's criterion have been proved for a general multiset of complex numbers [4], automorphic *L*-functions [39], the Selberg class [55, 56] and a certain subclass of the extended Selberg class [70].

Furthermore, defining

$$\lambda_F(n, \tau) = \lim_{t \to \infty} \sum_{\substack{\rho \\ |\Im(\rho)| \leq t}} \left(1 - \left(\frac{\rho}{\rho - \tau} \right)^n \right), \qquad n \in \mathbb{Z}_+, \qquad (3.3)$$

where the sum runs over the non-trivial zeros ρ of the function *F*, similar results as Freitas proved can be generalized to other functions. The numbers $\lambda_F(n, \tau)$ are called *τ-Li coefficients*. For certain functions *F* all (of the real parts) of the τ-Li coefficients are non-negative if and only if all non-trivial zeros ρ satisfy the condition $\Re(\rho) \leq \tau/2$. In addition to the Riemann zeta function, this type of results have been proved, for example, for certain subclasses of the extended Selberg class [9, 17], Rankin-Selberg *L*-functions [8], a very broad class of *L*-functions [24] and automorphic *L*-functions [45]. There are also different growth conditions for the Li and the τ-Li coefficients (see e.g. [25, 39, 44, 54, 80]).

Now we are ready to move on to the third question. We consider it for the Li and the τ-Li coefficients. F. C. Brown [7] investigated connections between the Li coefficients and certain zero-free regions for a certain set of functions including the Riemann zeta function. He was able to prove [7, Theorem 3] that if a finite number of the Li coefficients are non-negative, then the critical strip contains (certain) zero-free regions. Brown also tried to prove [7, Theorem 2] that if there exist certain zero-free regions, then certain Li coefficients are non-negative. Unfortunately, his proof of Lemma 5 contains two errors which affect to Theorem 2. A. D. Droll [17] was able to fix one of errors but not the another one.

In article [B], we investigate the same type of questions as Brown did. Furthermore, instead of considering only the Li coefficients, we consider also the τ-Li coefficients. More detailed explanations can be seen in the next sections and in article [B].

3.2 Results

In article [B], we work with similar problems related to the τ-Li coefficients as Brown worked with in the theory of the Li coefficients. We

investigate τ-Li coefficients, where $\tau > e^{-1}$ is a real number, defined as

$$\lambda_F(n,\tau) = \lim_{t\to\infty} \sum_{\substack{\rho \\ |\Im(\rho)|\leq t \\ 0\leq\Re(\rho)\leq\tau}} \left(1 - \left(\frac{\rho}{\rho-\tau}\right)^n\right). \tag{3.4}$$

With condition (i) definition (3.4) coincides with the classical definition (3.3) of the τ-Li coefficient. Furthermore, we assume that a function $F(s)$ satisfies the following conditions:

(i) **Location of the zeros, 1**: The function $F(s)$ does not have zeros ρ with $\Re(\rho) > \tau$.

(ii) **Location of the zeros, 2**: The function $F(s)$ does not have a zero $\rho = \tau$.

(iii) **Number of the zeros**: For some real numbers $A_F > 0$ and B_F and for a real number $T_0 > 0$ which is large enough, we have the following two properties: First, we have

$$|N_F(T) - A_F T \log T - B_F T|$$
$$< C_{F,1}(T_0) \log T + C_{F,2}(T_0) + \frac{C_{F,3}(T_0)}{T}, \tag{3.5}$$

where $T \geq T_0$ is a real number and the numbers $C_{F,j}(T_0)$ $(j = 1,2,3)$ are non-negative real numbers which depend on the function F and the number T_0. Furthermore, we also have

$$|N_F(T,2T) - A_F T \log T - (A_F \log 4 + B_F)T|$$
$$< c_{F,1}(T_0) \log T + c_{F,2}(T_0) + \frac{c_{F,3}(T_0)}{T}, \tag{3.6}$$

where the terms $c_{F,j}(T_0)$ $(j = 1,2,3)$ are non-negative real numbers which depend on the function F and the number T_0.

(iv) **Computation**: The numbers $\lambda_F(n,\tau)$ can be computed without knowing the zeros of the function $F(s)$.

Condition (i) is not necessary for proving the results. We want that definition (3.4) of the term $\lambda_F(n,\tau)$ coincides with classical definition (3.3) and thus we assume condition (i). Furthermore, condition (iv) is not needed for deriving the results but it is useful for applying the result for considering the zero-free regions. Further, the assumption $\tau > e^{-1}$ is used because of technical, computational purposes i.e. mainly to be able to estimate $\log(e\tau) > 0$.
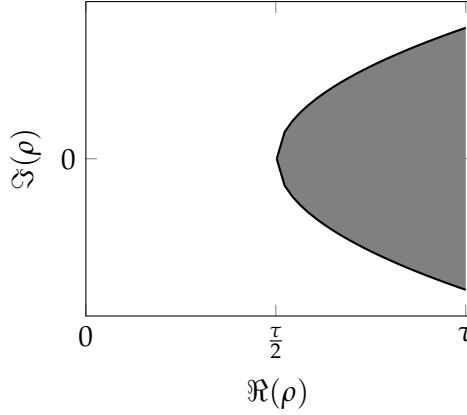
Figure 3.1: Region $\left|\frac{\rho}{\rho-\tau}\right| < R$ is in white.

It may be worth noticing that we do make any assumptions about the order of the function *F* whereas Brown did. Furthermore, in general, the assumptions are not too restrictive. For example, the Riemann zeta function satisfies conditions (i)–(iv) [43, 78, 79].

