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The present value of a perpetuity with stochastic discounting

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Abstract

Simple market models typically include a constant rate of risk-free interest, simplifying present value calculations and asset pricing significantly. In this thesis, present value calculations with a stochastic rate of interest is treated for random cash flows, annuities, and in particular, perpetuities. The literature on the finiteness and central moments of the present value of a perpetual cash flow is discussed and higher-order moment formulas are derived for several special cases, in particular for a continuous finite or infinite constant cash flow subject to an interest rate given by a Lévy process. The main part of the work is a long list of cases, some being new, when the density of the present value of a perpetuity can be found or a simple expression in terms of independent stochastic variables can be derived. Applications to stock valuation, approximation methodology, and risk theory are discussed briefly at the end.

Keywords: finance, perpetuities, probability

Swedish summary

Nuvärdet av en perpetuitet med stokastisk diskontering

I finansiell ekonomi är *nuvärdet* ett centralt begrepp. Kort sagt är nuvärdet av ett penningflöde den summa pengar som skulle krävas i nuläget för att kunna bekosta det framtida penningflödet. Till exempel kan man tänka sig en annuitet som, under $N \in \mathbb{N}$ år, årligen betalar en konstant penningsumma $c > 0$, vilket med en årlig räntegrad på $r > 0$ skulle ge upphov till nuvärdet

$$Z(N) = \sum_{k=1}^{N} c(1+r)^{-k}.$$

Den föreliggande avhandlingen behandlar främst *perpetuiteter*, det vill säga en variant av annuiteter vars regelbundna utbetalningar aldrig upphör. Den motsvarande perpetuiteten har nuvärdet

$$Z(\infty) = \sum_{k=1}^{\infty} c(1+r)^{-k}.$$

Tack vare formeln för en geometrisk summa är det lätt att förenkla nuvärdena; det gäller således att

$$Z(N) = c\frac{1-(1+r)^{-N}}{r}, \quad Z(\infty) = \frac{c}{r}.$$

Tyvärr är det i den riktiga världen ytterst sällsynt att räntegraden bevaras från år till år. Av detta skäl vore det gynnsamt att ha möjligheten att bruka matematiska modeller där räntan är stokastisk, eller till och med där räntan och utbetalningarna bägge är stokastiska. Det stora problemet är att det inte existerar någon variant av formeln för en geometrisk summa som gäller för stokastiska variabler. Låt oss betrakta nuvärdet av en perpetuitet vars utbetalningar och ränte- eller diskonteringsfaktorer ges av två oberoende processer av stokastiska variabler:

$$Z(\infty) = \sum_{k=1}^{\infty} C_k \prod_{j=1}^{k} V_j,$$

där $C_k, k = 1, 2, \ldots$ är betalningsprocessen och $V_k, k = 1, 2, \ldots$ är diskonteringsprocessen. Eftersom $Z(\infty)$ nu är en stokastisk variabel, behövs en metod för att beräkna dess fördelnings- eller frekvensfunktion. Man kan erhålla en grov skattning genom att simulera perpetuiteten fram till något ändligt antal utbetalningar och upprepa simulationen ett par hundra gånger, men detta medför ytterligare risker för fel som kunde undvikas om man kände till en metod för att analytiskt beräkna frekvensfunktionen för $Z(\infty)$. I avhandlingen visas att detta är möjligt åtminstone i vissa specialfall.

Även villkor för att nuvärdet är ändligt behandlas. Med stöd av tidigare forskning – särskilt W. Vervaats forskning – erhålls goda villkor som kan användas för att besvara frågan även i fall då nuvärdets fördelning är okänd. Dessutom härleds formler för nuvärdets centrala origomoment av alla ordningar; dessa formler kan brukas även då nuvärdets fördelning är okänd.

Avhandlingens huvuddel behandlar de ovanstående frågorna för perpetuiteter och penningflöden i kontinuerlig tid. För nuvärdet är den kontinuerliga motsvarigheten till den oändliga summan en stokastisk integral,

$$Z_\infty = \int_0^\infty e^{-X_s} dY_s,$$

där X är processen som representerar räntegraden och Y representerar betalningar. I det kontinuerliga fallet är mycket av teorin betydligt mer arbetsdryg, men då X och Y är oberoende Lévy processer kan ändlighetsvillkor och momentformler härledas, och explicita fördelningar för nuvärdet kan hittas för flera specialfall. Detta är särskilt gynnsamt eftersom det är betydligt mera besvärligt att simulera en kontinuerlig perpetuitet än en diskret sådan.

I det kontinuerliga fallet presenteras visserligen villkor för nuvärdets ändlighet relativt kortfattat, men momentformler behandlas utförligt i avsnitt 4.2, där bland annat tidigare forskning av professor P. Salminen och hans kollega L.

Vostrikova presenteras. Det kanske huvudsakliga resultatet i detta stycke är, under antagandet att Y är en deterministisk drift, alla ordningars momentformler för Z_∞, och den motsvarande ändliga integralen

$$Z_t = \int_0^t e^{-X_s} ds,$$

varav det senare fallets formler är motiverade med en ny härledning.

Forskning rörande fördelningen av en kontinuerlig perpetuitets nuvärde inleddes 1990 då aktuarien D. Dufresne publicerade en ytterst innovativ härledning av fördelningen för

$$\int_0^\infty e^{-\gamma s - \sigma W_s} ds,$$

där W är en standard Brownsk rörelse. Dufresne bevisade att integralen följer en invers gammafördelning och hans resultat återges även i denna avhandling, men beviset som presenteras här använder en enklare metod som senare upptäcktes av forskarna J. Bertoin och M. Yor.

Större delen av de återstående fördelningarna som behandlas är hämtade ur en artikel av H.K. Gjessing och J. Paulsen, men de flesta av härledningarna är nya. Det visade sig för det första att Bertoin och Yors metod kunde utnyttjas för några av Gjessing och Paulsens exempel, och för det andra är en – till min kännedom – ny innovation i denna avhandling att identifiera integraler med tidigare kända diskreta perpetuiteter. Således överförs många av bevisena på utsagor som redan tidigare bevisats med enklare metoder, och genom denna metod erhålls även några nya resultat.

Avhandlingens sista kapitel behandlar några intressanta tillämpningar av materialet. En orealistisk, men intressant, stokastisk modell för aktiepriser demonstreras i ett avsnitt, medan det följande avsnittet är en redogörelse för hur man kan approximera ett kontinuerligt penningflöde med en diskret modell (eller vice versa). Slutligen presenteras en av J. Paulsen upptäckt riskteoretisk ekvation som relaterar sannolikheten för konkurs till nuvärdet av en kontinuerlig perpetuitet.

Denna avhandling utgör således en rätt utförlig blanding av litteraturöversikt, ny forskning, och tillämpningar. Presentationen är till stor del inspirerad av D. Dufresnes år 1990 publicerade artikel, där han behandlade en blandning av konvergensresultat, momentformler, tillämpningar, och explicita fördelningar för nuvärdet. Dufresnes artikel är dessutom en av de få artiklar som behandlar både modeller med diskret och kontinuerlig tid, varför man kunde se denna avhandling som något av en uppföljare till Dufresnes artikel. I så fall har avhandlingen en stor föregångare att leva upp till.

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Chapter 1

Introduction

In finance, an *annuity* is a contract giving its owner a sequence of payments at regular intervals. In a financial portfolio the annuity payments will, as they arrive, be invested in other assets, such as bonds or stocks. Of course, another possibility is to buy an annuity with the intention of using the payments directly to pay regular fees, like e.g. property taxes.

As a basic example, consider an annuity with payments $c > 0$ arriving monthly, for $N \in \mathbb{N}$ months. As they arrive, the annuity payments start accruing monthly interest from bonds, at the constant rate $r > 0$. Then the cash in the portfolio at month n is

$$S(n) = \begin{cases} \sum_{k=1}^{n} c(1+r)^{k-1}, & \text{if } n < N, \\ \sum_{k=1}^{N} c(1+r)^{n-N+k-1}, & \text{if } n \geq N. \end{cases} \tag{1.1}$$

Valuation of a financial asset relies on calculating its *present value*. The present value is a concept from economics, defined as the cash amount of bonds you would need if you had to pay all the future payments of the asset. The present value at month zero of the above annuity is

$$Z(N) = \sum_{k=1}^{N} c(1+r)^{-k} = c\frac{1-(1+r)^{-N}}{r}. \tag{1.2}$$

The factor $(1+r)^{-1}$ is also called the *discount factor*, while $Z(N)$ may be called a *discounted cash flow*. The justification for this is that you would need $(1+r)^{-1}$ now in order to have 1 next month.

Pricing an annuity by its present value is also called *rational* pricing, because it can easily be shown that any price deviating from the present value would lead to a situation where either the seller or buyer could gain money with no risk – a situation called *arbitrage*. The *no-arbitrage principle* is a cornerstone of financial theory.

The present Master's thesis mainly concerns the pricing of a financial asset called a *perpetuity* – a variant of an annuity with payments continuing forever (or, one might say, the payments continue in perpetuity). The above example is a perpetuity if we let $N = +\infty$, and the rational price of this perpetuity can be found by taking the limit of $Z(N)$ as $N \to \infty$. As such,

$$Z(\infty) = \frac{c}{r}. \tag{1.3}$$

Of course, the deterministic case with constant payments is trivial. For this reason we generalize and study a perpetuity with a random payment process $(C_k)_{k\in\mathbb{N}}$ and random discount process $(V_k)_{k\in\mathbb{N}}$. Thus,

$$Z(\infty) = \sum_{k=1}^{\infty} C_k \prod_{j=1}^{k} V_j. \tag{1.4}$$

There is no counterpart to the geometric series formula in the stochastic case, but we shall see that in some special cases $Z(\infty)$ has a distribution with a well-known density. In such cases putting a price on the perpetuity is as simple as calculating the expected value of its distribution. Alternatively, a risk averse trader may calculate the expected utility using any utility function of choice.

The bulk of the thesis is devoted perpetuities in a continuous-time setting, i.e. a continuous cash flow subject to continuously varying discount factors. In this case, the sum (1.4) has an integral counterpart, namely

$$Z_\infty = \int_0^\infty e^{-X_t} dY_t, \tag{1.5}$$

where X is a process representing the discount factors and Y is the cash flow. In several cases, an explicit distribution for Z_∞ is found. We shall also study cases when the continuous cash flow is a diffusion type process.

Aside from finding explicit densities for Z_∞, two other subjects are studied. First of all, under which conditions can we even say that Z_∞ is finite? Second, when possible, formulas for the moments of Z_∞ are derived, as well as moments for cash flows lasting only a finite time. A trader could potentially use the moments to make purchasing decisions by relying on methods from Markowitz portfolio theory. Both the moments and the question of finiteness are treated in discrete and continuous time.

A secondary theme that is touched upon is financial modelling with jump processes. For instance, compound Poisson processes belong to the class of Lévy processes and have quite nice properties, allowing for several interesting results. Despite these processes not being diffusions, we find several explicit distributions for the discounted perpetuity Z_∞ as in (1.5), with X or Y having jump components.

Although the primary application discussed in this thesis is valuation of perpetuities, the mathematical content can easily turn out useful in other domains. For example, the integral $\int_0^T e^{-X_s} dY_s$ has an obvious interpretation as the present value of a portfolio, with dividends or fees arriving according to the process Y. Varying discount factors can either be interpreted as a changing interest rate, or a prediction of the varying time preference of the trader. We shall also see that distributions of a perpetuity are necessarily solutions to a stochastic equation.

Moreover, stochastic processes with jumps are potentially a useful component in fixing some alleged deficiencies in standard financial models. Traditional financial models cannot capture the market impact of a CEO getting caught smoking a cannabis joint on a public podcast, or of a country unexpectedly voting to leave the European Union. More mundanely, a left-wing government may suddenly cause an interest rate hike by beginning a major expansion of government programs, or alternatively a right-wing government may contract government spending, tanking interest rates in the process. Unexpected events occur when there is a mismatch between common knowledge and reality, but predicting this kind of mismatch remains difficult. This being the case, the prudent trader needs to consider the risk of sudden shocks and plan accordingly.

The main part of the thesis is divided into two chapters – one dedicated to the discrete-time setting and the other to continuous time. Preceding those is chapter 2, a brief overview of theorems and definitions that are necessary in the later chapters. Thus, chapters 3 and 4 treat the discrete and continuous-time settings, respectively. Finally, chapter 5 contains a short, partial overview of applications related to the theory.

Despite the finance-oriented theme, this is still a Master's thesis in mathematics. Due to this, the reader is naturally expected to have prior familiarity with analysis, probability theory, stochastic processes and stochastic calculus. Throughout the text are a few novel proofs and results, but the bulk of the thesis does still rely on the work of others, most importantly Dufresne [13, 12], Yor [2], Vervaat [33], Gjessing and Paulsen [15] and Salminen and Vostrikova [29]. Finally, I wish the reader a good time while studying the contents and thank for any interest shown.

Chapter 2

Preliminary theory

This chapter serves as a reference for theorems and definitions used in the later chapters. First, some notation and conventions are introduced. Afterwards follow sections on probability theory, stochastic processes, and finance.

If $x \in \mathbb{R}$, we use $[x]$ as notation for the largest integer smaller than or equal to x, i.e.

$$[x] = \max\{n \in \mathbb{Z} : n \leq x \in \mathbb{R}\}.$$

If $x = (x_1, x_2, \dots)$ is a sequence in $\mathbb{R}$, the notation $\#y(x)$ is used for the number of elements in x that are equal to $y \in \mathbb{R}$, i.e.

$$\#y(x) = \Big|\{n \in \mathbb{N} : x_n = y\}\Big|.$$

The limit superior and inferior of sequences of sets are defined by

$$\limsup_{n\to\infty} A_n = \bigcap_{k\geq 1} \bigcup_{n\geq k} A_n,$$

$$\liminf_{n\to\infty} A_n = \bigcup_{k\geq 1} \bigcap_{n\geq k} A_n.$$

We adopt the convention of using X_{t-} to denote the limit when s approaches $t > 0$ from the left, i.e.

$$X_{t-} := \lim_{s\uparrow t} X_s, \quad t > 0.$$

2.1 Elementary probability and stochastics

Lemma 2.1 (Borel-Cantelli Lemma). *If $\{A_n,\ n = 1, 2, \dots\}$ is a sequence of independent events in a given probability space, then*

$$\mathbf{P}\left(\limsup_{n\to\infty} A_n\right) = 1, \quad \text{if } \sum_{n=1}^{\infty} \mathbf{P}(A_n) = \infty,$$

$$\mathbf{P}\left(\limsup_{n\to\infty} A_n\right) = 0, \quad \text{if } \sum_{n=1}^{\infty} \mathbf{P}(A_n) < \infty.$$

The second assertion also holds without the assumption of independence.

Proof. See [19]. □

A basic fact of probability is that the moment-generating function (when it exists) uniquely determines the distribution of a random variable. In cases when it is difficult to calculate the moment-generating function, it may be possible to instead identify the distribution of a random variable by its positive integer moments.

Theorem 2.2 (Billingsley [3]). *Let X be a random variable on some probability space and suppose $\alpha_n = \mathbf{E}(X^n)$ is finite for every $n \in \mathbb{N}$. If the power series*

$$s(r) = \sum_{k=0}^{\infty} \frac{\alpha_k r^k}{k!}$$

converges within some neighbourhood of zero, then it holds for every random variable Y that

$$\mathbf{E}(Y^n) = \mathbf{E}(X^n) \quad \text{for every } n \in \mathbb{N} \Rightarrow X \stackrel{d}{=} Y.$$

Proof. See Billingsley [3, Ch. 30]. □

When the power series in Theorem 2.2 converges, it is said that the distribution is *determined by its (positive integer) moments.* Note that if the moment-generating function of X exists, then X is determined by its moments.

Definition 2.3. If $x = (x_0, x_1, x_2, \dots)$ is a real sequence, then $\theta(x) = (x_1, x_2, x_3, \dots)$ defines the *shift operator* θ. A set of sequences A is called *shift-invariant*, if $\theta(x) \in A$ if and only if $x \in A$.

Definition 2.4 (Stationary process)**.** Let $X = (X_t)_{t\in\mathbb{R}}$ be a stochastic process. Then the process X is called *stationary*, if

$$(X_{t_1}, X_{t_2}, \dots, X_{t_n}) \overset{d}{=} (X_{t_1+\tau}, X_{t_2+\tau}, \dots, X_{t_n+\tau}),\ \forall \tau, t_1, \dots, t_n \in \mathbb{R},\ \forall n \in \mathbb{N}.$$

Definition 2.5 (Ergodic stationary process in discrete time)**.** Let $X = (X_n)_{n\in\mathbb{Z}}$ be a stationary stochastic process. Then it is *ergodic*, if every shift-invariant event is trivial.

Theorem 2.6 (Birkhoff's pointwise ergodic theorem)**.** *Let* $(X_n)_{n\geq 1}$ *be stationary and ergodic with* $\mathbf{E}(|X_1|) < \infty$*. Then*

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} X_k = \mathbf{E}(X_1) \quad a.s.$$

Proof. See e.g. [21] □

2.2 Lévy processes

Definition 2.7. Let $X = (X_t)_{t\geq 0}$ be a stochastic process on some probability space. X is an *additive* process if it has independent increments.

An additive process is *homogeneous* if it has stationary increments.

A *Lévy process* is a stochastically continuous homogeneous additive process with $X_0 = 0$ a.s.

Theorem 2.8. *If X is a Lévy process, its characteristic function is for all $t \geq 0$ given by*

$$\mathbf{E}\left(e^{i\theta X_t}\right) = e^{-t\Psi(\theta)},\ \theta \in \mathbb{R},$$

where Ψ is the characteristic exponent *of X, given by the* Lévy-Khintchine formula

$$\Psi(\theta) = ia\theta + \frac{1}{2}\sigma^2\theta^2 + \int_{\mathbb{R}} (1 - e^{i\theta x} + i\theta x \mathbf{1}_{\{|x|<1\}})\Pi(dx),$$

where $a, \sigma \in \mathbb{R}$ and Π is a measure on $\mathbb{R}\setminus\{0\}$ satisfying

$$\int_{-\infty}^{+\infty} \min\{1, |x|^2\}\Pi(dx) < \infty.$$

Proof. See e.g. [31]. □

It follows that every valid characteristic exponent Ψ uniquely determines a corresponding Lévy process. The triple (a, σ^2, Π) is called the *Lévy-Khintchine triplet* and can be used to uniquely characterise the characteristic exponent of a Lévy process. We now state a few examples.

Example 2.9. (a) Let X be a Brownian motion with drift, i.e. $X_t = \gamma t + \sigma W_t$, where W is a standard Brownian motion. The characteristic function of X_t is

$$\mathbf{E}\left(e^{i\theta X_t}\right) = e^{-i\gamma\theta t - \frac{\sigma^2\theta^2}{2}t} = e^{-t(i\gamma\theta + \frac{\sigma^2\theta^2}{2})},$$

so X is a Lévy process with Lévy-Khintchine triplet (γ, σ^2, Π), where $\Pi \equiv 0$.

(b) Let Y be a compound Poisson process with intensity λ, i.e.

$$Y_t = \sum_{k=1}^{N_t} Z_k,$$

where N_t is a λ-intensity Poisson process and $(Z_k)_{k\in\mathbb{N}}$ is an i.i.d. sequence of jumps with density f_Z. The characteristic function of Y_t is given by

$$\begin{aligned}\varphi_{Y_t}(\theta) = e^{\lambda t(\varphi_Z(\theta)-1)} &= \exp\left\{-t\left(\lambda\left(1 - \int_{\mathbb{R}} e^{i\theta x} f_Z(x)dx\right)\right)\right\} \\ &= \exp\left\{-t\left(\int_{\mathbb{R}} \lambda f_Z(x)dx - \int_{\mathbb{R}} e^{i\theta x}\lambda f_Z(x)dx\right)\right\} \\ &= \exp\left\{-t\left(\int_{\mathbb{R}} \left(1 - e^{i\theta x}\right)\Pi(dx)\right)\right\},\end{aligned}$$

where $\Pi(dx) = \lambda f_Z(x)dx$ and f_Z is the density of Z. Then it is easy to see that Y is a Lévy process with Lévy-Khintchine triplet

$$\left(\; -\lambda\int_{(-1,1)} x f_Z(x)dx \; , \quad 0 \; , \quad \lambda f_Z(x)dx \; \right).$$

(c) Let X be a jump-diffusion, i.e.

$$X_t = \gamma t + \sigma W_t + \sum_{k=1}^{N_t} Z_k,$$

where W is a standard Brownian motion, N is a Poisson process with intensity λ independent of W and $(Z_k)_{k\in\mathbb{N}}$ is a sequence of i.i.d. jumps. It follows from (a) and (b) that X is a Lévy process with Lévy-Khintchine triplet

$$\left(\; \gamma - \lambda\int_{(-1,1)} x f_Z(x)dx \; , \quad \sigma^2 \; , \quad \lambda f_Z(x)dx \; \right).$$

2.3 Stochastic calculus

Definition 2.10. Let $(\mathcal{F}_t)_{t\geq 0}$ be a filtration on a given probability space $(\Omega, \mathcal{F}, \mathbf{P})$. Then, $\mathbb{L}^2(0,T)$ is the space of stochastic processes $G = (G_t)_{t\geq 0}$ such that G is progressively measurable with respect to $(\mathcal{F}_t)_{t\geq 0}$ and

$$\mathbf{E}\left(\int_0^T G_s^2 ds\right) < \infty$$

holds.

Similarly, $\mathbb{L}^1(0,T)$ is the space of stochastic processes $F = (F_t)_{t\geq 0}$ such that F is progressively measurable with respect to $(\mathcal{F}_t)_{t\geq 0}$ and

$$\left|\mathbf{E}\left(\int_0^T F_s ds\right)\right| < \infty$$

holds.

If $T = \infty$ we shall write only $\mathbb{L}^1$ and $\mathbb{L}^2$.

Definition 2.11 (Itô process)**.** Let X be a stochastic process on the probability space $(\Omega, \mathcal{F}, \mathbf{P})$ that is progressively measurable with respect to a filtration $(\mathcal{F}_t)_{t\geq 0}$. Then X is called an *Itô process* if X can be written in the form

$$X_t = X_0 + \int_0^t F_s ds + \int_0^t G_s dW_s,$$

where $F \in \mathbb{L}^1, G \in \mathbb{L}^2$.

Lemma 2.12 (Itô's formula)**.** *Let X be an Itô process with differential $dX_t = F_t dt + G_t dW_t$. Suppose $f(t,x) \in \mathbb{C}^{1,2}(\mathbb{R}_+ \times \mathbb{R})$ and that $f(t, X_t) \in \mathbb{L}^2$. Then $Y_t = f(t, X_t)$ is also an Itô process, with differential*

$$dY_s = \left(\frac{\partial f}{\partial t}(s, X_s) + \frac{\partial f}{\partial x}(s, X_s)F_s + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(s, X_s)G_s^2\right) ds + \frac{\partial f}{\partial x}(s, X_s)G_s dW_s.$$

Proof. See e.g. [20]. □

Lemma 2.13 (Itô's formula in multiple dimensions)**.** *Let W be a vector of d independent standard Brownian motions. Let $dX_t = F_t dt + G_t dW_t$, where the vector $F = (F_1, \ldots, F_d)$ and matrix $G = (G_{ij})_{1\leq i,j\leq d}$ have components in $\mathbb{L}^1$ and*

$\mathbb{L}^2$, *respectively. Further, let* $f(t,x) \in \mathbb{C}^{1,2}(\mathbb{R}_+ \times \mathbb{R}^d)$. *Then* $Y_t = f(t, X_t)$ *is also an Itô process and*

$$dY_s = \left(\frac{\partial f}{\partial t}(s, X_s) + \sum_{k=1}^{d} \frac{\partial f}{\partial x_k}(s, X_s) F_{k,s} + \sum_{k=1}^{d} \frac{1}{2} \frac{\partial^2 f}{\partial x_k^2}(s, X_s) G_{kk,s}^2 \right) ds$$

$$+ \sum_{n=1}^{d} \sum_{k=1}^{d} \frac{\partial f}{\partial x_k}(s, X_s) G_{kn,s} dW_{n,s}.$$

Proof. See e.g. [20]. □

Definition 2.14 (Quadratic variation)**.** Let $X = (X_t)_{t \in \mathbb{R}}$ be a stochastic process on a probability space $(\Omega, \mathcal{F}, \mathbf{P})$. The *quadratic variation* of X is a process denoted by $\langle X \rangle_t$, defined by

$$\langle X \rangle_t = \lim_{\|P\| \to 0} \sum_{k=1}^{n} (X_{t_k} - X_{t_{k-1}})^2,$$

where P belongs to the set of partitions of the interval $[0, t]$ and

$$\|P\| = \max\{(t_k - t_{k-1}); [t_{k-1}, t_k] \text{ subinterval in } P\}.$$

Definition 2.15 (Cross-variation)**.** Let there be two processes X and Y defined on a probability space $(\Omega, \mathcal{F}, \mathbf{P})$. The *cross-variation* of X and Y is a process denoted by $\langle X, Y \rangle_t$ and is given by

$$\langle X, Y \rangle_t = \frac{1}{4} \left(\langle X + Y \rangle_t - \langle X - Y \rangle_t \right).$$

Note in particular that $\langle X, X \rangle_t = \langle X \rangle_t$.

Proposition 2.16. *Let* $W = (W^1, W^2, \ldots, W^d)_{t \geq 0}$ *be a standard d-dimensional Brownian motion adapted to the filtration* $(\mathcal{F}_t)_{t \geq 0}$, *let* X, Y *and* Z *be Itô processes and let* $F \in \mathbb{L}^1$, *and* $G, G' \in \mathbb{L}^2$ *be processes adapted to* $(\mathcal{F}_t)_{t \geq 0}$. *The cross-variation, as defined in 2.15, satisfies the following properties.*

1. $\langle X, Y \rangle_t = \langle Y, X \rangle_t$.

2. $\langle \alpha X + Y, Z \rangle_t = \alpha \langle X, Z \rangle_t + \langle Y, Z \rangle_t$.

3. $\langle \int_0^{\cdot} F_s ds, X \rangle_t = 0$.

4. $\langle \int_0^\cdot G_s dW_s^i, \int_0^\cdot G'_s dW_s^j \rangle_t = 0$, *if* $i \neq j$.

5. $\langle \int_0^\cdot G_s dW_s^i, \int_0^\cdot G'_s dW_s^i \rangle_t = \int_0^t G_s G'_s ds$.

Proposition 2.17 (Itô integration by parts formula)**.** *Let X and Y be two Itô processes. Then*

$$d(X_t Y_t) = X_t dY_t + Y_t dX_t + d\langle X, Y \rangle_t. \tag{2.1}$$

Proof. See e.g. [23]. □

We shall now extend some of the above results for Poisson jump processes. For a better exposition of stochastic calculus with jump processes, the reader is referred to [7].

Let $Y = (Y_t)_{t\geq 0}$ be a compound Poisson process, written as

$$Y_t = \sum_{k=1}^{N_t} Z_k,$$

with $N = (N_t)_{t\geq 0}$ a Poisson process with intensity $\lambda > 0$, and $(Z_k)_{k\in\mathbb{N}}$ an i.i.d. sequence of jump sizes. We denote the jump size at time $t > 0$ by $\Delta Y_t \coloneqq Y_t - Y_{t-}$, which leads to a relation between the jump sizes of Y and N,

$$\Delta Y_t = Z_{N_t} \Delta N_t, \tag{2.2}$$

where $\Delta N_t = N_t - N_{t-}$ equals 1 only for the jump times $T_1, T_2, \ldots$ of N.

Based on the relation (2.2), we define the stochastic integral with respect to Y, which is in fact a Lebesgue-Stieltjes integral, by

$$X_t = \int_0^t G_s dY_s = \int_0^t G_s Z_{N_s} dN_s \coloneqq \sum_{k=1}^{N_t} G_{T_k} Z_k. \tag{2.3}$$

We also express the integral $X_t = \int_0^t G_s dY_s$ equivalently as

$$dX_t = G_t dY_t = G_t Z_{N_t} dN_t.$$

We shall next present some formulas for stochastic integrals with respect to a compound Poisson process, including a version of the Itô isometry and Itô's formula. These formulas are also familiar from the theory of Poisson processes.

Proposition 2.18 (Privault [28]). *Let $(G_t)_{t\geq 0}$ be a stochastic process progressively measurable with respect to the filtration generated by $(Y_t)_{t\geq 0}$, a compound Poisson process with intensity λ and i.i.d. jumps Z. Then, if $G \in \mathbb{L}^1$,*

$$\mathbf{E}\left(\int_0^t G_{s-} dY_s\right) = \mathbf{E}\left(\int_0^t G_{s-} Z_{N_s} dN_s\right) = \lambda \mathbf{E}(Z)\, \mathbf{E}\left(\int_0^t G_s ds\right). \qquad (2.4)$$

If $G \in \mathbb{L}^2$,

$$\mathbf{E}\left[\left(\int_0^t G_{s-}(dY_s - \lambda \mathbf{E}(Z)\, ds)\right)^2\right] = \lambda \mathbf{E}\left(Z^2\right) \mathbf{E}\left(\int_0^t G_s^2 ds\right). \qquad (2.5)$$

Proof. See [28]. □

Lemma 2.19 (Itô's Formula). *Let X be a stochastic integral with respect to a jump-diffusion, i.e.*

$$dX_t = F_t dt + G_t dW_t + H_t dY_t,$$

with $F, H \in \mathbb{L}^1, G \in \mathbb{L}^2$, and where Y is a compound Poisson process,

$$Y_t = \sum_{k=1}^{N_t} Z_k,$$

with N a Poisson process with intensity λ, and $(Z_k)_{k\in\mathbb{N}}$ an i.i.d. sequence of jumps.

Suppose $f(t,x) \in \mathbb{C}^{1,2}(\mathbb{R}_+ \times \mathbb{R})$ and that $f(t, X_t) \in \mathbb{L}^2$. Then, the process $U_t = f(t, X_t)$ is also a stochastic integral with respect to a jump-diffusion, and in particular

$$\begin{aligned} dU_t = &\left(\frac{\partial f}{\partial t}(s, X_s) + \frac{\partial f}{\partial x}(s, X_s) F_s + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(s, X_s) G_s^2\right) ds + \frac{\partial f}{\partial x}(s, X_s) G_s dW_s \\ &+ \left(f(t, X_t) - f(t, X_{t-})\right) dN_t. \end{aligned}$$

Proof. See [28]. □

Furthermore, we note that several more properties of the cross-variation 2.15 hold. In the following, let $W = (W^1, W^2, \ldots, W^d)_{t\geq 0}$ be a standard d-dimensional Brownian motion and $N = (N^1, N^2, \ldots, N^k)_{t\geq 0}$ be a vector of k independent Poisson processes. The processes $F, H, H' \in \mathbb{L}^1$ and $G \in \mathbb{L}^2$ are adapted to the filtration generated by $(W, N)_{t\geq 0}$.

6. $\langle \int_0^\cdot F_s ds, \int_0^\cdot H_s dN_s^i \rangle_t = 0$.

7. $\langle \int_0^\cdot G_s dW_s^i, \int_0^\cdot H_s dN_s^j \rangle_t = 0$.

8. $\langle \int_0^\cdot H_s dN_s^i, \int_0^\cdot H'_s dN_s^j \rangle_t = 0$, if $i \neq j$.

9. $\langle \int_0^\cdot H_s dN_s^i, \int_0^\cdot H'_s dN_s^i \rangle_t = \int_0^t H_s H'_s dN_s^i$.

Proposition 2.20. *The Itô integration by parts formula* (2.1) *holds also for stochastic integrals with respect to a jump-diffusion.*

Proof. See [28]. □

Chapter 3

Discrete-time models

Consider a sequence of rates of return, R_k, constant over the time period $[k-1, k)$, and a sequence of random cash payments C_k, $k = 1, 2, \dots$, that are always made at the beginning of time period $[k, k+1)$. We will call the initial capital held S_0, and consider the accumulated capital sequence $S = (S_k)_{k\geq 1}$ as well as the present value sequence $Z = (Z_k)_{k\geq 1}$ of payments up until $t = k$.

A recursive equation for S_k, $k \geq 1$, is

$$S_k = U_k S_{k-1} + C_k, \tag{3.1}$$

where $U_k := 1 + R_k$. This equation is called the *annuity equation* because in finance, an *annuity* is a finite sequence of payments made at regular intervals, and so satisfy the annuity equation with initial value $S_0 = 0$.

In finance, cash flows and annuities are priced using their present value Z_k, representing the present value of all payments made up to and including period k. With $V_k := U_k^{-1}$, the evolution through time can easily be derived from the relation $Z_k = S_k \prod_{j=1}^{k} V_j$, leading to the recursive equation

$$Z_k = Z_{k-1} + C_k \prod_{j=1}^{k} V_j. \tag{3.2}$$

In particular, starting from $Z_0 = S_0$,

$$Z_1 = Z_0 + V_1 C_1,$$

and if we assume that

$$Z_{k-1} = S_0 + \sum_{i=1}^{k-1} C_i \prod_{j=1}^{i} V_j,$$

then insertion into (3.2) yields

$$Z_k = \left(S_0 + \sum_{i=1}^{k-1} C_i \prod_{j=1}^{i} V_j \right) + C_k \prod_{j=1}^{k} V_j,$$

which simplifies to

$$Z_k = S_0 + \sum_{i=1}^{k} C_i \prod_{j=1}^{i} V_j. \tag{3.3}$$

By induction it follows that (3.3) holds for all $k \in \mathbb{N}$.

The present value of a perpetuity can be defined as the limit of the present value of an annuity, i.e. the limit $Z_\infty = \lim_{k\to\infty} Z_k$ of (3.3), with $S_0 = 0$ so that Z_k is the present value of an annuity. Therefore, in the rest of the chapter we shall set $S_0 = 0$ in (3.3), so that

$$Z_k = \sum_{i=1}^{k} C_i \prod_{j=1}^{i} V_j. \tag{3.4}$$

We also define the corresponding infinite sum,

$$Z_\infty := \sum_{i=1}^{\infty} C_i \prod_{j=1}^{i} V_j.$$

Due to the importance of the discounting factors, we also define

$$\mu := \mathbf{E}\left(\log|V_1|\right),$$

when it exists.

Like annuities, the pricing of a perpetuity is carried out using its present value Z_∞. If the distributions of every Z_k are known, which is a very special case, it may be possible to compute the distribution of Z_∞ by elementary methods. In other cases the distribution has to be inferred from the distribution of $(C_k, V_k)_{k\geq 1}$, if possible.

3.1 Finiteness of perpetuities

3.1.1 Sufficient conditions in general setting

While studying the convergence criteria of the stochastic difference equation $Y_n = A_n Y_{n-1} + B_n$, Vervaat [33] and Brandt [4] also found some convergence criteria for the perpetuity $(Z_k)_{k\geq 1}$. Here we present a proof with Vervaat's argument applied to the conditions in Brandt's paper. But before the theorem, one part of Vervaat's argument shall be presented as a separate lemma.

Lemma 3.1. *Let $(C_k, V_k)_{k\geq 1}$ be an i.i.d. sequence and assume that $\mu = \mathbf{E}(\log|V_1|)$ exists and is finite. Then,*

(a) if $\mathbf{E}\left(\log|C_1|^+\right) = \infty$, then $\limsup_{n\to\infty} |C_n V_1 \cdots V_n|^{1/n} = \infty$ a.s.