The first main result is of the same type as Brown's Theorem 3 but it is for the τ-Li coefficients:

Theorem 3.1 (Article [B]) *Let $R > 1$ be a real number. If all real parts of the τ-Li coefficients of F are non-negative in a certain interval, which is given in Theorem 3.1 in article [B], then all zeros ρ satisfy the condition $\left|\frac{\rho}{\rho-\tau}\right| < R$.*

The interval mentioned in Theorem 3.1 depends on the numbers R and τ and the constants defined in formula (3.5). Notice also that we have $|\rho/(\rho-\tau)| = 1$ if and only if $\Re(\rho) = \tau/2$. The region $|\rho/(\rho-\tau)| < R$ can be seen in Fig. 3.1. We investigate these kind of regions since we use lower and upper bounds of the term $|\rho/(\rho-\tau)|$ in our proofs and Brown also considered similar regions.

The second main result is similar Brown's Theorem 2, but again, it is for the τ-Li coefficients:

Theorem 3.2 (Article [B]) *Let $R > 1$ be a real number. If at least one of the real parts of the τ-Li coefficients of F is negative in a certain interval, which is given in Theorem 3.2 in article [B], then there is at least one zero ρ with $\left|\frac{\rho}{\rho-\tau}\right| \geq R$.*

The interval mentioned in Theorem 3.2 depends on the numbers R and τ as well as the constants defined on formulas (3.5) and (3.6). It does not necessarily overlap with the interval mentioned in Theorem 3.1.

Combining the results from Theorems 3.1 and 3.2 it *may* be possible to make conclusions about the locations of the zeros. Indeed, if all real parts of the coefficients defined in Theorem 3.1 are non-negative, then all zeros lie in a certain region. Furthermore, if at least one of the real parts of the τ-Li coefficients obtained from Theorem 3.2 is negative, we know that there exists at least one zero in a certain region. The problematic case is when some of the real parts obtained from Theorem 3.1 are negative but all real parts obtained from Theorem 3.2 are non-negative. In this case, we cannot say whether there exist zeros ρ with $|\rho/(\rho-\tau)| \geq R$ or not with a given number R. On the other hand, it is clear that in this case there must be at least one zero with $|\rho/(\rho-\tau)| > 1$ since otherwise we can not obtain negative terms from Theorem 3.1.

In addition to Theorems 3.1 and 3.2, the following equivalent condition is proved:

Theorem 3.3 (Article [B]) *Let $R > 1$ be a real number. There exists exactly one zero ρ with $\left|\frac{\rho}{\rho-\tau}\right| > 1$, and for it holds $\left|\frac{\rho}{\rho-\tau}\right| \geq R$, if and only if the term $|\Re(\lambda_F(n,\tau))|$ of F is large enough for at least one integer n in a certain interval. The interval and the lower bound for the term $|\Re(\lambda_F(n,\tau))|$ are given in Theorem 4.1 in article [B].*

The interval and the lower bound mentioned in the Theorem 3.3 depend on the numbers R and τ as well as the constants defined on formulas (3.5) and (3.6).

The advantages of Theorem 3.3 compared to Theorems 3.1 and 3.2 are that Theorem 3.3 gives an equivalent condition and it allows us to compute less τ-Li coefficients than Theorems 3.1 and 3.2 do. On the other hand, with Theorem 3.3 we can consider only the existence of exactly one zero inside certain regions.

At the end of paper [B] we also give some numerical examples of the bounds obtained from Theorems 3.1–3.3. See Tables 1–3 and Corollaries 5.1, 5.2 and 6.1 in article [B].

3.3   Main steps of the proofs

The proofs of Theorems 3.1–3.3 are straightforward. They are based on considering contributions coming from the zeros with absolute values large enough, small enough and possible exceptional zeros for the τ-Li coefficients. The reason behind this is that a single zero, whose imaginary part has an absolute value large enough, contributes less to a τ-Li coefficient than a zero with an absolute value small enough. Furthermore, the behavior of the terms $\rho/(\rho-\tau)$ depends also on the real parts

of the zeros. Thus it is also taken under consideration.

Let us start with the proof of Theorem 3.1. The main idea is to prove that if there exists a zero ρ with $|\rho/(\rho - \tau)| \geq R$, then at least one of the coefficients $\Re(\lambda_F(n, \tau))$ is negative when the integer n is on a certain interval. First, we recognize that when the absolute values of the imaginary parts of the zeros are large enough, then by the binomial theorem and the inequality $\binom{n}{2k} < \left(\frac{ne}{2k}\right)^{2k}$ [13, inequality (2)], we obtain

$$\left| \Re \left(1 - \left(\frac{\rho}{\rho - \tau} \right)^n \right) \right| \leq \frac{c(\tau)n^2}{|\Im(\rho)|^2}. \tag{3.7}$$

Here the number $c(\tau)$ is a certain explicit real number which depends on the constant τ. With formula (3.6), we can deduce that the contribution coming from the zeros with the absolute values of the imaginary parts large enough is of size $\mathcal{O}(n \log n)$ and we know an explicit version of it.

The contribution coming from the zeros with the absolute values of the imaginary parts small enough, follows from considering of two cases. First, we consider the zeros with the absolute values of the imaginary parts up to a certain number N. All zeros ρ with $|\rho/(\rho - \tau)| \geq R$ are included in this case. To derive the contribution, we apply the following theorem proved in [48, Chapter 5, Theorem 11]:

Theorem 3.4 *Let $M \geq 1$ be an integer and let $z_1, z_2, \ldots, z_M$ be complex numbers which satisfy the condition $\max\limits_{j} |z_j| = 1$. Then*

$$\max_{1 \leq n \leq 5M} \Re \left(\sum_{j=1}^{M} z_j^n \right) \geq \frac{1}{20}.$$

Applying the previous theorem, we bound the contribution

$$\Re \left(\sum_{\substack{\rho \\ |\Im(\rho)| < N}} \left(1 - \left(\frac{\rho}{\rho - \tau} \right)^n \right) \right) < \sum_{\substack{\rho \\ |\Im(\rho)| < N}} 1 - \frac{1}{20} R^n. \tag{3.8}$$

Now we have estimated the contributions coming from the zeros with the absolute values of the imaginary parts large and small enough. We still have to estimate the contribution coming from the zeros with the absolute values of the imaginary parts between the previous bounds. This is our next step.