(b) if $\mathbf{E}\left(\log|C_1|^+\right) < \infty$, then $\limsup_{n\to\infty} |C_n V_1 \cdots V_n|^{1/n} \leq e^\mu$ a.s.

The second assertion holds even if $(C_k, V_k)_{k\geq 1}$ is stationary and ergodic but not necessarily i.i.d. It also holds if $\mu = -\infty$ (with the interpretation $e^\mu = 0$).

Proof. We take an arbitrary $a > 1$ and write

$$\begin{aligned}\mathbf{E}\left(\log|C_1|^+\right) &= \int_0^\infty \mathbf{P}\left((\log|C_1|)^+ > x\right) dx \\ &= \log a \int_0^\infty \mathbf{P}\left((\log|C_1|)^+ > x \log a\right) dx.\end{aligned}$$

Define $f(x) = \mathbf{P}\left((\log|C_1|)^+ > x \log a\right)$ and note that since f is bounded and non-increasing,

$$\sum_{n=1}^\infty f(n) < \infty \quad \Longleftrightarrow \quad \int_0^\infty f(x)dx < \infty,$$

by Cauchy's integral test for series convergence (note that f need not be continuous).

The number $f(n)$ is now interpreted as the probability of the event

$$\xi_n = \left\{\omega \in \Omega : (\log|C_n|)^+ > n \log a\right\}.$$

The events are independent and so the Borel-Cantelli Lemma (Lemma 2.1) implies that

(i) if $\mathbf{E}\left(\log|C_1|^+\right) = \infty$, then $\mathbf{P}\left(\limsup_{n\to\infty} \xi_n\right) = 1$.

(ii) if $\mathbf{E}\left(\log|C_1|^+\right) < \infty$, then $\mathbf{P}\left(\limsup_{n\to\infty} \xi_n\right) = 0$.

The event

$$\limsup_{n\to\infty} \xi_n = \left\{\omega \in \Omega : (\log|C_n|)^+ > n \log a \text{ i.o.}\right\}$$

is clearly equivalent to

$$\left\{\omega \in \Omega : \log\left(|C_n|^{1/n}\right) > \log a \text{ i.o.}\right\}.$$

In case (i), this implies that, since $a > 1$ was arbitrary,

$$\forall a > 1 : \quad \mathbf{P}\left(\log\left(|C_n|^{1/n}\right) > \log a \text{ i.o.}\right) = 1,$$

that is, $\limsup_{n\to\infty}|C_n|^{1/n} > a$ a.s. for any $a > 1$. As such, with μ finite according to assumption and applying the Law of Large Numbers,

$$\begin{aligned}
\limsup_{n\to\infty} \log|C_n V_1 V_2 \cdots V_n|^{1/n} &= \limsup_{n\to\infty} \frac{\log|C_n|}{n} + \limsup_{n\to\infty} \frac{1}{n}\sum_{j=1}^{n} \log|V_j| \\
&= \limsup_{n\to\infty} \log\left(|C_n|^{1/n}\right) + \lim_{n\to\infty} \frac{1}{n}\sum_{j=1}^{n} \log|V_j| \\
&= \limsup_{n\to\infty} \log\left(|C_n|^{1/n}\right) + \mu \\
&> \log a + \mu, \quad \text{a.s.} \quad \forall a > 1.
\end{aligned}$$

Thus,

$$\limsup_{n\to\infty}|C_n V_1 V_2 \cdots V_n|^{1/n} = \infty.$$

So case (i) implies the assertion (a) of the lemma.

In the case (ii), for $a > 1$ arbitrarily close to 1,

$$\mathbf{P}\left(\log\left(|C_n|^{1/n}\right) > \log a \text{ i.o.}\right) = 0,$$

which means $|C_n|^{1/n} > 1$ occurs only for a finite number of $n \in \mathbb{N}$, and so, $\limsup_{n\to\infty}|C_n|^{1/n} \leq 1$ a.s. An analogous argument to the one for case (i) leads to

$$\limsup_{n\to\infty} \log\left(|C_n V_1 V_2 \cdots V_n|^{1/n}\right) = \limsup_{n\to\infty} \log\left(|C_n|^{1/n}\right) + \mu \leq \mu,$$

which implies that, almost surely,

$$\limsup_{n\to\infty} |C_n V_1 V_2 \cdots V_n|^{1/n} \le e^{\mu}.$$

Finally, the argument in case (ii) does not rely on the independence of each element in the sequence $(C_k, V_k)_{k\ge1}$, since the second part of the Borel-Cantelli Lemma does not require the events ξ_n to be independent and the Law of Large Numbers can be replaced by an application of the Birkhoff Ergodic Theorem (Theorem 2.6) if $(C_k, V_k)_{k\ge1}$ is a stationary ergodic sequence.

Moreover, if $\mu = -\infty$ in case (ii), then $\limsup_{n\to\infty} \frac{1}{n}\sum_{j=1}^{n} \log|V_j| = -\infty$ a.s. and so

$$\limsup_{n\to\infty} |C_n V_1 \cdots V_n|^{1/n} = 0 \quad \text{a.s.}$$

□

Theorem 3.2. *The sequence* $(Z_k)_{k\ge1}$*, where* Z_k *is defined as in* (3.4)*, converges absolutely almost surely, if one of the following holds:*

(a) $(V_k)_{k\ge1}$ *is stationary and ergodic, and* $\mathbf{P}(V_1 = 0) > 0$.

(b) $(C_k, V_k)_{k\ge1}$ *is stationary and ergodic, with*

$$\mu < 0 \text{ and } \mathbf{E}\left(\log|C_1|^+\right) < \infty.$$

Proof. Case (a). By the definition of ergodicity, for a stationary ergodic process every shift-invariant event must be trivial (i.e. its probability is either 0 or 1). Let

$$A = \{x = (x_0, x_1, \dots) : x_j \in [0, \infty[, \#0(x) = \infty\}.$$

It is easy to see that A is shift-invariant, so it must be trivial. Further observe that due to stationarity, $\mathbf{P}(V_k = 0) = \mathbf{P}(V_1 = 0) > 0$ for every $k \in \mathbb{N}$. Using this fact, we prove that $\mathbf{P}((V_1, V_2, \dots) \in A) > 0$,

Assume there exists $N \in \mathbb{N}$ such that $\mathbf{P}(V_k > 0, \ \forall k > N) > 0$. Since the set of sequences such that every element after the N:th one is non-zero is also shift-invariant, this probability must in fact be 1. Then the process cannot be stationary. In particular, from the definition of stationarity with $\tau = N$, we would get

$V_1 \overset{d}{=} V_{N+1}$, which is a contradiction since we have both $\mathbf{P}(V_{N+1} > 0) = 1$ and $\mathbf{P}(V_1 = 0) > 0$. Hence, no such N can exist and we have

$$\mathbf{P}((V_1, V_2, \dots) \in A) = 1,$$

that is $\mathbf{P}(V_k = 0 \text{ infinitely often}) = 1$.

This clearly implies that, upon taking the limit $\lim_{k\to\infty} Z_k$ in (3.4), with probability 1 the sum will only contain a finite number of terms before the first $V_k = 0$ occurs. Clearly, Z_k is a finite sum.

Case (b). The proof that $Z_\infty = \lim_{k\to\infty} Z_k$ is well-defined and converges is by Cauchy's root criterion. Given the conditions, case (b) of Lemma 3.1 says that

$$\limsup_{n\to\infty} |C_n V_1 V_2 \cdots V_n|^{1/n} \leq e^{\mu} < 1,$$

since it was assumed that $\mu < 0$. This implies that that $(Z_k)_{k\geq 0}$ converges absolutely almost surely. □

Naturally there is no guarantee that real-world time series will be stationary and ergodic. However, a perpetuity may be finite even despite neither $(V_k)_{k\geq 1}$ nor $(C_k)_{k\geq 1}$ being stationary; all that is required is that the discounting factors approach zero at a high enough rate. Consider the following example:

Example 3.3 (Dufresne [12]). Let $(C_k)_{k\geq 0}$ and $(\log V_k)_{k\geq 0}$ be random walks, defined by $C_0 = \log V_0 = 0$ and for $k \geq 1$,

$$C_k = c + C_{k-1} + e_k$$
$$\log V_k = a + \log V_{k-1} + f_k,$$

where $c \in \mathbb{R}, a < 0$, and $(e_k)_{k\geq 1}, (f_k)_{k\geq 1}$ are two i.i.d. sequences of random variables with mean zero.

First, assume $c \neq 0$. Because $a < 0$, we have $\log V_k \to -\infty$ a.s. as $k \to \infty$. Then, $k^{-1} \sum_{j=1}^{k} \log V_j \to -\infty$ a.s., which implies $|V_1 \cdots V_k|^{1/k} \to 0$ a.s.

If $c \neq 0$,

$$k^{-1} \log|C_k| = k^{-1} \log\left| ck + \sum_{j=1}^{k} e_j \right|$$

$$= k^{-1}\log|ck| + k^{-1}\log\left|1 + (ck)^{-1}\sum_{j=1}^{k} e_j\right| \to 0 \quad \text{a.s.}$$

Whereas if $c = 0$, since the logarithm is concave,

$$k^{-1}\log C_k \leq k^{-1}\sum_{j=1}^{k} e_k - k^{-1} \to 0 \quad \text{a.s.}$$

As such, in both cases

$$\limsup_{k\to\infty} |C_k V_1 \cdots V_k|^{1/k} < 1,$$

and thus the perpetuity is finite a.s. Note that it was not assumed that the processes $(e_k)_{k\geq 1}$ and $(f_k)_{k\geq 1}$ are mutually independent, and neither was it assumed that they have finite variance. We can conclude that there are a.s. finite perpetuities with $\mathbf{Var}\,(C_k) = \infty$ and $C_k \to \infty$ a.s.

3.1.2 Payments and interest as i.i.d. process

It is often difficult or even impossible to explicitly compute the distribution of Z_∞. To this end, we make the restricting assumption of i.i.d. $(C_k, V_k)_{k\geq 1}$ in order to use some known methods of computing the distribution of the perpetuity. Of course Theorem 3.2 still applies in the i.i.d. case. Here is an argument used by Dufresne [12].

We defined the process $(Z_k)_{k\geq 1}$ so that

$$Z_k = C_1V_1 + C_2V_1V_2 + \cdots + C_kV_1V_2\cdots V_k.$$

If $(C_k, V_k)_{k\geq 1}$ is an i.i.d. sequence of random pairs, we can reverse both the payments and discounting factors without changing the distribution. That is, if $(B_k)_{k\geq 1}$ is a stochastic process such that

$$B_k = C_kV_k + C_{k-1}V_kV_{k-1} + \cdots + C_1V_kV_{k-1}\cdots V_1,$$

then

$$Z_k \overset{d}{=} B_k$$

holds for every $k \in \mathbb{N}$. Note that this does not require $(C_k)_{k\geq 1}$ and $(V_k)_{k\geq 1}$ to be mutually independent, only that, for any $k \neq n$, $(C_k, V_k) \perp (C_n, V_n)$. This means that the stochastic equation

$$Z_k \stackrel{d}{=} V_k(Z_{k-1} + C_k) \tag{3.5}$$

holds, so the distribution of Z_k has the same structure as the annuity equation studied by Brandt [4] and Vervaat [33], but only in the i.i.d. case. Because $B_k \xrightarrow{d} Z$ would imply that $Z_k \xrightarrow{d} Z$ also holds, this fact has the interesting implication that we are able to apply much of Vervaat's research on the study of discrete perpetuities.

Remark 3.4. Note the distinction between the processes $(Z_k)_{k\geq 1}$ and $(B_k)_{k\geq 1}$; it is only the case that $Z_k \stackrel{d}{=} B_k$, for every $k \in \mathbb{N}$, and only in the case with i.i.d. discounting and payments. The distinction between them is particularly important when computing autocovariances $\mathbf{Cov}\,(Z_k, Z_{k+m})$.

Vervaat's line of research is about the relationship between the Equation (3.5) and the convergence of $(Z_k)_{k\geq 1}$ when $(C_k, V_k)_{k\geq 1}$ is an i.i.d. sequence. We begin our exposition of this research with an elementary lemma.

Lemma 3.5. *If $Z_k \xrightarrow{d} Z$, then the stochastic equation*

$$Z \stackrel{d}{=} V(Z + C), \quad Z \perp (V, C) \tag{3.6}$$

where $V \stackrel{d}{=} V_1, C \stackrel{d}{=} C_1$, holds.

Proof. Recall the stochastic equation (3.5), in which the left-hand side converges in distribution to Z. For the right-hand side, $(Z_{k-1}, V_k, C_k) \xrightarrow{d} (Z, V, C)$, where (V, C) is independent of Z because every (V_k, C_k) is independent of Z_{k-1}. As such, the right-hand side converges in distribution to $V(Z+C)$ and the assertion holds. □

Since we are mainly interested in actuarial applications, we will typically be working with discounting factors $V = (1 + R)^{-1}$ with $0 < \mathbf{E}\,(V) < 1$, which by Jensen's inequality implies that $\mu = \mathbf{E}\,(\log|V|) < 0$, although we will see that merely assuming $\mu < 0$ is often enough. A more thorough study without this assumption can be found in [16]. We will now show how an explicit distribution can sometimes be derived given the assumption on μ. From Vervaat [33] we gain the following result. The arguments presented below are original except for the first part, which is due to Vervaat.

Lemma 3.6. *Let $-\infty < \mu < 0$. Suppose that, for some pair (V, C), Equation (3.6) has a solution Z. Then the solution is unique in distribution and $(Z_k)_{k\geq 1}$ converges to it a.s.*

Proof. The process $(B_k)_{k\geq 1}$ has a recursive structure like that in (3.5). Moreover, if $B_k \xrightarrow{d} Z$ then also $Z_k \xrightarrow{d} Z$ must hold. We show the uniqueness in distribution by showing that $(B_k)_{k\geq 1}$ can only converge to one random variable, unique in distribution.

We introduce an arbitrary initial variable B_0, independent of every (V_k, C_k). Then the sequence $(B_n)_{n\geq 0}$ is given by

$$B_n(B_0) = B_0 \prod_{k=1}^{n} V_k + \sum_{j=1}^{n} C_j \prod_{k=j}^{k} V_k.$$

If $B_0 = Z$ is a valid solution to (3.6), then $B_1 = V_1(Z + C_1) \stackrel{d}{=} Z$, and so $Z_1 \stackrel{d}{=} Z$. So then for every $n \in \mathbb{N}$, $Z_n \stackrel{d}{=} Z$. Now, let B_0 and B_0' be two distinct random variables, both independent of $(V_k, C_k)_{k\in\mathbb{N}}$. Then,

$$B_n(B_0) - B_n(B_0') = (B_0 - B_0') \prod_{k=1}^{n} V_k.$$

However,

$$(B_0 - B_0') \prod_{k=1}^{n} V_k \xrightarrow{d} 0,$$

which follows from Lemma 3.1 with constant payments $C_j = 1$ for all j and $\mu < 0$. As such, because $B_n(Z) \xrightarrow{d} Z$, it must also be the case that $B_n(B_0) \xrightarrow{d} Z$ for all B_0, and so it must also hold that $Z_n \xrightarrow{d} Z$. In other words, the solution Z is unique in distribution and Z_n converges in distribution to it.

Recall that $(C_k, V_k)_{k\geq 1}$ i.i.d. implies that it is also stationary and ergodic. We prove that the convergence is actually a.s. by proving that the conditions in Theorem 3.2 hold. This is done using proof by contradiction. Thus, we assume the hypothesis $\mathbf{E}\left(\log|C|^{+}\right) = \infty$ and show that it leads to a contradiction.

First, we pick an arbitrary δ such that $0 < \delta < \frac{1}{2}$. Then, we pick two real numbers $s > t$ such that

$$F_Z(s) - F_Z(t) > \frac{1}{2} + \delta.$$

Now, pick large enough natural numbers N_1, N_2 such that $k \geq \max(N_1, N_2)$ implies that $|F_Z(s) - F_{Z_k}(s)| < \delta/2$ and $|F_Z(t) - F_{Z_k}(t)| < \delta/2$, respectively (recall that it was earlier shown that $Z_k \xrightarrow{d} Z$). Then, for such numbers k,

$$\frac{1}{2} + \delta < \frac{\delta}{2} + \frac{\delta}{2} + |F_{Z_k}(s) - F_{Z_k}(t)|, \tag{3.7}$$

because by the triangle inequality,

$$\begin{aligned} |F_Z(s) - F_Z(t)| &= |F_Z(s) - F_{Z_k}(s) + F_{Z_k}(s) - F_{Z_k}(t) + F_{Z_k}(t) - F_Z(t)| \\ &\leq |F_Z(s) - F_{Z_k}(s)| + |F_{Z_k}(s) - F_{Z_k}(t)| + |F_{Z_k}(t) - F_Z(t)|. \end{aligned}$$

So for any such k, from (3.7) it follows that

$$|F_{Z_k}(s) - F_{Z_k}(t)| > \frac{1}{2}.$$

Also due to the convergence in distribution and the completeness of $\mathbb{R}$, the sequence of $F_{Z_n}(t)$ must be a Cauchy sequence. As such, if we let $\epsilon = 1/2$, there exists $N_3 \in \mathbb{N}$ such that for all $m, n \geq N_3$,

$$|F_{Z_m}(s) - F_{Z_n}(s)| < \epsilon. \tag{3.8}$$

Now, we define a stopping time τ with respect to the filtration $(\mathcal{F}_n)_{n=1}^{\infty}$, where $\mathcal{F}_n$ is the σ-algebra generated by $(C_n, V_n)_{n=1}^{\infty}$, by

$$\tau := \min\{n \in \mathbb{N} : n > \max(N_1, N_2, N_3), |C_n V_1 \cdots V_n| > s - t\}.$$

Recall that it was assumed that $\mathbf{E}\left(\log|C|^+\right) = \infty$. Since we know $-\infty < \mu < 0$, by part (a) of Lemma 3.1,

$$\limsup |C_n V_1 \cdots V_n| = \infty \quad \text{a.s.}$$

Thus it is easy to see that the stopping time τ is finite a.s. This fact makes it possible to identify two disjoint cases based on the properties of the process $(C_k, V_k)_{k\geq 1}$. Two new stopping times, τ_+ and τ_- are defined by

$$\tau_+ := \min\left\{n \in \mathbb{N} : n \geq \tau, C_n V_1 \cdots V_n > s - t\right\},$$

$$\tau_- := \min\left\{n \in \mathbb{N} : n \geq \tau, C_n V_1 \cdots V_n < t - s\right\}.$$

The idea is that in every case either τ_+ or τ_- is equal to τ, but we use an alternative partition for the sake of simplicity. Since $\tau < \infty$ a.s. either (a) $\tau_+ < \infty$ a.s. or (b) $\tau_+ = \infty$ a.s. but $\tau_- < \infty$ a.s. Cases (a) and (b) are disjoint, so we finish the proof separately for these two cases.