For the rest of the zeros we have $|\rho/(\rho - \tau)| < R$. Since we have

$$\left| \Re \left(\left(\frac{\rho}{\rho - \tau} \right)^n \right) \right| \leq \left(1 + \frac{\tau^2}{\Im(\rho)^2} \right)^{\frac{n}{2}},$$

remembering that both n and $|\Im(\rho)|$ lie in a certain interval, an explicit upper bound for the term $\left|\Re\left((\rho/(\rho-\tau))^n\right)\right|$ is obtained. Together with formulas (3.8) and (3.5) we obtain that the contribution coming from the zeros with quite small absolute values of the imaginary parts is

$$K_{F,1}(\tau)N^{2+\epsilon_F(\tau)}(\log N)^2 - \frac{1}{20}R^N. \tag{3.9}$$

Here the terms $K_{F,1}(\tau)$ and $\epsilon_F(\tau)$ are explicit, positive constants which depend on the function F and the number τ.

Remembering the upper bound of size $\mathcal{O}(n\log n)$ for the contribution coming from the zeros with the absolute values of the imaginary parts large enough, and that the number n lies in a certain interval, we have obtained an explicit, type (3.9) upper bound for the term $\Re(\lambda_F(n,\tau))$. Since it is always negative when N is large enough, we have proved Theorem 3.1.

Let us now take a look at the proof of Theorem 3.2. The idea is to prove that if all zeros ρ satisfy the condition $|\rho/(\rho-\tau)| < R$, then certain terms $\Re(\lambda_F(n,\tau))$ are non-negative. As before, we divide the consideration to the two cases depending on the absolute values of the imaginary parts of the zeros. When the absolute values are small enough, we just use the inequality $|\rho/(\rho-\tau)| < R$. By formula (3.5) these zeros give a contribution which is greater than $-K_{F,2}(R^n-1)$ for some real number $K_{F,2} > 0$ which depends on the function F.

When the absolute values of the imaginary parts of the zeros are large enough, we write

$$\Re\left(1 - \left(\frac{\rho}{\rho-\tau}\right)^n\right) = -n\tau\Re\left(\frac{1}{\rho-\tau}\right)$$
$$- \frac{n(n-1)\tau^2}{2}\Re\left(\frac{1}{(\rho-\tau)^2}\right) \tag{3.10}$$
$$- \sum_{j=3}^{n}\binom{n}{j}\Re\left(\left(\frac{\tau}{\rho-\tau}\right)^j\right).$$

The first term on the right-hand side of formula (3.10) is always non-negative. Furthermore, we can easily prove that the second term is at least $c(\tau)n(n-1)/|\Im(\rho)|^2$. Here the constant $c(\tau)$ is a positive real number which depends on the constant τ and we know an explicit version of it. The estimate for the third term on the right-hand side of formula (3.10) follows from the observation

$$\left|\binom{n}{j}\left(\frac{\tau}{\rho-\tau}\right)^j\right| \le \frac{\tau^3 n(n-1)(n-2)}{e^{j-3}j!|\Im(\rho)|^3}.$$

It is explicit and greater than $-c(\tau)n(n-1)(n-1)/|\Im(\rho)|^3$ for some constant $c(\tau)$.

Combining the previous three estimates for the terms defined on the right-hand side of equality (3.10), we find an explicit lower bound which can be described as

$$\Re\left(1-\left(\frac{\rho}{\rho-\tau}\right)^n\right) \geq \frac{c(\tau)n(n-1)}{|\Im(\rho)|^2}, \quad \text{for some } c(\tau) > 0,$$

when the absolute values of the imaginary parts are large enough. Using formula (3.6), the contribution coming from the zeros with the absolute values of the imaginary parts large enough, is of size $\Omega(n\log n)$ and given explicitly. Together with the contribution coming from the zeros with the absolute values of the imaginary parts small enough, we obtain

$$\Re(\lambda_F(n,\tau)) > -K_{F,2}(R^n-1) + K_{F,3}(\tau)n\log n.$$

Here the number $K_{F,3}(\tau)$ is a positive real number which depends on the function F and the constant τ. When the number R is close enough to 1, we always find an integer n for which the right-hand side of the previous inequality is greater than zero. This proves Theorem 3.2.

Let us now move on to the proof of Theorem 3.3. For all zeros with $|\rho/(\rho-\tau)| \leq 1$ it holds

$$\left|\Re\left(1-\left(\frac{\rho}{\rho-\tau}\right)^n\right)\right| \leq 2.$$

Together with estimate (3.7) for the zeros with the absolute values of the imaginary parts large enough and formulas (3.5), (3.6), this leads to an explicit contribution of size $\mathcal{O}(n\log n)$. Furthermore, for the zero ρ with $|\rho/(\rho-\tau)| \geq R$ and n large enough, we have

$$\left|\Re\left(1-\left(\frac{\rho_1}{\rho_1-\tau}\right)^n\right)\right| \geq \frac{1}{20}R^n - 1.$$

Since R^n grows faster than $n\log n$, this completes Theorem 3.3.

3.4 Discussion

In Section 3.2, we provide some relations between the τ-Li coefficients and the zero-free regions. The main steps of the proofs are described in Section 3.3. Even though the methods of the proofs are quite elementary, the results give interesting and useful connections between the τ-Li coefficients and certain zero-free regions for a very large set of functions. Furthermore, they also generalize and correct some results proved by Brown [7, Theorem 2, Theorem 3].

Since very few assumptions are made for the functions under consideration, the results may not be very sharp. Thus, it would be interesting to prove similar results for specific functions, for example, Dirichlet *L*-functions or for the Selberg class. For instance, the existence of certain functional equations, complete functions etc. have been used for determining the Li and the τ-Li coefficients (see e.g. [7, 9, 25, 28, 54, 80]). Thus more precise conditions may also be helpful. This way it may be possible to find equivalent conditions to hold instead of implications. Furthermore, since many of the considerations seem to rely on the number of the zeros [7, 9, 17], better bounds for them may also be useful.

One obvious application of the results is to actually determine zero-free regions. Several computations for the Li and the τ-Li coefficients have already been made (see e.g. [9, 11, 24, 25, 43, 52, 53]). Comparing them to the results proved in article [B], it may be possible to make some conclusions about the zero-free regions.

Furthermore, instead of considering similar regions as described in Fig. 3.1, it may be interesting to assume that all zeros with real parts in $[0, \tau]$ up to height T satisfy the condition $\Re(s) = \tau/2$ and then estimate the behavior of the τ-Li coefficients. This kind of result has already been published for the Riemann zeta function [57] but it, unfortunately, contained errors [58].

In Theorem 3.3 we prove an equivalent condition between the τ-Li coefficients and existence of a certain zero. In [7, Theorem 5] Brown proves a similar, but stronger, result for the Li coefficients. He shows that the non-negativity of the term $\lambda_F(2, 1)$ for certain functions leads to the non-existence of a Siegel zero. It would be interesting to prove a similar result for a more general context and to the τ-Li coefficients.

4 On primes in arithmetic progressions under GRH

In article [C], we study an explicit bound for the number of primes in arithmetic progressions assuming the GRH. First we take a look at the number of primes and then at the number of primes in arithmetic progressions.

4.1 On the number of primes

In his famous paper [63], G. F. B. Riemann stated that the number of primes up to x satisfies the condition

$$\pi(x) \sim \frac{x}{\log x}. \tag{4.1}$$

The result was proved almost forty years later independently by J. Hadamard [30] and C.-J. de la Vallée Poussin [15]. A couple of years later de la Vallée Poussin [16] also showed a more precise result. Namely, there exists a real number $c > 0$ such that

$$\pi(x) = \text{Li}(x) + \mathscr{O}\left(xe^{-c\sqrt{\log x}}\right).$$

Again, a couple of years later, H. Koch [38] improved the result showing the RH is true if and only if the error term is $\mathscr{O}(\sqrt{x}\log x)$.

Thus, many useful asymptotic results were already known by the beginning of the 20th century. But what do we know about explicit estimates for the number of primes?

The explicit bounds can be divided to two different categories: To the so called *de la Vallée Poussin-type* results, where the error term is significantly smaller than the main term, and to the *Chebyshev-type* estimates, where the error term is a small multiple of the main term. J. B. Rosser and L. Schoenfeld [65, 66, 67, 68] started the work with the de la Vallée Poussin-type estimates. For the latest improvements for the de la Vallée Poussin-type bounds, see [10, 22, 61, 76]. There are also several Chebyshev-type bounds for the number of primes and some functions related to it (see e.g. [22, 51, 67, 68]).

For further survey, see [14, 31]. In this text, we concentrate on the de la Vallée Poussin-type bounds for the number of primes in arithmetic progressions.

4.2   On number of primes in arithmetic progressions

In 1896 de la Vallée Poussin [15] proved that the number of primes in an arithmetic progression up to x is

$$\pi(x;q,a) \sim \frac{x}{\varphi(q) \log x}.$$

Keeping in mind formula (4.1) for the number of primes, this is a very natural result since the number of primitive residue classes modulo q is $\varphi(q)$. By A. Walfisz [81] the result can be improved to the following: For all positive real numbers A there exists a positive real number $C(A)$ such that

$$\left| \pi(x;q,a) - \frac{\mathrm{Li}(x)}{\varphi(q)} \right| < C(A) \frac{x}{(\log x)^A}, \quad \text{when} \quad x \geq 3.$$

The constant $C(A)$ is *ineffective* meaning that it is not possible to find a numerical value for it going through the proof. As in the case of the number of primes, assuming the GRH leads to a better error term for the number of primes in arithmetic progressions. Indeed, then we have (see e.g. [14, Chapter 20])

$$\pi(x;q,a) = \frac{\mathrm{Li}(x)}{\varphi(q)} + \mathcal{O}\left(\sqrt{x} \log x\right). \tag{4.2}$$

There are many Chebyshev-type explicit estimates for the number of primes in arithmetic progressions (see e.g. [2, 46, 47, 62]) but de la Vallée Poussin-type results are much more rare. The only de la Vallée Poussin-type results without assuming the GRH have been proved by P. Dusart [21], who considered the case $q = 3$, and by M. A. Bennett, G. Martin, K. O'Bryant and A. Rechnitzer [3] who generalized the result for all numbers q and various numbers x. They did not assume the GRH and thus the error terms are of sizes $\mathcal{O}(x/\log x)$. Furthermore, in his PhD-thesis, P. Dusart derived [20, Theorem 3.7] sharp explicit estimates for the term $\psi(x;q,a)$ (see Notation) where $x \geq 10^{10}$ and $q \leq \frac{4}{5} \log x$ or $q \leq 432$. Even though this function is closely related to the number of primes in arithmetic progressions (see e.g. Chapter 1, [14, Chapters 19, 20]), the result is not sufficient for estimating the number of primes in arithmetic progressions up to some x.

In the next sections we describe the explicit result for the number of primes in arithmetic progressions, assuming the GRH, proved in article [C]. This gives an error term of the same size as the one described in formula (4.2).