Recall from (3.2) that

$$Z_{\tau+1} = Z_\tau + C_\tau V_1 \cdots V_\tau.$$

Due to how τ is defined, we know from (3.8) that

$$\epsilon > \left|F_{Z_{\tau+1}}(s) - F_{Z_\tau}(s)\right|,$$

and this holds also for τ_+ and τ_- since they are larger than or equal to τ. Consequently, in case (a):

$$\begin{aligned} F_{Z_{\tau_+ +1}}(s) &= \mathbf{P}\left(Z_{\tau_+} + C_{\tau_+} V_1 \cdots V_{\tau_+} \leq s\right) \\ &= \mathbf{P}\left(Z_{\tau_+} + C_{\tau_+} V_1 \cdots V_{\tau_+} \leq s, C_{\tau_+} V_1 \cdots V_{\tau_+} > s - t\right) \\ &\leq \mathbf{P}\left(Z_{\tau_+} \leq t\right) = F_{Z_{\tau_+}}(t). \end{aligned}$$

Thus,

$$\begin{aligned} \frac{1}{2} = \epsilon &< \left|F_{Z_{\tau_+}}(s) - F_{Z_{\tau_+}}(t)\right| = F_{Z_{\tau_+}}(s) - F_{Z_{\tau_+}}(t) \\ &\leq F_{Z_{\tau_+}}(s) - F_{Z_{\tau_+ +1}}(s) = \left|F_{Z_{\tau_+}}(s) - F_{Z_{\tau_+ +1}}(s)\right|, \end{aligned}$$

but since $\tau_+ > N_3$,

$$\frac{1}{2} = \epsilon < \left|F_{Z_{\tau_+}}(s) - F_{Z_{\tau_+ +1}}(s)\right| < \epsilon = \frac{1}{2},$$

that is, a contradiction.

An analogous argument in case (b) leads to $F_{Z_{\tau_- +1}}(t) \geq F_{Z_{\tau_-}}(s)$. A consequence is

$$\begin{aligned} \frac{1}{2} = \epsilon &< \left|F_{Z_{\tau_-}}(s) - F_{Z_{\tau_-}}(t)\right| = F_{Z_{\tau_-}}(s) - F_{Z_{\tau_-}}(t) \\ &\leq F_{Z_{\tau_- +1}}(t) - F_{Z_{\tau_-}}(t) = \left|F_{Z_{\tau_- +1}}(t) - F_{Z_{\tau_-}}(t)\right| < \epsilon = \frac{1}{2}, \end{aligned}$$

again a contradiction.

Since both case (a) and (b) lead to contradictions, the hypothesis must be false, i.e.

$$\mathbf{E}\left(\log|C|^+\right) < \infty.$$

Since the conditions of Theorem 3.2 part (b) hold, it follows that the $(Z_k)_{k\geq 1}$ must converge absolutely almost surely. □

3.1.3 Main result

From the proof of Lemma 3.6 and Theorem 3.2 we can summarize the main convergence result as the theorem below.

Theorem 3.7. *Let $-\infty < \mu < 0$. Then Equation (3.6) has a solution Z, which is unique in distribution, if and only if*

$$\mathbf{E}\left(\log|C|^+\right) < \infty.$$

On the other hand, when $\mu = -\infty$, $\mathbf{E}\left(\log|C|^+\right) < \infty$ is only a sufficient condition for the existence of a solution to Equation (3.6). In both cases $(Z_k)_{k\geq 1}$ converges a.s. to the solution Z.

Proof. The proof of Lemma 3.6 showed that a solution unique in distribution exists and Z_k converges to it a.s. as long as $\mu < 0$ and $\mathbf{E}\left(\log|C|^+\right) < \infty$. The proof of the converse fails only when $\mu = -\infty$. □

The case when $\mu = -\infty$ is more difficult, and moreover, if one desires to model rates of return with a Cauchy distribution, then μ will not exist. Both cases are solved by Goldie and Maller in [16], where a somewhat complicated integral criterion is derived.

3.2 The moments of a cash flow

This section is concerned with formulas for moments of i.i.d. cash flows and their present values. The accumulated value of i.i.d. payments $C_1, C_2, \ldots, C_k$ that earn i.i.d. returns $R_1 = U_1 - 1, R_2 = U_2 - 1, \ldots, R_k = U_k - 1$, is governed by the recursive formula (cf. (3.1), the annuity equation)

$$S_k = U_k(S_{k-1} + C_k).$$

In this section we also assume that $(U_k)_{k\geq 1}$ and $(C_k)_{k\geq 1}$ are mutually independent.

As a separate case, we define a present value process

$$Z_k = V_k(Z_{k-1} + C_k),$$

which has the same structure. As such, all formulas are applicable to both discounted and non-discounted cash flows. Most of the credit for the work in this section goes to Dufresne [13, 12], who has studied moments of annuities extensively with difference equations. The coefficient calculations are my own, however.

3.2.1 Difference equations for integer moments

Let $u_j = \mathbf{E}\left(U^j\right)$ and $c_j = \mathbf{E}\left(C^j\right)$. Then an application of the Binomial theorem yields

$$\mathbf{E}\left(S_k^m\right) = u_m \mathbf{E}\left((S_{k-1} + C_k)^m\right) = u_m \sum_{j=0}^{m} \binom{m}{j} c_{m-j} \mathbf{E}\left(S_{k-1}^j\right). \tag{3.9}$$

Equation (3.9) can be used to recursively compute the higher moments. Moving on, we bring out the last term of the sum, getting a difference equation

$$\mathbf{E}\left(S_k^m\right) - u_m \mathbf{E}\left(S_{k-1}^m\right) = u_m \sum_{j=0}^{m-1} \binom{m}{j} c_{m-j} \mathbf{E}\left(S_{k-1}^j\right). \tag{3.10}$$

This is an example of a *non-homogeneous first-order difference equation* satisfied by $\mathbf{E}\left(S_k^m\right)$. The corresponding *homogeneous* equation is

$$\mathbf{E}\left(S_k^m\right) - u_m \mathbf{E}\left(S_{k-1}^m\right) = 0,$$

which is solved by $\mathbf{E}\left(S_k^m\right) = K \cdot u_m^k$, where K is a constant. To characterise the whole set of solutions to a non-homogeneous difference equation, one needs only find a particular solution and then add the particular solution to the solution of the corresponding homogeneous equation. For example, consider the case $m = 1$. Then,

$$\mathbf{E}\left(S_k\right) - u_1 \mathbf{E}\left(S_{k-1}\right) = u_1 c_1$$

is the non-homogeneous equation. This equation can in fact be solved by a constant if $u_1 \neq 1$, since $x - u_1 x = u_1 c_1$ implies that $x = u_1 c_1/(1-u_1)$. In other words, assuming that $u_1 \neq 1$, the general solution of the equation is

$$\mathbf{E}\left(S_k\right) = K_1 u_1^k + \frac{u_1 c_1}{1-u_1},$$

where K_1 is a constant again. We can find the value of K_1 by applying the initial condition $S_0 = 0$, which leads to $K_1 = -u_1 c_1/(1-u_1)$ and so,

$$\mathbf{E}\left(S_k\right) = \frac{u_1 c_1 - u_1^{k+1} c_1}{1-u_1},$$

is the full solution to the difference equation. For higher moments, the calculations and applying boundary conditions quickly becomes very difficult. However, some information about the structure of solutions can still be gained. The solution to the $m = 1$ case can be written as,

$$\mathbf{E}\left(S_k\right) = K_0 + K_1 \cdot u_1^k,$$

where K_0, K_1 are constants. The case when $u_1 = 1$ has the solution $\mathbf{E}\left(S_k\right) = k c_1$, as is easy to see.

Now, consider the $m = 2$ case. Then the difference equation becomes,

$$\mathbf{E}\left(S_k^2\right) - u_2 \mathbf{E}\left(S_{k-1}^2\right) = 2u_2 c_1 \mathbf{E}\left(S_{k-1}\right) + u_2 c_2.$$

However, we already have a full solution to the $m = 1$ case, so we insert that into the right-hand side (assuming first that $u_1 \neq 1$), yielding

$$\mathbf{E}\left(S_k^2\right) - u_2 \mathbf{E}\left(S_{k-1}^2\right) = 2u_2 c_1 \left(\frac{u_1 c_1 - u_1^k c_1}{1-u_1}\right) + u_2 c_2.$$

We try a solution of the form $\mathbf{E}\left(S_k^2\right) = K_0 + K_1 \cdot u_1^k$. Then the left-hand side evaluates to $K_0 - u_2 K_0 + K_1(u_1 - u_2) u_1^{k-1}$. For a while, we will assume $u_1 \neq 1$ and $u_2 \neq 1$. Rearranging the right-hand side, we get

$$(1-u_2)K_0 + (u_1-u_2)K_1 \cdot u_1^{k-1} = (1-u_2)\left[\frac{u_2 c_2(1-u_1) + 2u_1 u_2 c_1^2}{(1-u_2)(1-u_1)}\right]$$
$$+(u_1-u_2)\left[-\frac{2u_1 u_2 c_1^2}{(u_1-u_2)(1-u_1)}\right] \cdot u_1^{k-1}.$$

This expression allows us to identify the values of the constants as

$$K_0 = \frac{u_2 c_2(1-u_1) + 2u_1 u_2 c_1^2}{(1-u_2)(1-u_1)}, \quad K_1 = -\frac{2u_1 u_2 c_1^2}{(u_1-u_2)(1-u_1)}.$$

The homogeneous equation's solution added to this yields a solution

$$\mathbf{E}\left(S_k^2\right) = K_0 + K_1 u_1^k + K_2 u_2^k,$$

and to get the value of K_2 one needs only apply the boundary condition $S_0 = 0$ in order to get $K_2 = -(K_0 + K_1)$.

On the other hand, if $u_1 = 1$ were the case, then the equation to solve would have been

$$\mathbf{E}\left(S_k^2\right) - u_2 \mathbf{E}\left(S_{k-1}^2\right) = 2u_2 c_1 \left(k c_1\right) + u_2 c_2.$$

In this case it simplifies the solution method if one tries a particular solution $\mathbf{E}\left(S_k^2\right) = K_0 + K_1 \cdot (k+1)$. No calculations will be shown, but the particular solution is

$$K_0 = \frac{u_2 c_2}{1-u_2} - \frac{2c_1^2 u_2}{(1-u_2)^2}, \quad K_1 = \frac{2c_1^2 u_2}{1-u_2}.$$

From this the general solution can be found as usual. We will not elaborate on the further special cases when $u_2 = 1$ or $u_2 = u_1$, as they tend to be less interesting, but the reader should keep these methods in mind. Dufresne formulates the conditions

$$u_i \neq u_j, \quad 0 \leq i < j \leq m, \tag{3.11}$$

that guarantee that no complications arise (note that $u_0 = 1$ so none of the moments can be 1). "Normally" these conditions will hold, because if the very standard conditions $U \geq 0$ and $\mathbf{E}(U) > 1$ hold, then for $m \geq 2$ it holds that $\mathbf{E}(U^m) > \mathbf{E}(U^{m-1})$. To see this, recall that $\|X\|_p = \left(\mathbf{E}\left(|X|^p\right)\right)^{1/p}$ is non-decreasing with respect to p. Then the fact follows from

$$\left(\mathbf{E}\left(U^m\right)\right)^{\frac{1}{m}} \geq \left(\mathbf{E}\left(U^{m-1}\right)\right)^{\frac{1}{m-1}} \Rightarrow \mathbf{E}\left(U^m\right) \geq \left(\mathbf{E}\left(U^{m-1}\right)\right)^{1+\frac{1}{m-1}} > \mathbf{E}\left(U^{m-1}\right).$$

In general, the following theorem holds.

Proposition 3.8. *Provided conditions* (3.11) *hold up to m, then the m-moments have the form*

$$\mathbf{E}\left(S_k^m\right) = \sum_{j=0}^{m} d_{mj} u_j^k, \tag{3.12}$$

where the $\{d_{mj},\ 0 \leq j \leq m\}$ *are constant with respect to* k *and* $u_j = \mathbf{E}\left(U^j\right)$.

Proof. It has already been shown that the solution has the desired form when $m = 1, 2$. Moreover, the solution to the homogeneous difference equation is always the part with u_m^k. We only need to show that a particular solution has the appropriate form, and this can be done inductively. As such, we assume that

$$\mathbf{E}\left(S_k^{m-i}\right) = \sum_{j=0}^{m-i} d_{(m-i)j} u_j^k,$$

for all $1 \leq i \leq (m-1)$. Then, clearly the right-hand side of (3.10) can be rearranged to have the form $\sum_{j=0}^{m-1} K_j u_j^k$, while the left-hand side becomes

$$(1 - u_m)K_0' + \sum_{j=1}^{m-1} (u_j - u_m) K_j' u_k^{k-1}.$$

From there it is only a matter of trying a particular solution of the same form, matching the expressions, and solving the system of equations. Then by adding the solution to the homogeneous equation, the solution has the desired form. □

Finally, Dufresne [12] has calculated a recursive formula for the constants d_{mj} by taking formula (3.12) and inserting it into (3.10). This yields the recursive relation

$$\begin{aligned} d_{mj} &= \frac{u_m}{u_j - u_m} \sum_{i=j}^{m-1} \binom{m}{i} c_{m-i} d_{ij}, \ 0 \leq j \leq m-1, \\ d_{mm} &= -\sum_{j=0}^{m-1} d_{mj}. \end{aligned} \tag{3.13}$$

No more efficient formulas are currently known.

3.2.2 Moments of a discounted perpetuity

Now, some remarks about discounted cash flows and perpetuities. If $(C_k, V_k)_{k \geq 1}$ are i.i.d., then the discounted cash flow Z_k satisfies (3.6), and so the moments can be calculated in the same way. However, particular care that conditions (3.11) hold must be taken, since a discounted cash flow implies that $\mathbf{E}(V) < 1$.

On the other hand, if a discounted cash flow is a convergent perpetuity with $C_k \perp V_k, \forall k \in \mathbb{N}$, the calculation of the moments at infinity are simplified significantly.

If the conditions in Theorem (3.7) hold, then the moments of the perpetual cash flow must satisfy

$$\mathbf{E}\left(Z_\infty^m\right) = \mathbf{E}\left(V^m\right)\mathbf{E}\left((Z_\infty + C)^m\right) = v_m \sum_{j=0}^{m} \binom{m}{j} c_{m-j} \mathbf{E}\left(Z_\infty^j\right),$$

where $v_k = \mathbf{E}\left(V^k\right)$. This leads to a particularly simple equation of

$$\mathbf{E}\left(Z_\infty^m\right) - v_m \mathbf{E}\left(Z_\infty^m\right) = v_m \sum_{j=0}^{m-1} \binom{m}{j} c_{m-j} \mathbf{E}\left(Z_\infty^j\right).$$

If conditions (3.11) hold, this can always be solved:

$$\mathbf{E}\left(Z_\infty^m\right) = \frac{v_m}{1 - v_m} \sum_{j=0}^{m-1} \binom{m}{j} c_{m-j} \mathbf{E}\left(Z_\infty^j\right). \tag{3.14}$$

Equation (3.14) is an excellent formula for recursive computation of higher moments for perpetuities. Despite this a general formula for direct computation would be ideal, yet no such formula has, to my knowledge, been discovered.

3.3 The distribution of a perpetuity

This section is dedicated to examples where distributions for Z_∞ can be derived. In the following the sequences $(C_k)_{k\geq 1}, (V_k)_{k\geq 1}$ are, unless otherwise specified, assumed to be i.i.d. and mutually independent. Most of the examples are due to Vervaat [33] and Dufresne [12].