4.3 Results

In article [C] we prove the following bound for the number of primes in arithmetic progressions:

Theorem 4.1 (Article [C]) *Assume the GRH. Then the number of primes in an arithmetic progression $\pi(x; q, a)$ for integers $x \geq e$, $q \geq 3$ and a satisfies the bound*

$$\left| \pi(x; q, a) - \frac{\text{li}(x)}{\varphi(q)} \right| \leq \left(\frac{1}{8\pi\varphi(q)} + \frac{1}{6\pi} \right) \sqrt{x} \log x - 75.306$$
$$+ \left(47.270 \log^2 q + 1199.553 \log q + \frac{1}{4\pi\varphi(q)} + 6808.840 \right) \sqrt{x}.$$

Please notice that the coefficients in front of the term $\sqrt{x}$ can be slightly sharpened using Lemma 9 instead of Corollary 10 from [C]. On the other hand, this would lead to a longer error term. Furthermore, by [68, Corollary 1], under the RH, the error term for the number of primes is $\sqrt{x} \log x / (8\pi)$ when the number x is large enough. Thus it may be natural to assume that the main error term in Theorem 4.1 can be improved to $\frac{1}{8\pi\varphi(q)}$. This topic is discussed more in Section 4.5.

In order to prove Theorem 4.1, we prove the following result which is also interesting in itself:

Theorem 4.2 (Article [C]) *Let $x \geq 2$, $q \geq 3$ and a be integers and the function $\psi(x; q, a)$ defined as in Notation. Assume the GRH for Dirichlet characters modulo q and for any modulo dividing q. We have*

$$\left| \psi(x; q, a) - \frac{x}{\varphi(q)} \right| < \left(\frac{1}{8\pi\varphi(q)} + \frac{1}{6\pi} \right) \sqrt{x} (\log x)^2 + 1.363 x^{0.423} (\log x)^2$$
$$+ (0.319 \log q + 15.931) \sqrt{x} \log x$$
$$+ (7.433 \log q + 84.472) \sqrt{x} + R_1(x)$$

where the term $R_1(x)$ describes the contribution coming from the terms which are asymptotically at most $\mathcal{O}\left(\frac{x^{0.423}}{\log x} \right)$. The term $R_1(x)$ is explicitly given in Theorem 1 in [C].

In addition to the number x, the term $R_1(x)$ also depends on the number q. Furthermore, as in Theorem 4.1, the coefficient in front of the term $\sqrt{x} (\log x)^2$ can most probably be sharpened.

Besides Theorems 4.1 and 4.2, several other interesting explicit estimates, obtained assuming the GRH, are proved in article [C]. For example, in Lemma 7 there is an explicit upper bound for the term $|b(\chi)|$, where the term comes from the Laurent series expansion of the term

$\frac{L'(s,\chi)}{L(s,\chi)} - \frac{1}{s}$ at $s = 0$. Here $L(s, \chi)$ is a Dirichlet L-function associated with a primitive nonprincipal character modulo q. Furthermore, in Lemma 9 and Corollary 10, upper bounds for the terms $|L'(0, \chi)/L(s, \chi)|$ with $\chi(-1) = -1$ are mentioned.

4.4   Main steps of the proofs

By partial summation, we can write

$$\pi(x; q, a) = \frac{\psi(x; q, a)}{\log x} + \int_2^x \frac{\psi(t; q, a)}{t(\log t)^2} dt$$
$$+ \frac{\theta(x; q, a) - \psi(x; q, a)}{\log x} + \int_2^x \frac{\theta(t; q, a) - \psi(t; q, a)}{t(\log t)^2} dt.$$
(4.3)

(To remember the definition of the function $\theta(x; q, a)$, please see Notation.) Thus it is sufficient to estimate the functions $\psi(x; q, a)$ and $\theta(x; q, a) - \psi(x; q, a)$. We follow the same steps as described in H. Davenport's book [14, Chapters 19, 20]. Furthermore, we also keep in mind that in order to obtain the wanted main and error terms (see Theorem 4.1), we need to get the main term $x/\varphi(q)$ and the error terms to the size of at most $\mathcal{O}(\sqrt{x}(\log x)^2)$ for the terms $\psi(x; q, a)$ and $\theta(x; q, a) - \psi(x; q, a)$.

First we derive an estimate for the difference $\theta(x; q, a) - \psi(x; q, a)$. The estimate follows straightforwardly from the definitions of the functions and the explicit estimate [66, Theorem 13] proved for the difference $\psi(x) - \theta(x)$. It is explicit and of size $\mathcal{O}(\sqrt{x})$.

Now we are ready to move on to estimate the term $\psi(x; q, a)$. This case is a little bit more complicated than the previous one. First, using the definition of the function $\psi(x; q, a)$ and some properties of character sums, we can write

$$\psi(x; q, a) = \frac{\overline{\chi_0}(a)\psi(x, \chi_0)}{\varphi(q)} + \frac{1}{\varphi(q)} \sum_{\chi \neq \chi_0} \overline{\chi}(a)\psi(x, \chi). \qquad (4.4)$$

Here χ_0 is a principal character modulo q. The reason why we separate the principal character from the sum is that later we are going to apply the following formula (see [75, Theorem 1]) for the number of non-trivial zeros of the Dirichlet L-functions and it is only proved for primitive nonprincipal characters. More detailed explanations how the formula for the number of the zeros is used can be seen in those parts of this section where we actually use the following formula:

Theorem 4.3 *Let $T \geq 1$ be a real number and χ be a primitive nonprincipal*

character modulo q. Then

$$\left| N(T,\chi) - \frac{T}{\pi} \log \frac{qT}{2\pi e} \right| \leq 0.317 \log(qT) + 6.401,$$

where $N(T,\chi)$ counts the number of the zeros ρ of a function $L(s,\chi)$ with $0 < \Re(s) < 1$ and $|\Im(\rho)| \leq T$.

To prove the result, we need to estimate the contributions coming from the principal and nonprincipal characters.