3.3.1 Limit of characteristic functions

Example 3.9 (Compound geometric, Vervaat [33], Dufresne [12]). Let C be arbitrary and $V \sim Ber(p), q = 1 - p > 0$. This models a situation where there is some chance after every time period that all payments will stop, or that their worth will become zero from that point forward. At time $k = 1$ this is a compound Bernoulli probability, that is, $Z_1 = C_1$ with probability p and $Z_1 = 0$ with probability q. For Z_2, it's as a branch from Z_1, i.e. if Z_1 had value 0 then

$Z_2 = 0$. On the other hand, if $Z_1 = C_1$, then $Z_2 = C_1 + C_2$ with probability p and $Z_2 = C_1$ with probability q. Generalizing,

$$Z_k = \begin{cases} \sum_{j=1}^m C_j & \text{with probability } p^m q, \quad \text{for } 0 \le m \le k-1 \\ \sum_{j=1}^k C_k & \text{with probability } p^k \qquad \text{otherwise.} \end{cases}$$

Let the characteristic function of C be $\phi(s) = \mathbf{E}\left(e^{isC}\right)$. In terms of ϕ, we have

$$\begin{aligned} \mathbf{E}\left(e^{isZ_k}\right) &= \sum_{m=0}^{k-1} \left[qp^m \mathbf{E}\left(e^{is\sum_{j=1}^m C_j}\right)\right] + p^k \mathbf{E}\left(e^{is\sum_{j=1}^k C_j}\right) \\ &= q\sum_{m=0}^{k-1} \left[p^m \phi(s)^m\right] + p^k \phi(s)^k \\ &= q\frac{1 - p^k\phi(s)^k}{1 - p\phi(s)} + p^k\phi(s)^k. \end{aligned}$$

Recall that for every $s \in \mathbb{R}$, $|\phi(s)| \le 1$, and so $p^k\phi(s)^k \to 0$ as $k \to \infty$. As such,

$$\mathbf{E}\left(e^{isZ_\infty}\right) = \frac{q}{1 - p\phi(s)}.$$

This is the characteristic function of a compound geometric distribution.

Example 3.10 (Random geometric series)**.** Consider a case where the discounting factor $V = v \in (0, 1)$ is constant. In most cases sums of random variables are difficult to compute, but if C is finite a.s. this is a convergent perpetuity. For some particular choices of C we can easily compute the distribution. For example, let $C \sim N(\mu, \sigma^2)$. Then

$$\begin{aligned} \mathbf{E}\left(e^{isZ_k}\right) = \mathbf{E}\left(e^{is\sum_{j=1}^k v^j C_j}\right) = \prod_{j=1}^k \phi(sv^j) = \prod_{j=1}^k e^{isv^j\mu - \sigma^2(v^j s)^2/2} \\ = \exp\left\{ is\mu \sum_{j=1}^k v^j - \frac{s^2}{2}\sigma^2 \sum_{j=1}^k (v^2)^j \right\}. \end{aligned}$$

As $k \to \infty$, we use the geometric series formula and get

$$\mathbf{E}\left(e^{isZ_\infty}\right) = \exp\left\{ is\frac{v\mu}{1-v} - \frac{s^2}{2}\frac{v^2\sigma^2}{1-v^2} \right\},$$

which corresponds to a $N\left(\frac{v\mu}{1-v}, \frac{v^2\sigma^2}{1-v^2}\right)$-distribution.

In the above two examples, please note that since we know the examples are perpetuities it constitutes proof that the convergence is actually almost sure (except if C has a distribution with infinite or non-existent mean, like e.g. Lévy or Cauchy distributions). In general pointwise convergence of characteristic functions can only be used as an argument for convergence in distribution.

3.3.2 Vervaat-class perpetuities

In some cases it's possible to use Equation (3.6) to derive a differential equation for the characteristic function of Z. In this section a class of such perpetuities is discussed.

Example 3.11 (Vervaat perpetuities, Vervaat [33], Dufresne [12])**.** Let the discounting factor V be an exponential function of the rate of return R, which is assumed to be exponentially distributed with parameter $\alpha > 0$. Vervaat represents this by writing $V = e^{-R} \stackrel{d}{=} X^{1/\alpha}$, where $X \sim U(0,1)$, which results in a density

$$f_V(x) = \alpha x^{\alpha-1}\mathbf{1}_{[0,1]}(x).$$

Note that this is the density function of the $\beta(\alpha,1)$ distribution defined on the closed interval $[0,1]$; the general $\beta(a,b)$ distribution has density

$$f(x) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}x^{a-1}(1-x)^{b-1}, \quad \text{for } 0 < x < 1.$$

It holds that $-\infty < \mu < 0$, so if also $\mathbf{E}\left(\log|C|^+\right) < \infty$ we can use Equation (3.6) to derive the distribution of Z_∞. With $\eta(s) := \mathbf{E}\left(e^{isZ}\right)$ and $\phi(s) := \mathbf{E}\left(e^{isC}\right)$, we get

$$\begin{aligned}\eta(s) = \mathbf{E}\left(e^{isV(Z+C)}\right) &= \int_{-\infty}^{\infty} f_V(x)\mathbf{E}\left(e^{isx(Z+C)}\right)dx \\ &= \int_0^1 \alpha x^{\alpha-1}\mathbf{E}\left(e^{isxZ}\right)\mathbf{E}\left(e^{isxC}\right)dx = \int_0^1 \alpha x^{\alpha-1}\eta(sx)\phi(sx)dx,\end{aligned}$$

where substituting $u = sx$ yields

$$\eta(s) = \alpha s^{-\alpha}\int_0^s u^{\alpha-1}\eta(u)\phi(u)du.$$

Multiplying both sides by s^α and taking the derivative gives a differential equation

$$s^\alpha\eta' + \alpha s^{\alpha-1}\eta = \alpha s^{\alpha-1}\eta\phi.$$

Dividing both sides by $s^\alpha \eta$ and rearranging yields

$$\frac{\eta'}{\eta} = \alpha s^{-1}(\phi - 1),$$

which is easily recognized as an elementary differential equation with the solution

$$\eta(s) = \exp\left\{\alpha \int_0^s \frac{\phi(u) - 1}{u} du\right\}. \tag{3.15}$$

This defines the *Vervaat-class perpetuities.* For some distributions of C the integral in the characteristic function can be easy to calculate, as the next few examples show.

Example 3.12 (Vervaat gamma, Vervaat [33], Dufresne [12])**.** Consider a Vervaat perpetuity with $C \sim Exp(\lambda)$. Then $\phi(s) = \lambda(\lambda - is)^{-1}$. With this, the expression for η(s) (3.15) becomes

$$\exp\left\{\alpha \int_0^s \frac{1}{u} \cdot \frac{\lambda - (\lambda - iu)}{\lambda - iu} du\right\} = e^{\alpha \log(\lambda) - \alpha \log(\lambda - is)} = \left(\frac{\lambda}{\lambda - is}\right)^\alpha.$$

This is the characteristic function of the $\Gamma(\alpha, \lambda)$-distribution.

Example 3.13 (Vervaat symmetric VG)**.** This is a slight modification of the above example, and to the author's knowledge this approach provides a new proof for this explicit solution to (3.6). Consider a Vervaat perpetuity with $C \sim$ *Laplace*$(0, \lambda)$. The situation should be thought of as exponentially distributed size of payments, but without knowledge of whether we will *receive* or *pay* the amount. It is an infinite series of coin flips determining whether we pay or get paid (in other words, not much unlike trading in securities). In this case $\phi(s) = \lambda^2(\lambda^2 + s^2)^{-1}$ and the expression for $\eta(s)$ (3.15) becomes

$$\exp\left\{\alpha \int_0^s \frac{1}{u} \cdot \frac{\lambda^2 - (\lambda^2 + u^2)}{\lambda^2 + u^2} du\right\} = \exp\left\{-\alpha \int_0^s \frac{u\,du}{\lambda^2 + u^2}\right\}$$

$$= \exp\left\{-\frac{\alpha}{2}\log(\lambda^2 + s^2) + \frac{\alpha}{2}\log(\lambda^2)\right\} = \left(\frac{\lambda^2}{\lambda^2 + s^2}\right)^{\alpha/2}.$$

This is the characteristic function of a symmetric *Variance-gamma distribution*, or *VG distribution* for short, with mean zero, which arises among other places as the distribution of the difference between two i.i.d. χ^2-distributed random variables, although it can also be defined as a generalization of the Laplace distribution. Note that if $\alpha = 2$ then Z would follow a *Laplace*$(0, \lambda)$ distribution.

The variance-gamma distribution has four parameters in general. We show its density below. If $X \sim VG(m, a, b, \lambda)$, with $\lambda > 0$ and $\gamma = \sqrt{a^2 - b^2} > 0$, then its density function is defined for any $x \in \mathbb{R}$ and is given by

$$f_X(x) = \frac{\gamma^{2\lambda}|x-m|^{\lambda-1/2}K_{\lambda-1/2}(a|x-m|)}{\sqrt{\pi}\Gamma(\lambda)(2a)^{\lambda-1/2}}e^{b(x-m)},$$

where K denotes a function, given by

$$K_\alpha(x) = \frac{\pi}{2} \cdot \frac{I_{-\alpha}(x) - I_\alpha(x)}{\sin(\alpha\pi)},$$

where

$$I_\alpha(x) = \sum_{k=0}^{\infty} \frac{1}{k!\Gamma(k+\alpha+1)} \left(\frac{x}{2}\right)^{2k+\alpha}.$$

The functions K_α and I_α are called *modified Bessel functions* of the *first* and *second kind*, respectively. With the same parameters, the characteristic function of a VG-distributed variable is given by

$$\mathbf{E}\left(e^{isX}\right) = e^{mis}\left(\gamma/\sqrt{a^2-(b+is)^2}\right)^{2\lambda}.$$

The next example will be the final Vervaat perpetuity brought up in this section and unlike the prior examples, this is only expressed as a difference of two random variables. Despite leading to difficulties for moment calculations, it is still a useful expression for numerical computations.

Example 3.14. The idea in this example is to have C follow a distribution that has a similar two-way exponential character as the Laplace distribution, but skewed. Such a distribution can be constructed as follows. Let $a, b > 0$, and

$$C = XF - (1-X)U, \ X \sim Ber\left(\frac{b}{a+b}\right), \ F \sim Exp(a), U \sim Exp(b), \quad (3.16)$$

where $F \perp U$ and X is independent of F, U. Then the characteristic function ϕ of C is easily calculated.

$$\begin{aligned}\phi(s) = \mathbf{E}\left(e^{isC}\right) = \mathbf{E}\left(e^{is(XF-(1-X)U)}\right) &= \frac{b}{a+b}\mathbf{E}\left(e^{isF}\right) + \frac{a}{a+b}\mathbf{E}\left(e^{-isU}\right) \\ &= \frac{b}{a+b} \cdot \frac{a}{a-is} + \frac{a}{a+b} \cdot \frac{b}{b+is}.\end{aligned}$$

Plugging this into the integral in (3.15) yields

$$
\begin{aligned}
I(s) &:= \int_0^s \frac{\phi(u)-1}{u}du = \int_0^s \frac{1}{u}\left[\frac{b}{a+b}\cdot\frac{a}{a-is} + \frac{a}{a+b}\cdot\frac{b}{b+is} - \frac{a+b}{a+b}\right]du \\
&= \frac{b}{a+b}\int_0^s \frac{\frac{a}{a-iu}-1}{u}du + \frac{a}{a+b}\int_0^s \frac{\frac{b}{b+iu}-1}{u}du \\
&= \frac{b}{a+b}\log\left(\frac{a}{a-is}\right) + \frac{a}{a+b}\log\left(\frac{b}{b+is}\right).
\end{aligned}
$$

As such, from (3.15) we get

$$\eta(s) = e^{\alpha I(s)} = \left(\frac{a}{a-is}\right)^{\frac{\alpha b}{a+b}}\left(\frac{b}{b+is}\right)^{\frac{\alpha a}{a+b}}.$$

Let $G_1 \sim \Gamma(\alpha b/(a+b), a)$ and $G_2 \sim \Gamma(\alpha a/(a+b), b)$, and $G_1 \perp G_2$. Then,

$$\eta(s) = \mathbf{E}\left(e^{isG_1}\right)\cdot\mathbf{E}\left(e^{-isG_2}\right) = \mathbf{E}\left(e^{is(G_1-G_2)}\right).$$

It is now established that $Z \overset{d}{=} G_1 - G_2$.

There is a slight modification of the Vervaat perpetuity technique that is easily applied to finding solutions for the stochastic equation

$$Z \overset{d}{=} VZ + C, \quad V, Z, C \text{ independent}.$$

Consider V as before, so that

$$
\begin{aligned}
\eta(s) = \mathbf{E}\left(e^{isZ}\right) = \mathbf{E}\left(e^{isVZ+isC}\right) = \mathbf{E}\left(e^{i}sC\right)\mathbf{E}\left(e^{isVZ}\right) &= \phi(s)\int_0^1 \alpha x^{\alpha-1}\eta(sx)dx \\
&= \alpha\phi(s)s^{-\alpha}\int_0^s u^{\alpha-1}\eta(u)du,
\end{aligned}
$$

leading to the equation

$$s^\alpha\frac{\eta(s)}{\phi(s)} = \alpha\int_0^s u^{\alpha-1}\eta(u)du.$$

Taking the derivative and multiplying by $s^{-\alpha}\phi(s)/\eta(s)$ yields the differential equation

$$\frac{\eta'}{\eta} = \frac{\alpha}{s}(\phi-1) + \frac{\phi'}{\phi},$$

which has the solution

$$\eta(s) = \phi(s)\exp\left\{\alpha\int_0^s \frac{\phi(u)-1}{u}du\right\}. \tag{3.17}$$

This form can be useful when a solution to (3.6) is already known but a solution to $Z \overset{d}{=} VZ + C$ is desired instead. We illustrate on the prior example, 3.14.

Corollary 3.15. *Let $G_1 \sim \Gamma(1+\alpha b/(a+b), a)$ and $G_2 \sim \Gamma(1+\alpha a/(a+b), b)$, and $G_1 \perp G_2$. Assume we have a Vervaat setting with $V = e^{-R}$, where $R \sim Exp(\alpha)$ and that C is as in Example 3.14. Then*

$$Z = G_1 - G_2$$

solves the stochastic equation

$$Z \overset{d}{=} VZ + C, \quad V, Z, C \text{ independent.}$$

Proof. In Example 3.14, we had a Vervaat perpetuity with

$$\phi(s) = \frac{b}{a+b} \cdot \frac{a}{a-is} + \frac{a}{a+b} \cdot \frac{b}{b+is}$$

and

$$\eta(s) = \left(\frac{a}{a-is}\right)^{\frac{\alpha b}{a+b}} \left(\frac{b}{b+is}\right)^{\frac{\alpha a}{a+b}}.$$

Using the modification (3.17) to get a solution to

$$Z \overset{d}{=} VZ + C, \quad V, Z, C \text{ independent},$$

we get

$$\mathbf{E}\left(e^{isZ}\right) = \left(\frac{b}{a+b} \cdot \frac{a}{a-is} + \frac{a}{a+b} \cdot \frac{b}{b+is}\right) \left(\frac{a}{a-is}\right)^{\frac{\alpha b}{a+b}} \left(\frac{b}{b+is}\right)^{\frac{\alpha a}{a+b}}$$

$$= \frac{b}{a+b} \left(\frac{a}{a-is}\right)^{1+\frac{\alpha b}{a+b}} \left(\frac{b}{b+is}\right)^{\frac{\alpha a}{a+b}} + \frac{a}{a+b} \left(\frac{b}{b+is}\right)^{1+\frac{\alpha a}{a+b}} \left(\frac{a}{a-is}\right)^{\frac{\alpha b}{a+b}}$$

$$= \left(\frac{a}{a-is}\right)^{1+\frac{\alpha b}{a+b}} \left(\frac{b}{b+is}\right)^{1+\frac{\alpha a}{a+b}} \left[\frac{b}{a+b} \cdot \frac{b+is}{b} + \frac{a}{a+b} \cdot \frac{a-is}{a}\right]$$

$$= \mathbf{E}\left(e^{is(G_1-G_2)}\right) \left[\frac{b+is+a-is}{a+b}\right] = \mathbf{E}\left(e^{is(G_1-G_2)}\right),$$

where $G_1 \sim \Gamma(1+\alpha b/(a+b), a)$ and $G_2 \sim \Gamma(1+\alpha a/(a+b), b)$, and $G_1 \perp G_2$. This proves the assertion. □

3.3.3 Beta-gamma algebra

There are also methods for finding solutions to (3.6) that rely on other stochastic identities. A big interest of Daniel Dufresne [11, 9, 10] is properties of Gamma and Beta distributions that lead to equations in distribution. We list some well-known such properties without proofs.

$$X \sim \Gamma(a,k), Y \sim \Gamma(b,k), X \perp Y \Rightarrow X+Y \sim \Gamma(a+b,k) \tag{3.18}$$

$$X \sim \Gamma(a,1), Y \sim \Gamma(b,1), X \perp Y \Rightarrow X/(X+Y) \sim \beta(a,b) \tag{3.19}$$

$$X_1, X_2 \sim \Gamma(a,1), Y_1, Y_2 \sim \Gamma(b,1), \text{ all independent}$$
$$\Rightarrow \frac{X_1}{X_1+Y_1} \cdot (X_2+Y_2) \stackrel{d}{=} X_1. \tag{3.20}$$

Together with Theorem 3.7, various distributions of perpetuities can be derived from these properties. For example, (3.20) immediately leads to a solution with $V \sim \beta(a,b), C \sim \Gamma(b,1)$, and $Z \sim \Gamma(a,1)$. Checking, one easily sees that the conditions in Theorem 3.7 hold, so this is a convergent perpetuity with a known distribution. In fact, this is a slight generalization of Example 3.12. Moreover, if every payment is multiplied by a constant k, then the perpetuity Z would also be multiplied by that k. Since, with the parametrization of the Gamma distribution we have used so far, $kX \sim \Gamma(a,1/k)$ whenever $X \sim \Gamma(a,1)$, it follows that when $C \sim \Gamma(b,1/k)$ we have $Z \sim \Gamma(a,1/k)$.

In the following it will be useful to introduce an extra distribution, with density

$$f(x) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} x^{a-1}(1+x)^{-a-b}\mathbf{1}_{(0,\infty)}(x), \quad a,b>0.$$

This distribution is called a *Beta distribution of the second kind* and denoted by $\beta_2(a,b)$. The Beta distribution of the second kind is of special interest since it arises most naturally as a transformed Beta distribution where if $X \sim \beta(a,b)$, then

$$Y = \frac{X}{1-X} \sim \beta_2(a,b).$$

Although it may look like a stochastic version of the geometric series formula, this is not how the distribution arises. Here is an example where it arises as a perpetuity.