Next we estimate the contribution coming from the principal character in formula (4.4)**.** The goal is to obtain the main term $x/\varphi(q)$ and error terms which are small enough. We keep in mind that we already know how to estimate the function $\psi(x)$ [68, Theorem 13] and thus the goal is to use this information. Hence, we estimate the function $\psi(x,\chi_0)$ with the function $\psi(x)$. Using the definitions of the functions $\psi(x,\chi_0)$ and $\psi(x)$, it follows that the difference $|\overline{\chi_0}(a)\psi(x,\chi_0) - \psi(x)|$ is of size $\mathcal{O}(\log x)$ and we find an explicit upper bound of that size. Furthermore, using [68, Theorem 13] and verifying the result also for the small numbers x, we obtain an explicit upper bound of size $\mathcal{O}(\sqrt{x}(\log x)^2)$ with the leading coefficient $1/(8\pi)$ for the difference $|\psi(x) - x|$. Thus we have obtained the wanted main term $x/\varphi(q)$ for the function $\psi(x;q,a)$ and are ready to move on.

Now we estimate the contribution coming from the nonprincipal characters in formula (4.4)**.** Since the function $\psi(x,\chi)$ has discontinuities when the number x is a prime power and we want to use some properties of continuous functions later, we estimate the function $\psi(x,\chi)$ with the function

$$\psi_0(x,\chi) = \begin{cases} \psi(x,\chi) - \frac{1}{2}\Lambda(x)\chi(x) & \text{if } x \text{ is a prime power} \\ \psi(x,\chi) & \text{otherwise.} \end{cases}$$

This leads to an explicit error term of size $\mathcal{O}(\log x)$. Furthermore, as we have already mentioned, we are going to apply such a formula for the number of zeros which is only proved for primitive nonprincipal characters. Thus we estimate the terms $\psi_0(x,\chi)$ with the terms $\psi_0(x,\chi^\star)$ where $\chi^\star$ is a primitive character which induces the character χ. This is a very easy computation and leads to an explicit error term of size $\mathcal{O}(\log x)$.

Because of the previous two paragraphs, it is sufficient to estimate the term $\psi_0(x,\chi)$ for primitive nonprincipal characters. By [14, Section

19, formulas (2) and (3)] we have

$$\psi_0(x,\chi) = - \sum_{\substack{\rho \\ \text{non-trivial}}} \frac{x^\rho}{\rho} - \mathfrak{a}\frac{L'(0,\chi)}{L(0,\chi)}$$

$$- (1-\mathfrak{a})(\log x + b(\chi)) + \sum_{m=1}^{\infty} \frac{x^{\mathfrak{a}-2m}}{2m-\mathfrak{a}},$$

(4.5)

where

$$\mathfrak{a} = \begin{cases} 0 & \text{if} \quad \chi(-1)=1 \\ 1 & \text{if} \quad \chi(-1)=-1 \end{cases}$$

and $b(\chi)$ comes from the Laurent series of $\frac{L'(s,\chi)}{L(s,\chi)} = \frac{1}{s} + b(\chi) + \ldots$. Thus the term $\psi_0(x,\chi)$ can be estimated by estimating the terms on the right-hand side of formula (4.5) separately.

Let us start with the last term on the right-hand side of formula (4.5). Writing this term as an integral and doing some trivial estimates, we shortly obtain an explicit error term of a constant size.

Next we estimate the term $b(\chi)$ and thus the third term on the right-hand side of formula (4.5). First we want to derive such a formula for it that we can (easily) handle. Since the function $b(\chi)$ is related to the function $L'(s,\chi)/L(s,\chi)$, we consider this function. By the functional equation for the Dirichlet L-functions and logarithmic differentiation we have

$$\frac{L'(s,\chi)}{L(s,\chi)} = -\frac{1}{2}\log\frac{q}{\pi} - \frac{1}{2}\frac{\Gamma'\left(\frac{s}{2}\right)}{\Gamma\left(\frac{s}{2}\right)} + B(\chi) + \sum_{\substack{\rho \\ \text{non-trivial}}} \left(\frac{1}{s-\rho} + \frac{1}{\rho}\right),$$

(4.6)

where the term $B(\chi)$ is a constant which depends on the term χ. Since we want to avoid estimating the constant $B(\chi)$, we subtract formula (4.6) with $s=2$ from formula (4.6) with s. Furthermore, using formula

$$-\frac{\Gamma'(z)}{\Gamma(z)} = \gamma + \frac{1}{z} + \sum_{n=1}^{\infty} \left(\frac{1}{z+n} - \frac{1}{n}\right)$$

(see [14, Section 12, formula (9)]) for the logarithmic derivative of the gamma function, we obtain

$$b(\chi) = \frac{L'(2,\chi)}{L(2,\chi)} - \sum_{\substack{\rho \\ \text{non-trivial}}} \frac{2}{\rho(2-\rho)}.$$

(4.7)

An explicit estimate for the last term on formula (4.7) follows easily from assuming the GRH, applying Theorem 4.3 and using partial summation. It is of constant size. So we move on to estimate the first term

on the right-hand side of formula (4.7). The idea is to derive the estimate from an explicit estimate proved for the Riemann zeta function [6, Lemma 2.2]:

Theorem 4.4 *For $\sigma > 1$ let*

$$f(\sigma) := \frac{\zeta'(s)}{\zeta(s)} + \frac{1}{\sigma - 1} + \gamma.$$

Then there exists a positive absolute constant c_2 such that

$$-c_2(\sigma - 1) < f(\sigma) < 0,$$

and c_2 can be taken to be $\gamma^2 - 2\gamma_1$ where

$$\gamma_1 = -\lim_{N \to \infty} \left(\sum_{n=2}^{N} \frac{\log m}{m} - \frac{(\log N)^2}{2} \right) = -0.0723\dots$$

so $c_2 = 0.47789\dots$.