Example 3.16 (Dufresne [12, 11]). Let $a,b>0$, $V \sim \beta_2(a,a+b)$ and $C \equiv 1$. Then $Z \sim \beta_2(a,b)$ is a solution to (3.6). This is most easily shown by an application of property (3.20). We introduce a variable $B \sim \beta(a,b)$, and see that if $Z = B/(1-B)$, then

$$(Z+1)^{-1} = (1-B) \sim \beta(b,a).$$

Introducing variables $X \sim \Gamma(a,1), Y \sim \Gamma(b,1)$ such that $B = X/(X+Y)$, we get

$$(Z+1) = (X+Y)/Y.$$

But on the other hand, if $V = B'/(1-B')$ where $B' \sim \beta(a, a+b)$ and $B' = X'/(X'+Y')$ with $X' \sim \Gamma(a,1), Y' \sim \Gamma(a+b,1)$, then

$$V = \frac{B'}{1-B'} = \frac{X'/(X'+Y')}{Y'/(X'+Y')} = \frac{X'}{Y'}.$$

Since X' and Y' are independent we can separate them and get

$$V(Z+1) = \frac{X'}{Y'} \cdot \frac{X+Y}{Y},$$

and since $Y' = G_a + G_b$ for some independent $\Gamma(a,1), \Gamma(b,1)$ distributed random variables, we can apply property (3.20) and get

$$V(Z+1) \stackrel{d}{=} \frac{X'}{Y} \sim \beta_2(a,b),$$

which is the same distribution as that of Z. In other words, Z solves Equation (3.6). What remains to be seen is that V satisfies the conditions of Theorem 3.2. Consider

$$\log|V| = \log|X'| - \log|Y'|.$$

In other words, $\mathbf{E}(\log|V|) < 0$ if and only if $\mathbf{E}(\log|X'|) < \mathbf{E}(\log|Y'|)$. It can be shown that the logarithmic expectation of gamma distributions can be expressed with the digamma function ψ. For X',

$$\mathbf{E}(\log X) = \psi(a) - \log(1) = -\gamma + \sum_{k=0}^{\infty} \left(\frac{1}{k+1} - \frac{1}{k+a} \right)$$

$$= -\gamma + \sum_{k=0}^{\infty} \left(\frac{a-1}{(k+1)(k+a)} \right),$$

which is an (absolutely) convergent series since

$$\sum_{k=0}^{\infty} \left(\frac{1}{(k+1)(k+a)} \right) < \sum_{k=0}^{\infty} \frac{1}{k^2} < \infty.$$

As such,

$$\mathbf{E}\left(\log|V|\right) = \psi(a) - \psi(a+b) = \sum_{k=0}^{\infty}\left(\frac{1}{k+a+b} - \frac{1}{k+a}\right) < 0,$$

since clearly $a+b > a$. Thus, $\mu < 0$ and by theorem 3.7 the solution Z is unique in distribution and the value of the perpetuity converges to Z a.s.

Some more examples can be found in [11, 9, 10].

Chapter 4

Continuous-time models

Consider first a discrete-time model (as in Chapter 3). A typical way of introducing continuous time is to write a discrete-time process $(S_k)_{k\geq 0}$ as a step function $S(t) \coloneqq S_{[t]}$, for $t \geq 0$. Taking it one step further, such models can be made more precise by shortening the intervals, e.g. by setting $S(t) \coloneqq S_{[tn]}$ for some $n \in \mathbb{N}$. Then the number of discrete steps taken until time t were multiplied by n, so the terms of the model also need to be scaled appropriately. This procedure is well represented by Dufresne [12], as a prerequisite for his convergence theorem (for which we will present an alternative proof in Section 4.3). The bottom line is that the discrete-time processes of Chapter 3 have continuous-time analogues as integral processes and that this can be rigorously proven using techniques related to weak convergence in Skorokhod space. This shall not be done here, however; instead the interested reader is referred to Dufresne's articles [13, 12], but see also section 5.2.

A continuous-time cash flow with random return on investment is

$$S_t = \int_0^t e^{X_t - X_s} dY_s, \tag{4.1}$$

where X is a return-on-investment generating process and Y is a payment generating process. In our case, we are most interested in the integral representing the present value of a continuous cash flow,

$$Z_t = e^{-X_t} S_t = \int_0^t e^{-X_s} dY_s, \tag{4.2}$$

and the present value of the corresponding perpetuity is thus

$$Z_\infty = \int_0^\infty e^{-X_s} dY_s. \tag{4.3}$$

For most of this chapter we shall use the assumption that X and Y are some types of Lévy processes. Dufresne [12] restricts himself to Brownian motion with drift, while also defining the processes S and Z differently so as to preserve consistency with his chapter on discrete-time models.

The next proposition is one that Dufresne proves as a consequence of his weak convergence of discrete-time models. The proof supplied here is instead by elementary stochastic calculus.

Proposition 4.1. *Let X and Y be standard independent Brownian motions and let*

$$\begin{aligned} \widetilde{X}_t &:= \gamma t + \sigma X_t, \\ \widetilde{Y}_t &:= \mu t + \zeta Y_t, \end{aligned} \tag{4.4}$$

where $\mu, \gamma, \sigma, \zeta \in \mathbb{R}$.

Then the process S, defined by

$$S_t = \int_0^t e^{\tilde{X}_t - \tilde{X}_s} d\widetilde{Y}_s, \tag{4.5}$$

satisfies the SDE

$$dS_t = (\alpha S_t + \mu)dt + \sigma S_t dX_t + \zeta dY_t, \tag{4.6}$$

where $\alpha = \gamma + \sigma^2/2$.

Proof. The SDE (4.6) can be solved with integration by parts. First, recall that

$$d\left(e^{-\tilde{X}_t}\right) = \left(-\gamma + \frac{\sigma^2}{2}\right) e^{-\tilde{X}_t} dt - \sigma e^{-\tilde{X}_t} dX_t, \tag{4.7}$$

according to the geometric Brownian motion equation. Then, by the Itô integration by parts formula of equation (2.1)

$$d\left(e^{-\tilde{X}_t} S_t\right) = e^{-\tilde{X}_t} dS_t + S_t d\left(e^{-\tilde{X}_t}\right) + d\langle e^{-\tilde{X}}, S\rangle_t.$$

Using Proposition 2.16, we see that the cross-variation

$$d\langle e^{-\tilde{X}}, S\rangle_t = -\sigma^2 e^{-\tilde{X}_t} S_t dt.$$

We insert the cross-variation and equations (4.6) and (4.7) into the integration by parts formula, yielding

$$\begin{aligned} d\left(e^{-\tilde{X}_t}S_t\right) &= (\alpha S_t+\mu)e^{-\tilde{X}_t}dt+\sigma e^{-\tilde{X}_t}S_t dX_t+\zeta e^{-\tilde{X}_t}dY_t \\ &+\left(-\gamma+\frac{\sigma^2}{2}\right)e^{-\tilde{X}_t}S_t dt-\sigma e^{-\tilde{X}_t}S_t dX_t+d\langle e^{-\tilde{X}},S\rangle_t, \end{aligned}$$

which simplifies to

$$d\left(e^{-\tilde{X}_t}S_t\right)=\mu e^{-\tilde{X}_t}dt+\zeta e^{-\tilde{X}_t}dY_t. \tag{4.8}$$

Equivalently expressed,

$$e^{-\tilde{X}_t}S_t=\mu\int_0^t e^{-\tilde{X}_s}ds+\zeta\int_0^t e^{-\tilde{X}_s}dY_s=\int_0^t e^{-\tilde{X}_s}d\widetilde{Y}_s,$$

and so the solution to equation (4.6) is

$$S_t=\int_0^t e^{\tilde{X}_t-\tilde{X}_s}d\widetilde{Y}_s,$$

which wraps up the proof. □

Remark 4.2 (Norberg [26])**.** The solution of Equation (4.6) can be rewritten as a one-dimensional diffusion if one recognizes that

$$W_t:=\int_0^t (S_s^2\sigma^2+\zeta^2)^{-1/2}(S_s\sigma dX_s+\zeta dY_s)$$

is a standard Brownian motion. It is justified by the fact that

$$\begin{aligned} \langle W,W\rangle_t &= \left\langle \int_0^\cdot (S_s^2\sigma^2+\zeta^2)^{-1/2}S_s\sigma dX_s, \int_0^\cdot (S_s^2\sigma^2+\zeta^2)^{-1/2}S_s\sigma dX_s\right\rangle_t \\ &+\left\langle \int_0^\cdot (S_s^2\sigma^2+\zeta^2)^{-1/2}\zeta dY_s, \int_0^\cdot (S_s^2\sigma^2+\zeta^2)^{-1/2}\zeta dY_s\right\rangle_t \\ &= \int_0^t (S_s^2\sigma^2+\zeta^2)^{-1}(S_s^2\sigma^2+\zeta^2)dt=t \end{aligned}$$

and Theorem 6.1 in [6]. Thus, S is governed by the one-dimensional diffusion equation

$$dS_t=(\alpha S_t+\mu)dt+(S_t^2\sigma^2+\zeta^2)^{1/2}dW_t, \tag{4.9}$$

where $\alpha=\gamma+\sigma^2/2$.

The next section treats convergence criteria for perpetuities, very similarly to the conditions of Chapter 3.

4.1 Finiteness of perpetuities

4.1.1 A brief literature review

An overview of convergence criteria for perpetuities shall be presented below. In this section we prove only what is necessary; other proofs, that would require a significant amount of theory, are omitted for the sake of brevity. In all such cases, the reader is instead referred to a source that proves the theorem.

As a preliminary, we let

$$Z_t = \int_0^t e^{-X_s} ds,$$

for all $t > 0$, and present equivalent characterizations of the a.s. finiteness of $Z_\infty = \lim_{t\to\infty} Z_t$.

Theorem 4.3 (Bertoin and Yor [2]). *Let $X = (X_t)_{t\geq 0}$ be a Lévy process with Lévy-Khintchine triplet (a, σ^2, Π) and let*

$$Z_\infty = \int_0^\infty e^{-X_s} ds.$$

Then the following assertions are equivalent:

(i) $Z_\infty < \infty$ *a.s.*
(ii) $\mathbf{P}(Z_\infty < \infty) > 0$.
(iii) $\lim_{t\to\infty} X_t = +\infty$ *a.s.*
(iv) $\lim_{t\to\infty} t^{-1} X_t > 0$ *a.s.*
(v) $\int_1^\infty \mathbf{P}(X_t \leq 0)\, t^{-1} dt < \infty$.
(vi) Either

$$\int_{(-\infty,-1)} |x|\Pi(dx) < \infty \quad \text{and} \quad \int_{|x|>1} x\Pi(dx) > a,$$

or

$$\int_{(-\infty,-1)} |x|\Pi(dx) = \int_{(1,\infty)} x\Pi(dx) = \infty \quad \text{and} \quad \int_1^\infty \bar{\Pi}^-(x) d\big(x/J^+(x)\big) < \infty,$$

where for every $x > 0$

$$\bar{\Pi}^+(x) = \Pi\big((x,\infty)\big),\ \bar{\Pi}^- = \Pi\left((-\infty,-x)\right),\ J^+(x) = \int_0^x \bar{\Pi}^+(y)dy.$$

Proof. See [2]. □

Corollary 4.4. *If $X_t = \gamma t + \sigma W_t$, where $W = (W_t)_{t\geq 0}$ is a standard Brownian motion, then*

$$Z_\infty = \int_0^\infty e^{-X_s} ds < \infty$$

if and only if $\gamma > 0$.

Proof. From Theorem 4.3, assertions (i) and (iii). □

Consider the integral

$$\int_0^\infty f(X_s) ds,$$

where X is a Lévy process as before. Some results in the literature give a simple integral test for convergence of such an integral functional. The currently best published characterization is due to Döring and Kyprianou [8], which we shall not discuss further due to the restricted scope of this thesis.

In a forthcoming paper by Kolb and Savov [22], the integral test is improved upon by finding a criterion that extends to any Lévy process with $\lim_{t\to\infty} X_t = +\infty$. A similar integral test was also proved by Salminen and Yor [30] for Brownian motion with drift, and by Erickson and Maller [14] when restricting f to non-increasing functions. We state the integral test by Erickson and Maller.

Theorem 4.5 (Erickson and Maller [14])**.** *Let $X = (X_t)_{t\geq 0}$ be a Lévy process with $\lim_{t\to\infty} X_t = +\infty$, and let f be a positive and non-increasing function on $\mathbb{R}$. Then,*

$$\int_0^\infty f(X_s) ds < \infty \iff \int_0^\infty f(x) dx < \infty.$$

Proof. See [14]. □

4.1.2 Two short propositions on convergence

For reference we state a part of Theorem 4.3. We only prove the implication in one direction, but the argument is a simpler one than in [2].

Proposition 4.6. *Let*

$$Z_t = \int_0^t e^{-X_s} ds,$$

where X is a Lévy process. Then, $\lim_{t\to\infty} Z_t < \infty$ a.s. if and only if $\lim_{t\to\infty} X_t = +\infty$ a.s.

Proof. ($\Rightarrow$) This direction relies on advanced theory. See [2] for the details.

($\Leftarrow$) Assume $\lim_{t\to\infty} X_t = +\infty$ a.s. If $\mathbf{P}\left(\lim_{t\to\infty} \frac{X_t}{t} \leq 0\right) > 0$, then it would immediately contradict the assumption. Thus, $\lim_{t\to\infty} \frac{X_t}{t} > 0$ a.s.

This implies that there exists an a.s. finite stopping time τ and an $\epsilon > 0$ such that for all $t > \tau(\omega)$, $\frac{X_t(\omega)}{t} > \epsilon$. As such, (cf. Dufresne [12, Prop. 4.4.1])

$$Z_\infty = \int_0^\tau e^{-X_s} ds + \int_\tau^\infty e^{-X_s} ds,$$

where the first term clearly is finite a.s. and the other,

$$\int_\tau^\infty e^{-X_s} ds = \int_\tau^\infty e^{-s\frac{X_s}{s}} ds < \int_\tau^\infty e^{-s\epsilon} ds < \infty \text{ a.s.}$$

So $Z_\infty < \infty$ a.s. and thus the proof is complete. □

Next we consider the case where we integrate not with respect to $Y_t = t$, but a process with a Brownian and a jump component, in addition to deterministic drift. The previous proposition dealt with drift only, while among others Dufresne [12] has proved convergence for the Brownian component. The interesting part is the jump process, which we assume to be a compound Poisson process. I have discovered an interesting proof that the integral with respect to the compound Poisson process converges a.s. as $t \to \infty$.

Proposition 4.7. *Let $X = (X_t)_{t\geq 0}$ and $Y = (Y_t)_{t\geq 0}$ be independent Lévy processes. Assume Y is given by*

$$Y_t = \mu t + \sigma W_t + \sum_{k=1}^{N_t} K_k,$$

where W is a standard Brownian motion, N is a Poisson point process, and $K_1, K_2, \ldots,$ are i.i.d. random variables with $\mathbf{E}\left(\log|K_1|^+\right) < \infty$.

Then

$$Z_\infty = \int_0^\infty e^{-X_s} dY_s$$

is finite a.s., if $\lim_{t\to\infty} X_t = +\infty$ a.s.

Proof. Separate the integral Z_t into

$$Z_t = \mu \int_0^t e^{-X_s} ds + \sigma \int_0^t e^{-X_s} dW_s + \int_0^t e^{-X_s} d\left(\sum_{k=1}^{N_s} K_k\right).$$

The first term converges a.s. as $t \to \infty$ according to Proposition 4.6. The same holds for the second term according to Dufresne [12, Prop. 4.4.1]. That leaves the third integral.

By our assumptions, for every $\epsilon > 0$, there exists an a.s. finite random time τ such that for all $t > \tau(\omega)$, $\frac{X_t(\omega)}{t} > \epsilon$. As such,

$$\int_0^\infty e^{-X_s} d\left(\sum_{k=1}^{N_s} K_k\right) \leq \int_0^\tau e^{-X_s} d\left(\sum_{k=1}^{N_s} K_k\right) + \int_\tau^\infty e^{-s\epsilon} d\left(\sum_{k=1}^{N_s} K_k\right).$$

In a finite interval a Poisson process has a Poisson-distributed number of jumps, that is finite a.s., so the integral up to τ is finite a.s. For the latter integral,

$$\int_\tau^\infty e^{-s\epsilon} d\left(\sum_{k=1}^{N_s} K_k\right) = \sum_{k=N_\tau+1}^{\infty} e^{-\epsilon T_k} K_k = e^{-\epsilon T_{N_\tau}} \sum_{k=1}^{\infty} e^{-\epsilon T_k} K_k < \sum_{k=1}^{\infty} e^{-\epsilon T_k} K_k,$$

where $T_k, k = 1, 2, \ldots$ is the sequence of jump times for N, and, recalling that $\tau \perp N$, the memoryless property of the Poisson process N was used. Recall that for a Poisson process, for any $k \in \mathbb{N}$,

$$T_k = \sum_{j=1}^{k} \upsilon_j,$$

where $\upsilon_j \sim Exp(\lambda)$ is an i.i.d. sequence of sojourn times. As such, defining $V_k := e^{-\epsilon \upsilon_k}$, we have

$$\int_\tau^\infty e^{-s\epsilon} d\left(\sum_{k=1}^{N_s} K_k\right) < \sum_{k=1}^{\infty} K_k \prod_{j=1}^{k} V_j,$$

that is a discrete-time perpetuity, which is finite a.s. according to Theorem 3.7. This completes the proof. □

4.2 The moments of a continuous cash flow

4.2.1 Interest and payments as Brownian motion with drift

Like in chapter 3, this section shows how to compute moments of the processes $(S_t)_{t\geq 0}$ and $(Z_t)_{t\geq 0}$, defined as

$$S_t = \int_0^t e^{\tilde{X}_t - \tilde{X}_s} d\widetilde{Y}_s,$$

and

$$Z_t = \int_0^t e^{-\tilde{X}_s} d\widetilde{Y}_s,$$

where $\widetilde{X}$ and $\widetilde{Y}$ are two independent Brownian motions with drift, representing the rate of return on investment and the stream of payments received, respectively. For the sake of reference their definitions are

$$\widetilde{X}_t := \gamma t + \sigma X_t,$$
$$\widetilde{Y}_t := \mu t + \zeta Y_t.$$

Once again, much of the basic work is due to Dufresne [12, 13].