Thus we can also estimate the Dirichlet L-functions at that point:

Corollary 4.5 (Article [C]) *Let $s = 1 + \frac{1}{\log y} + it$, where $y > 1$ and t are real numbers. Then*

$$\left| \frac{L'(s, \chi)}{L(s, \chi)} \right| < \log y + \gamma + \frac{0.478}{\log y}.$$

Furthermore, if $s = 2 + it$, then $\left| \frac{L'(s,\chi)}{L(s,\chi)} \right| < 0.570$.

Using Corollary 4.5, the first term on the right-hand side of formula (4.7) is estimated with an explicit upper bound of a constant size. This also leads to an explicit upper bound of a constant size for the term $b(\chi)$.

Next we estimate the second term on the right-hand side of formula (4.5). First, according to the functional equation for the Dirichlet L-functions, we can estimate the logarithmic derivative at the point $s = 1$ instead of the point $s = 0$. By assuming the GRH, [3, Lemmas 6.4 and 6.5], [40, Theorem 1.5, Lemmas 2.3 and 2.5] and some numerical computations, we can estimate the terms $L'(1, \chi)$ and $L(1, \chi)$. The total, explicit error term in this case, is of constant size.

Now we estimate the first term on the right-hand side of formula (4.5). This case is a little bit trickier than the previous ones. First of all, we cannot use just the GRH, write $|x^\rho| = \sqrt{x}$ and straightforwardly apply Theorem 4.3, since the sum does not converge. Thus we divide the consideration to two cases: the absolute values of the imaginary parts of the zeros ρ are 1) large and 2) small enough.

**First we consider the contribution coming from the non-trivial ze-
ros with the absolute values of the imaginary parts large enough.** The
idea is to estimate the whole term $\psi_0(x, \chi)$ with the integral

$$J(x, T(x), \chi) = \frac{1}{2\pi i} \int_{c-iT(x)}^{c+iT(x)} \left(-\frac{L'(s,\chi)}{L(s,\chi)} \right) \frac{x^s}{s} ds.$$

Here $T(x)$ is a large enough real number and it depends on the num-
ber x and we denote $c = 1 + \log x$. First we estimate the difference
$\psi_0(x, \chi) - J(x, T(x), \chi)$ and then the function $J(x, T(x), \chi)$.

Now we estimate the difference $|\psi_0(x, \chi) - J(x, T(x), \chi)|$. Apply-
ing [14, Section 17, formula (3)] and the definitions of the functions, we
obtain

$$|\psi_0(x, \chi) - J(x, T(x), \chi)|$$
$$< \sum_{\substack{n=1 \\ n \neq x}}^{\infty} \Lambda(n) \left(\frac{x}{n} \right)^c \min \left\{ 1, (T(x))^{-1} \left| \log \frac{x}{n} \right|^{-1} \right\} + c(T(x))^{-1} \Lambda(x).$$

$$(4.8)$$

We divide the consideration to different cases depending on how close
the numbers n and x are to each other. By partial summation this leads
to an explicit error term of size $\mathcal{O}\left(x(\log x)^2/T(x) \right)$. Please notice that
the number $T(x)$ depends on the number x and thus the previous asymp-
totic upper bound depends (only) on the variable x.

Next we estimate the term $J(x, T(x), \chi)$**.** Again, we cannot just triv-
ially estimate $|x^s| = xe$ and $|L'(s, \chi)/L(s, \chi)| = \mathcal{O}(\log x)$, since this
leads to a too large error term. Thus we apply an integral over a (modi-
fied) rectangle with vertices

$$c + iT_2 \quad c + iT_1 \quad -U + iT_1 \quad \text{and} \quad -U + iT_2.$$

Here $U > 0$ is a real number and T_1 and T_2 are real numbers such that
there are no zeros with $\Im(\rho) \in T_1, T_2$ and we have $|T(x) - T_1| < 1$ and
$|-T(x) - T_2| < 1$. The existence of the numbers T_1, T_2 follows from
Theorem 4.3. If a Dirichlet L-function has a zero at $s = -U$, then we
avoid it with a half circle which has a very small radius and whose cen-
tre is at $(-U, 0)$. The idea is to estimate the integral over the whole
rectangle, then remove integrals over horizontal lines and the left verti-
cal line and lastly remove or add necessary parts to obtain the integral
$J(x, T(x), \chi)$.

We start with estimating the error term which comes from adding
or removing necessary integrals to the right vertical line. The estimate
follows easily from using Corollary 4.5 and the definitions of the terms
T_1, T_2, c. We obtain an explicit error term of size $\mathcal{O}\left(x \log x/T(x) \right)$.

For obtaining the contribution coming from the horizontal lines, we apply the functional equation for the Dirichlet *L*-functions, Corollary 4.5, partial summation and divide the consideration to different cases depending on the real part of the number *s*. That way we obtain an explicit error term of size $\mathscr{O}\left(\frac{x(\log T(x))^2}{T(x)\log x}\right)$.

Furthermore, by [14, Pages 116-117], the contribution coming from the left line goes to zero as U goes to infinity. Thus, we let U go to infinity and we have to only estimate the integral over the (modified) rectangle.

Using residue calculus we almost obtain formula (4.5) as U goes to infinity. The only exception is that instead of the first sum on the right-hand side of formula (4.5), we look at the term

$$\sum_{\substack{T_2<|\Im(\rho)|<T_1 \\ \text{non-trivial}}} \frac{x^\rho}{\rho}.$$

Since the absolute values of the terms T_1, T_2 are at most $T(x)+1$, we can estimate the previous formula with a sum running over the zeros with absolute values of the imaginary parts at most $T(x)+1$. This estimate leads to an explicit error term of size at most $\mathscr{O}\left(\frac{\sqrt{x}\log T(x)}{T(x)}\right)$.