In order to derive moment formulas for the process S, the Itô formula with $f(t, x, y) = x^m$ will be applied on the Itô SDE (4.6). Then,

$$\frac{\partial f}{\partial t} = 0,\ \frac{\partial f}{\partial x} = mx^{m-1},\ \frac{\partial^2 f}{\partial x^2} = m(m-1)x^{m-2}.$$

As such,

$$dS_t^m = \left(mS_t^{m-1}(\alpha S_t + \mu) + \frac{1}{2}m(m-1)S_t^{m-2}(\sigma^2 S_t^2 + \zeta^2) \right) dt$$
$$+ mS_t^{m-1}\sigma S_t dX_t + mS_t^{m-1}\zeta dY_t,$$

where $(X_t)_{t\geq 0}$ and $(Y_t)_{t\geq 0}$ are two independent, standard Brownian motions. We express this in integral form (as usual taking $S_0 = 0$), while letting $\alpha_m = m\gamma + m^2\sigma^2/2, \beta_m = m\mu$, and $\varepsilon_m = m(m-1)\zeta^2/2$,

$$S_t^m = \int_0^t \left(\alpha_m S_s^m + \beta_m S_s^{m-1} + \varepsilon_m S_s^{m-2}\right) ds + m\sigma \int_0^t S_s^m dX_s + m\zeta \int_0^t S_s^{m-1} dY_s.$$

Recall that $\mathbf{E}\left(\int_0^t G_s dW_s\right) = 0$ when the process $G \in \mathbb{L}^2$ and is progressively measurable with respect to the filtration generated by W. Thus, by taking the expectation and applying the Fubini-Tonelli theorem to change order of integration,

$$\mathbf{E}\left(S_t^m\right) = \int_0^t \left(\alpha_m \mathbf{E}\left(S_s^m\right) + \beta_m \mathbf{E}\left(S_s^{m-1}\right) + \varepsilon_m \mathbf{E}\left(S_s^{m-2}\right)\right) ds.$$

Then, taking the derivative yields the differential equation

$$\frac{d}{dt}\mathbf{E}\left(S_t^m\right) = \alpha_m \mathbf{E}\left(S_t^m\right) + \beta_m \mathbf{E}\left(S_t^{m-1}\right) + \varepsilon_m \mathbf{E}\left(S_t^{m-2}\right). \tag{4.10}$$

Compare to the approach in Section 3.2. Starting from $m = 1$, Equation (4.6) leads to

$$\frac{d}{dt}\mathbf{E}\left(S_t\right) = \alpha_1 \mathbf{E}\left(S_t\right) + \beta_1.$$

We note that $\alpha_1 = 0$ only in a few special cases better left for later, so we assume $\alpha_1 \neq 0$. Then,

$$\mathbf{E}\left(S_t\right) = \frac{\beta_1}{\alpha_1}\left(e^{\alpha_1 t} - 1\right)$$

is the general solution. For $m = 2$ we have the equation

$$\frac{d}{dt}\mathbf{E}\left(S_t^2\right) = \alpha_2 \mathbf{E}\left(S_t^2\right) + \beta_2 \mathbf{E}\left(S_t\right) + \varepsilon_2$$

$$= \alpha_2 \mathbf{E}\left(S_t^2\right) + \frac{\beta_1 \beta_2}{\alpha_1}\left(e^{\alpha_1 t} - 1\right) + \varepsilon_2.$$

This is also a linear first-order differential equation, which is easily solved e.g. by taking the general solution to the homogeneous equation and then using the method of variation of constants. When $0 \neq \alpha_1 \neq \alpha_2 \neq 0$ the general solution is

$$\mathbf{E}\left(S_t^2\right) = -\frac{\beta_1 \beta_2}{\alpha_1 \alpha_2} e^{\alpha_2 t} + \frac{\varepsilon_2}{\alpha_2}\left(e^{\alpha_2 t} - 1\right) + \frac{\beta_1 \beta_2}{\alpha_1(\alpha_1 - \alpha_2)}\left(e^{\alpha_1 t} - 1\right) + \frac{\beta_1 \beta_2}{\alpha_1 \alpha_2}.$$

Note that for $m = 1, 2$ the moments have the form

$$\mathbf{E}\left(S_t^m\right) = \sum_{j=0}^{m} d_{mj} e^{\alpha_j t}, \tag{4.11}$$

where d_{mj} are constants. It turns out that there is a continuous analogue to Proposition 3.8.

Proposition 4.8. *Let $m \in \mathbb{N}$. Provided that $\alpha_i \neq \alpha_j$, for all $0 \leq i < j \leq m$, Equation (4.11) holds, for some constants $\{d_{mj}, 0 \leq j \leq m\}$.*

Proof. The assertion was shown above for the cases $m = 1, 2$. We take an arbitrary $m > 2$ and assume it holds for $m - 1$ and $m - 2$, using induction to prove it holds in general. Inserting Equation (4.11) into Equation (4.10) gives

$$\frac{d}{dt}\mathbf{E}\left(S_t^m\right) = \alpha_m \mathbf{E}\left(S_t^m\right) + \beta_m \sum_{j=0}^{m-1} d_{(m-1)j} e^{\alpha_j t} + \varepsilon_m \sum_{j=0}^{m-2} d_{(m-2)j} e^{\alpha_j t}$$

$$= \alpha_m \mathbf{E}\left(S_t^m\right) + \sum_{j=0}^{m-2} (\beta_m d_{(m-1)j} + \varepsilon_m d_{(m-2)j}) e^{\alpha_j t} + \beta_m d_{(m-1)(m-1)} e^{\alpha_{m-1} t}.$$

This is a linear differential equation which can be solved by variation of constants. First, notice that the general solution to the homogeneous equation

$$\frac{d}{dt}\mathbf{E}\left(S_t^m\right) = \alpha_m \mathbf{E}\left(S_t^m\right)$$

is

$$\mathbf{E}\left(S_t^m\right) = K e^{\alpha_m t}.$$

Then a particular solution found by variation of constants is given by

$$\int_0^t g(s) e^{\alpha_m (t-s)} ds,$$

where

$$g(s) = \sum_{j=0}^{m-2} (\beta_m d_{(m-1)j} + \varepsilon_m d_{(m-2)j}) e^{\alpha_j s} + \beta_m d_{(m-1)(m-1)} e^{\alpha_{m-1} s}.$$

Computing the solution from this and adding the solution to the homogeneous equation yields the general solution

$$\begin{aligned} \mathbf{E}\left(S_t^m\right) = K e^{\alpha_m t} &+ \sum_{j=0}^{m-2} \frac{\beta_m d_{(m-1)j} + \varepsilon_m d_{(m-2)j}}{\alpha_j - \alpha_m} \left(e^{\alpha_j t} - e^{\alpha_m t}\right) \\ &+ \frac{\beta_m d_{(m-1)(m-1)}}{\alpha_{m-1} - \alpha_m} \left(e^{\alpha_{m-1} t} - e^{\alpha_m t}\right), \end{aligned} \tag{4.12}$$

which is clearly an expression that can be rearranged into form (4.11). □

It is also possible to find a recursive formula for the constants d_{mj} from Equation (4.12). First consider that the initial condition $S_0 = 0$ must hold, which leads to $K = 0$. Thus,

$$\begin{aligned} d_{mj} &= \frac{\beta_m d_{(m-1)j} + \varepsilon_m d_{(m-2)j}}{\alpha_j - \alpha_m}, \ 0 \le j \le m-2, \\ d_{m(m-1)} &= \frac{\beta_m d_{(m-1)(m-1)}}{\alpha_{m-1} - \alpha_m}, \ d_{mm} = -\sum_{j=0}^{m-1} d_{mj}. \end{aligned} \tag{4.13}$$

Restriction to the case $\widetilde{Y}_t = t$

In the special case where $\zeta = 0$ (i.e. payments are constant) there is a general formula for d_{mj}. Dufresne [13] has derived the formula with a clever argument based on interpolation theory; here we give an alternative proof relying on the Fundamental Theorem of Algebra.

Lemma 4.9. *Suppose $a_i \neq a_j$, whenever $0 \leq i < j \leq k$ holds. Then,*

$$\sum_{j=0}^{k} \prod_{\substack{i=0 \\ i \neq j}}^{k} (a_j - a_i)^{-1} = 0.$$

Proof. We consider the expression as a function of a_k, defining the function

$$f(x) = \sum_{j=0}^{k-1} (a_j - x)^{-1} \prod_{\substack{i=0 \\ i \neq j}}^{k-1} (a_j - a_i)^{-1} + \prod_{i=0}^{k-1} (x - a_i)^{-1}.$$

Multiplying both sides by $\prod_{i=0}^{k-1}(x - a_i)$ gives a polynomial with extended domain $\mathbb{R}$,

$$p(x) = f(x) \prod_{i=0}^{k-1} (x - a_i) = 1 - \sum_{j=0}^{k-1} \prod_{\substack{i=0 \\ i \neq j}}^{k-1} \frac{x - a_i}{a_j - a_i}.$$

Then p has k roots, in the points $a_0, \ldots, a_{k-1}$, but the degree of p is $k - 1$. By the Fundamental Theorem of Algebra, $p(x) = 0$, for all $x \in \mathbb{R}$. It then follows directly that for $x = a_k$ such that $a_k \neq a_i$ for all $i < k$,

$$0 = \frac{p(a_k)}{\prod_{i=0}^{k-1}(a_k - a_i)} = \sum_{j=0}^{k} \prod_{\substack{i=0 \\ i \neq j}}^{k} (a_j - a_i)^{-1},$$

which proves the result. □

Proposition 4.10. *When $\zeta = 0$ the constants in the recursive equations in* (4.13) *are given by*

$$d_{mj} = \mu^m m! \prod_{\substack{i=0 \\ i \neq j}}^{k} (\alpha_j - \alpha_i)^{-1}. \tag{4.14}$$

Proof. If $\zeta = 0$, Equations (4.13) hold but with $\varepsilon = 0$. From earlier we know that the assertion holds for $m = 1$ (recall that $\alpha_0 = 0$). What remains is to employ induction over m. From (4.13), when $j < m$

$$d_{mj} = \frac{\beta_m}{\alpha_j - \alpha_m} \mu^{m-1}(m-1)! \prod_{\substack{i=0 \\ i \neq j}}^{m-1} (\alpha_j - \alpha_i)^{-1} = \mu^m m! \prod_{\substack{i=0 \\ i \neq j}}^{m} (\alpha_j - \alpha_i)^{-1}.$$

And for the last constant,

$$d_{mm} = -\mu^m m! \sum_{j=0}^{m-1} \prod_{\substack{i=0 \\ i \neq j}}^{m} (\alpha_j - \alpha_i)^{-1}$$

$$= -\mu^m m! \sum_{j=0}^{m} \prod_{\substack{i=0 \\ i \neq j}}^{m} (\alpha_j - \alpha_i)^{-1} + \mu^m m! \prod_{\substack{i=0 \\ i \neq m}}^{m} (\alpha_j - \alpha_i)^{-1} = \mu^m m! \prod_{\substack{i=0 \\ i \neq m}}^{m} (\alpha_j - \alpha_i)^{-1},$$

where Lemma (4.9) has been used. □

Remark 4.11. The solutions are slightly different if $\alpha_j = \alpha_m$ for some $j < m$. When computing the particular solution for (4.12) there will be a point where one computes the integral

$$\int_0^t e^{(\alpha_j - \alpha_m)s} ds.$$

Thus, the solution will have a last term of the form $K \cdot te^{\alpha_m t}$, where K is a constant. Due to the recursive structure successive higher moments will also contain such a term.

Moments of a discounted cash flow

Recall that

$$Z_t = \int_0^t e^{-\tilde{X}_s} d\widetilde{Y}_s.$$

Define a new process $(B_t)_{t \in \mathbb{R}}$, by

$$B_t = \int_0^t e^{-(\tilde{X}_t - \tilde{X}_s)} d\widetilde{Y}_s.$$

Recalling that $\tilde{X}$ has stationary increments, i.e. for all $t \geq s \geq 0$, $\tilde{X}_{t-s} \stackrel{d}{=} \tilde{X}_t - \tilde{X}_s$, one sees that

$$B_t \stackrel{d}{=} \int_0^t e^{-\tilde{X}_{t-s}} d\widetilde{Y}_s \stackrel{d}{=} -\int_t^0 e^{-\tilde{X}_s} d\widetilde{Y}_s = \int_0^t e^{-\tilde{X}_s} d\widetilde{Y}_s,$$

provided that $\widetilde{X}$ and $\widetilde{Y}$ are independent. In this case it follows that

$$Z_t \stackrel{d}{=} B_t, \ \forall t \geq 0.$$

The moments of Z_t can therefore be found from the process B, which has the same structure as S. Essentially, B is a version of S where $\widetilde{X}_t$ gets mapped to $-\widetilde{X}_t$, or equivalently γ, σ get replaced by $-\gamma, -\sigma$. Thus the moments of Z_t can be computed from Equations (4.11) and (4.13) with only $\alpha_m = m\gamma + m^2\sigma^2/2$ replaced by $\alpha_m = -m\gamma + m^2\sigma^2/2$.

Dufresne [12] also makes this argument, and lists the constants $d_{mj}, 0 \leq j \leq m$ when possible.

4.2.2 Exponential functionals of a stochastic process

In this section moment formulas for the exponential functional of an additive process are derived using recent results due to Salminen and Vostrikova [29]. Prior work such as [32] and [5] have derived moment formulas for exponential functionals of subordinators or with an exponential stopping time. Lévy processes satisfying their conditions is an important special case of Salminen and Vostrikova's work.

Let $(X_t)_{t\geq 0}$ be an additive process. Define

$$Z_{s,t} := \int_s^t e^{-X_u} du, \quad 0 \leq s < t \leq \infty.$$

Assume

$$\mathbf{E}\left(e^{-\lambda X_t}\right) < \infty \text{ for all } t \geq 0, \lambda \geq 0. \tag{4.15}$$

Then Φ given by

$$\Phi(t, \lambda) := -\log \mathbf{E}\left(e^{-\lambda X_t}\right)$$

is well-defined, for $t \geq 0$ and $\lambda \geq 0$.

For $0 \leq s \leq t \leq \infty$ and $\alpha \geq 0$, define

$$m_{s,t}^{(\alpha)} := \mathbf{E}\left(Z_{s,t}^{\alpha}\right) = \mathbf{E}\left(\left(\int_s^t e^{-X_u} du\right)^{\alpha}\right), \quad m_t^{(\alpha)} := m_{0,t}^{(\alpha)}.$$

Theorem 4.12 (Salminen and Vostrikova [29])**.** *Let $0 \leq s \leq t < \infty$. Then, given the assumption (4.15), it holds for $\alpha \geq 1$ that $m_{s,t}^{(\alpha)} < \infty$ and*

$$m_{s,t}^{(\alpha)} = \alpha \int_s^t m_{u,t}^{(\alpha-1)} e^{-(\Phi(u,\alpha)-\Phi(u,\alpha-1))} du. \tag{4.16}$$

Proof. When $t > 0$, $Z_{s,t}$ is continuous and decreasing in s in the interval $[0, t]$. As such, for $\alpha \geq 1$

$$Z_{s,t}^{\alpha} - Z_{0,t}^{\alpha} = \alpha \int_{Z_{0,t}}^{Z_{s,t}} x^{\alpha-1} dx = \alpha \int_0^s Z_{u,t}^{\alpha-1} dZ_{u,t},$$

where the last integral is interpreted as a Riemann-Stieltjes integral for almost all ω, and the variable substitution formula for Riemann-Stieltjes integrals has been used (valid since $Z_{s,t}$ is continuous and monotonous in s, see Apostol [1, Thm. 7.7]). Moving on, from the definition of Z it follows that

$$Z_{s,t}^{\alpha} - Z_{0,t}^{\alpha} = -\alpha \int_0^s Z_{u,t}^{\alpha-1} e^{-X_u} du.$$

Now define

$$\widehat{Z}_{s,t} := \int_0^{t-s} e^{-(X_{u+s}-X_s)} du.$$

Then,

$$\widehat{Z}_{s,t} = e^{X_s} Z_{s,t} = e^{X_s} \int_s^t e^{-X_u} du.$$

As such,

$$Z_{s,t}^{\alpha} - Z_{0,t}^{\alpha} = -\alpha \int_0^s \widehat{Z}_{u,t}^{\alpha-1} e^{-\alpha X_u} du. \tag{4.17}$$

The independent increments property of X implies that, for $\alpha \geq 0$

$$\mathbf{E}\left(\widehat{Z}_{u,t}^{\alpha}\right) = \mathbf{E}\left(Z_{0,t}^{\alpha}\right) / \mathbf{E}\left(e^{-\alpha X_u}\right). \tag{4.18}$$

Now, let $\alpha \in [1,2]$. Taking the expectation in (4.17),

$$\mathbf{E}\left(Z_{s,t}^{\alpha} - Z_{0,t}^{\alpha}\right) = -\alpha \int_0^s \mathbf{E}\left(\widehat{Z}_{u,t}^{\alpha-1}\right) \mathbf{E}\left(e^{-\alpha X_u}\right) du > -\infty, \tag{4.19}$$

with finiteness holding due to, with $0 \leq p < 1$, $\mathbf{E}\left(Z_{u,t}^{p}\right) \leq \mathbf{E}\left(Z_{u,t}\right)^{p}$, from Jensen's inequality, and the assumption that $\mathbf{E}\left(e^{-\lambda X_u}\right) < \infty$ for every $t \geq 0, \lambda \geq 0$. With these conditions, the monotone convergence theorem as $s \uparrow t$ can be used on equation (4.19), yielding

$$\mathbf{E}\left(Z_{0,t}^{\alpha}\right) = \alpha \int_0^t \mathbf{E}\left(\widehat{Z}_{u,t}^{\alpha-1}\right) \mathbf{E}\left(e^{-\alpha X_u}\right) du.$$

Now insert this back into (4.19) to get

$$\mathbf{E}\left(Z_{s,t}^{\alpha}\right) = \alpha \int_s^t \mathbf{E}\left(\widehat{Z}_{u,t}^{\alpha-1}\right) \mathbf{E}\left(e^{-\alpha X_u}\right) du.$$

Now, since $\alpha \in [1,2]$ (4.18) can be applied in order to get the desired formula,

$$m_{s,t}^{(\alpha)} = \alpha \int_s^t m_{u,t}^{(\alpha-1)} e^{-(\Phi(u,\alpha)-\Phi(u,\alpha-1))} du.$$

The formula also holds for $\alpha > 2$ since (4.18) holds for any $\alpha \geq 0$. Finiteness is preserved since the moments were finite when $\alpha \in [1,2]$. □

A generalized formula holds also for $t = \infty$, although only for positive integer moments. No more practical formulas that can be applied on perpetuities based on an additive discounting process appear to be known currently.