Now it is sufficient to estimate the term

$$\sum_{\substack{|\Im(\rho)|\leq T(x)+1 \\ \text{non-trivial}}} \frac{x^\rho}{\rho}. \tag{4.9}$$

By partial summation and using Theorem 4.3, this leads to an explicit error term of size $\mathscr{O}(\sqrt{x}(\log T(x))^2)$.

Now it is time to put everything together and choose the size of the term $T(x)$**.** We keep in mind that we want an error term which is at most of size $\mathscr{O}(\sqrt{x}(\log x)^2)$. According to the previous proofs, we find an explicit upper bound of size $\mathscr{O}\left(x(\log x)^2/T(x)\right)$ for formula (4.8). Thus, we have to select the number $T(x)$ to be of size $\Omega(\sqrt{x})$. Let us write $T(x)\approx x^{0.5+\epsilon}$ where $\epsilon\geq 0$ is a real number and the term $T(x)$ can differ from the term $x^{0.5+\epsilon}$ by a constant. Since we want the largest error term to be as small as possible, we select $\epsilon>0$. The larger the number ϵ is, the smaller the contribution coming from formula (4.8) is.

Furthermore, by the previous proofs, we have mentioned that the explicit upper bound for term (4.9) is of size $\mathscr{O}(\sqrt{x}(\log T(x))^2)$. Hence, the larger the number ϵ is, the larger the coefficient of the largest term coming from term (4.9) is. Furthermore, this coefficient actually comes from the term $(\log(T(x)+1))^2/(2\pi)$. Since we have $(0.5+\epsilon)^2>0.25$,

the smallest positive integer k such that $(\log(T(x)+1))^2 < \log x/k$ is $k = 3$. Thus, using three decimals in our computations, we can select $\epsilon = 0.577$.

Using formula (4.3) and the results which are proved in the first two paragraphs of this section, **we obtain Theorem 4.1 from Theorem 4.2**.

The main differences between the proofs described in [14, Chapters 19, 20] and the explicit results described here are that we cite explicit results instead of the asymptotic ones and do everything more precisely. For example, we use explicit estimates for the terms $L(s, \chi)$ and $L'(s, \chi)$ instead of the asymptotic ones.

Sometimes proving a good explicit estimate requires a little bit more careful work than an asymptotic one. For example, let us take a look at the proof of the estimate for formula (4.8). In the asymptotic case [14, page 107], the proof is divided into the three different cases depending on how close the terms n and x are to each other. Even though we could do the same in order to prove the explicit bound, it would yield a quite big coefficient in front of the biggest error term. Thus we add two more cases and obtain a little bit sharper result.

Furthermore, in Davenport's book [14, page 127] the number $T(x)$ is selected to be $\sqrt{x}$ and the leading coefficient does not need to be optimized. Since we prove an explicit result, we select the number $T(x)$ differently and more carefully.

More detailed proofs can be seen in article [C].

4.5   Discussion

In Section 4.3, we give explicit estimates for the functions $\pi(x; q, a)$ and $\psi(x; q, a)$ assuming the GRH. The main steps of the proofs are described in Section 4.4. Please also notice that our results generalize and improve the results for the function $\psi(x; q, a)$ proved by Dusart [20, Theorem 3.7].

As we already mentioned in Section 4.3, the coefficients of the leading error terms in Theorems 4.1 and 4.2 are probably not the most optimal ones. The problematic coefficient $1/(6\pi)$ comes from term (4.9). Assuming the GRH we have $|x^\rho| = \sqrt{x}$ and by Theorem 4.3 the number of the non-trivial zeros up to $T(x)$ is asymptotically equivalent to $T(x) \log(T(x))/\pi$. Thus, in order to obtain better upper bounds than $\mathcal{O}(\sqrt{x}(\log x)^2)$, we need to have cancellations inside the sum or select the number $T(x)$ to be at most of size $o(x^\epsilon)$ for all $\epsilon > 0$. In order to have cancellations inside the sum, we may need to know something about the locations of the zeros and thus the first problem is difficult to solve. Furthermore, the later case causes problems with formula (4.8) since there, using the methods described in Section 4.4, the term $T(x)$

should be of size $\Omega(\sqrt{x})$. Hence, in order to prove the better estimates, we probably need to find a somehow different approach to this part.

Now we move on from improving the coefficient of the largest error term to the other topics of this discussion. It should also be noted that, even though we do not need the assumption of the GRH in all of the proofs, the assumption is used in the most parts of the proofs and especially in deriving the largest error term.

Furthermore, in Section 4.4 we applied a formula for the number of the non-trivial zeros of the Dirichlet *L*-functions. Obviously sharper results for the error term would improve Theorems 4.1 and 4.2. On the other hand, these improvements would not affect the coefficient of the largest error terms in Theorems 4.1 and 4.2 since they come from estimating the function $\psi(x)$ with the term x, from the main term for the number of the non-trivial zeros and from the formulas which are not that closely related to the number of the zeros. Please also notice that for $T(x) \geq 10$, where the number $T(x)$ is defined as in Section 4.4, we already know some slight improvements by [75, Table 1].

In addition to assuming the GRH, there are some results for the number of primes and related functions assuming the GRH up to some height [3, 62, 19]. It would be interesting to see if the results could be generalized or improved using same methods as described in Section 4.4.

Furthermore, the motivation of paper [C] came from A.-M. Ernvall-Hytönen's, T. Matala-aho's and L. Seppälä's work [23] with *p*-adic evaluations of Euler's divergent series. Theorem 4.1 may be used to improve Lemma 1 and the proof of Theorem 5 in Ernvall-Hytönen's, Matala-aho's and Seppälä's article. In addition to this problem, Theorem 4.1 may be beneficial for other problems related to the number of primes in arithmetic progressions assuming the GRH. For example, recent work with finding explicit intervals containing primes which are in certain arithmetic progressions [19, 20] may benefit from Theorem 4.1.

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