Theorem 4.13 (Salminen and Vostrikova [29]). *For $0 \leq s \leq t \leq \infty$ and $n \in \mathbb{N}$,*

$$\begin{aligned} m_{s,t}^{(n)} &= n! \int_s^t dt_1 \int_{t_1}^t dt_2 \cdots \\ &\cdots \int_{t_{n-1}}^t dt_n \exp\left\{ -\sum_{k=1}^n (\Phi(t_k, n-k+1) - \Phi(t_k, n-k)) \right\}. \end{aligned} \tag{4.20}$$

The moment of the perpetuity, $m_{s,\infty}^{(n)}$, is finite if and only if the integral is convergent.

Proof. First, let $t < \infty$. Then,

$$\begin{aligned} m_{s,t}^{(n)} &= \mathbf{E}\left(\left(\int_s^t e^{-X_u} du\right)^n\right) \\ &= \mathbf{E}\left(\int_s^t \cdots \int_s^t e^{-(X_{t_1}+\cdots+X_{t_n})} dt_1 \cdots dt_n\right) \\ &= n!\mathbf{E}\left(\int_s^t dt_1 e^{-X_{t_1}} \int_{t_1}^t dt_2 e^{-X_{t_2}} \cdots \int_{t_{n-1}}^t dt_n e^{-X_{t_n}}\right) \\ &= n! \int_s^t dt_1 \int_{t_1}^t dt_2 \cdots \int_{t_{n-1}}^t dt_n \mathbf{E}\left(e^{-(X_{t_1}+\cdots+X_{t_n})}\right). \end{aligned}$$

Note the use of a symmetry argument above: for a continuous function such that $g(x_1, \ldots, x_i, x_{i+1}, \ldots, x_n) = g(x_1, \ldots, x_{i+1}, x_i, \ldots, x_n)$ for every i, it holds that

$$\int_a^b \cdots \int_a^b g(x_1, \ldots, x_n) dx_1 \cdots dx_n = n! \int_a^b dx_1 \int_{x_1}^b dx_2 \cdots \int_{x_{n-1}}^b dx_n g(x_1, \ldots, x_n),$$

which is the claim used in the third step above.

Because X has independent increments,

$$\mathbf{E}\left(e^{-\alpha X_t}\right) = \mathbf{E}\left(e^{-\alpha(X_t - X_s) - \alpha X_s}\right) = \mathbf{E}\left(e^{-\alpha(X_t - X_s)}\right) \mathbf{E}\left(e^{-\alpha X_s}\right).$$

As such,

$$\mathbf{E}\left(e^{-\alpha(X_t - X_s)}\right) = \mathbf{E}\left(e^{-\alpha X_t}\right) / \mathbf{E}\left(e^{-\alpha X_s}\right) = e^{-(\Phi(t,\alpha) - \Phi(s,\alpha))}.$$

Next we notice that (with $t_0 = 0$)

$$X_{t_1} + \cdots + X_{t_n} = \sum_{k=1}^n (n-k+1)(X_{t_k} - X_{t_{k-1}}).$$

Using the equations above, we get

$$\begin{aligned} m_{s,t}^{(n)} &= n! \int_s^t dt_1 \int_{t_1}^t dt_2 \cdots \int_{t_{n-1}}^t dt_n \mathbf{E}\left(e^{-\sum_{k=1}^n (n-k+1)(X_{t_k} - X_{t_{k-1}})} \right) \\ &= n! \int_s^t dt_1 \int_{t_1}^t dt_2 \cdots \\ &\cdots \int_{t_{n-1}}^t dt_n \exp\left\{ -\sum_{k=1}^n (\Phi(t_k, n-k+1) - \Phi(t_{k-1}, n-k+1)) \right\}. \end{aligned}$$

Finally, recognize from the definition of Φ that $\Phi(0, \lambda) = \Phi(t, 0) = 0$. Then,

$$\sum_{k=1}^n (\Phi(t_k, n-k+1) - \Phi(t_{k-1}, n-k+1)) = \sum_{k=1}^n (\Phi(t_k, n-k+1) - \Phi(t_k, n-k)).$$

Therefore,

$$\begin{aligned} m_{s,t}^{(n)} &= n! \int_s^t dt_1 \int_{t_1}^t dt_2 \cdots \\ &\cdots \int_{t_{n-1}}^t dt_n \exp\left\{ -\sum_{k=1}^n (\Phi(t_k, n-k+1) - \Phi(t_k, n-k)) \right\}. \end{aligned}$$

The formula is further proved for the $t = +\infty$ case by taking the limit, using the monotone convergence theorem to justify $\lim_{t\to\infty} \mathbf{E}\left(m_{s,t}^{(n)} \right) = \mathbf{E}\left(m_{s,\infty}^{(n)} \right)$. □

Exponential functionals of a Lévy process

The above results are well applied to the case where X is a Lévy process. The assumption (4.15) is kept, but the Laplace exponent ϕ satisfies

$$\mathbf{E}\left(e^{-\lambda X_t} \right) = e^{-t\phi(\lambda)}, \quad \text{for } \lambda \geq 0.$$

Theorem 4.14 (Salminen and Vostrikova [29]). *Let X be a Lévy process satisfying (4.15). The moments of*

$$Z_t = \int_0^t e^{-X_u} du$$

are given by the recursive formula

$$m_t^{(\alpha)} = \alpha e^{-t\phi(\alpha)} \int_0^t m_u^{(\alpha-1)} e^{u\phi(\alpha)} du, \tag{4.21}$$

for $\alpha \geq 1$.

Proof. First, note that, with $\alpha \geq 1$,

$$m_{u,t}^{(\alpha-1)} = e^{-u\phi(\alpha-1)} m_{t-u}^{(\alpha-1)}.$$

This is because, keeping in mind the stationary of increments property,

$$\begin{aligned} m_{u,t}^{(\alpha-1)} &= \mathbf{E}\left(\left(\int_u^t e^{-X_v} dv\right)^{\alpha-1}\right) \\ &= \mathbf{E}\left(e^{-(\alpha-1)X_u}\left(\int_u^t e^{-(X_v - X_u)} dv\right)^{\alpha-1}\right) \\ &= e^{-u\phi(\alpha-1)} \mathbf{E}\left(\left(\int_0^{t-u} e^{-(X_{v+u} - X_u)} dv\right)^{\alpha-1}\right) \\ &= e^{-u\phi(\alpha-1)} \mathbf{E}\left(\left(\int_0^{t-u} e^{-X_v} dv\right)^{\alpha-1}\right) \\ &= e^{-u\phi(\alpha-1)} m_{t-u}^{(\alpha-1)}, \end{aligned}$$

as it should be.

Applying Theorem 4.12 with $s = 0$ yields

$$m_t^{(\alpha)} = \alpha \int_0^t m_{u,t}^{(\alpha-1)} e^{-u(\phi(\alpha)-\phi(\alpha-1))} du.$$

Applying the prior equation here leads to

$$m_t^{(\alpha)} = \alpha \int_0^t e^{-u\phi(\alpha-1)} m_{t-u}^{(\alpha-1)} e^{-u(\phi(\alpha)-\phi(\alpha-1))} du = \alpha \int_0^t m_{t-u}^{(\alpha-1)} e^{-u\phi(\alpha)} du.$$

Finally, a variable substitution of $v = t - u$ directly yields

$$m_t^{(\alpha)} = \alpha e^{-t\phi(\alpha)} \int_0^t m_v^{(\alpha-1)} e^{v\phi(\alpha)} dv,$$

which proves the theorem. □

Next, we present an explicit formula for positive integer moments of Z_t.

Theorem 4.15. *Let X be a Lévy process satisfying* (4.15). *If $\phi(i) \neq \phi(j)$, whenever $0 \leq i < j \leq n$, then the positive integer moments of*

$$Z_t = \int_0^t e^{-X_u} du, \ t > 0$$

are given by the formula

$$\mathbf{E}\left(Z_t^n\right) = n! \sum_{k=0}^{n-1} \frac{e^{-t\phi(k)} - e^{-t\phi(n)}}{\prod_{\substack{i=0 \\ i\neq k}}^{n} (\phi(i) - \phi(k))}, \quad n \in \mathbb{N}. \tag{4.22}$$

Proof. The proof is carried out by induction. For $n = 1$, recalling that $\phi(0) = 0$, compute the integral in (4.21),

$$m_t^{(1)} = e^{-t\phi(1)} \int_0^t e^{u\phi(1)} du = \frac{1 - e^{-t\phi(1)}}{\phi(1)},$$

which is consistent with (4.22).

Next, assume that formula (4.22) holds for some $n \in \mathbb{N}$. We shall show that this implies that it also holds for $n + 1$. Again, consider formula (4.21) but with $\alpha = n + 1$. Then, using the induction assumption,

$$\begin{aligned}
m_t^{(n+1)} &= (n+1)e^{-t\phi(n+1)} \int_0^t m_u^{(n)} e^{u\phi(n+1)} du \\
&= (n+1)e^{-t\phi(n+1)} \int_0^t n! \left[\sum_{k=0}^{n-1} \frac{e^{-u\phi(k)} - e^{-u\phi(n)}}{\prod_{\substack{i=0 \\ i\neq k}}^{n} (\phi(i) - \phi(k))} \right] e^{u\phi(n+1)} du \\
&= (n+1)! e^{-t\phi(n+1)} \sum_{k=0}^{n-1} \left[\frac{\int_0^t e^{u(\phi(n+1)-\phi(k))} du - \int_0^t e^{u(\phi(n+1)-\phi(n))} du}{\prod_{\substack{i=0 \\ i\neq k}}^{n} (\phi(i) - \phi(k))} \right] = (*),
\end{aligned}$$

where we need to insert expressions for the integrals in the sum. For $k \leq n$,

$$\int_0^t e^{u(\phi(n+1)-\phi(k))} du = \frac{e^{t(\phi(n+1)-\phi(k))} - 1}{\phi(n+1) - \phi(k)}.$$

Inserting this into $(*)$ yields

$$\begin{aligned}
(*) &= (n+1)! \sum_{k=0}^{n-1} \left[\frac{\frac{e^{-t\phi(k)} - e^{-t\phi(n+1)}}{\phi(n+1)-\phi(k)} - \frac{e^{-t\phi(n)} - e^{-t\phi(n+1)}}{\phi(n+1)-\phi(n)}}{\prod_{\substack{i=0 \\ i\neq k}}^{n} (\phi(i) - \phi(k))} \right] \\
&= (n+1)! \sum_{k=0}^{n-1} \frac{e^{-t\phi(k)} - e^{-t\phi(n+1)}}{\prod_{\substack{i=0 \\ i\neq k}}^{n+1} (\phi(i) - \phi(k))} - A,
\end{aligned}$$

where

$$\begin{aligned}
A &= (n+1)! \sum_{k=0}^{n-1} \frac{e^{-t\phi(n)} - e^{-t\phi(n+1)}}{(\phi(n+1) - \phi(n)) \prod_{\substack{i=0 \\ i\neq k}}^{n} (\phi(i) - \phi(k))} \\
&= (n+1)! \frac{e^{-t\phi(n)} - e^{-t\phi(n+1)}}{\phi(n+1) - \phi(n)} \cdot \sum_{k=0}^{n-1} \frac{1}{\prod_{\substack{i=0 \\ i\neq k}}^{n} (\phi(i) - \phi(k))}.
\end{aligned}$$

Now, applying Lemma 4.9 with $a_j = \phi(j)$ yields the equation

$$\sum_{k=0}^{n-1} \frac{1}{\prod_{\substack{i=0 \\ i\neq k}}^{n} (\phi(i) - \phi(k))} = - \frac{1}{\prod_{\substack{i=0 \\ i\neq n}}^{n} (\phi(i) - \phi(n))},$$

so that

$$A = -(n+1)! \frac{e^{-t\phi(n)} - e^{-t\phi(n+1)}}{\prod_{\substack{i=0\\i\neq n}}^{n+1} (\phi(i) - \phi(k))}.$$

It follows that

$$\begin{aligned} m_t^{(n+1)} &= (n+1)! \sum_{k=0}^{n-1} \frac{e^{-t\phi(k)} - e^{-t\phi(n+1)}}{\prod_{\substack{i=0\\i\neq k}}^{n+1} (\phi(i) - \phi(k))} + (n+1)! \frac{e^{-t\phi(n)} - e^{-t\phi(n+1)}}{\prod_{\substack{i=0\\i\neq n}}^{n+1} (\phi(i) - \phi(k))} \\ &= (n+1)! \sum_{k=0}^{n} \frac{e^{-t\phi(k)} - e^{-t\phi(n+1)}}{\prod_{\substack{i=0\\i\neq k}}^{n+1} (\phi(i) - \phi(k))}, \end{aligned}$$

which proves the assertion for the $n+1$ case. By induction, the formula holds for all $n \in \mathbb{N}$. □

The moments of Z_∞ also have an explicit formula in terms of the Lévy exponent.

Theorem 4.16 (Salminen and Vostrikova [29]). *Let X be a Lévy process with Laplace exponent ϕ. Define $N := \min\{n \in \mathbb{N} : \phi(n) \leq 0\}$. Then,*

$$\mathbf{E}\left(Z_\infty^n\right) = \begin{cases} n! \prod_{k=1}^{n} \phi(k)^{-1}, & \text{if } n < N, \\ +\infty, & \text{if } n \geq N. \end{cases} \tag{4.23}$$

Proof. We use the integral expressions in Theorem 4.13,

$$\begin{aligned} \mathbf{E}\left(Z_\infty^n\right) = n! \int_s^\infty dt_1 \int_{t_1}^\infty dt_2 \cdots \\ \cdots \int_{t_{n-1}}^\infty dt_n \exp\left\{-t_k \sum_{k=1}^{n} (\phi(n-k+1) - \phi(n-k))\right\}. \end{aligned}$$

The innermost integral is evaluated for $n > 1$,

$$\int_{t_{n-1}}^\infty e^{-t_n(\phi(1)-\phi(0))} dt_n = \frac{e^{-t_{n-1}\phi(1)}}{\phi(1)}.$$

Thus, the next integral becomes

$$\int_{t_{n-2}}^\infty e^{-t_{n-1}(\phi(2)-\phi(1))} \frac{e^{-t_{n-1}\phi(1)}}{\phi(1)} dt_{n-1} = \frac{e^{-t_{n-2}\phi(2)}}{\phi(1)\phi(2)},$$

provided that $N > 2$. Proceeding by induction, take a $k \in \mathbb{N}, k < n$, assume that $N > k$ and the k:th integral evaluates to $\frac{e^{-t_{n-k}\phi(k)}}{\phi(1)\cdots\phi(k)}$. Then the $k+1$:th integral evaluates to

$$\int_{t_{n-k-1}}^\infty e^{-t_{n-k}(\phi(k+1)-\phi(k))} \frac{e^{-t_{n-k}\phi(k)}}{\phi(1)\cdots\phi(k)} dt_{n-k} = \frac{e^{-t_{n-k-1}\phi(k+1)}}{\phi(1)\cdots\phi(k+1)},$$

if $N > k+1$. If $N = k+1$ the integral diverges to $+\infty$ either because, if $\phi(k+1) = 0$ the integral becomes $\int_{t_{n-k-1}}^{\infty} 1 dt_{n-k}$ or because if $\phi(k+1) < 0$ the sign of the integral evaluates to positive.

The assertion then follows from the principle of induction, stopped at $k = n$ where the outermost integral is evaluated to

$$\mathbf{E}\left(Z_\infty^n\right) = \begin{cases} n! \prod_{k=1}^n \phi(k)^{-1}, & \text{if } n < N, \\ +\infty, & \text{if } n \geq N. \end{cases}$$

□

In some cases all the integer moments of Z_∞ exist. When this holds it may be possible to find the distribution of Z_∞ by computing its moment-generating function or identifying a distribution with identical integer moments. The following proposition is an elementary consequence of the prior results of this section.

Proposition 4.17. *Assume that* $N := \min\{n \in \mathbb{N} : \phi(n) \leq 0\} = \infty$. *Further assume there exist* $\delta > 0$ *such that for all* $|s| < \delta$, *it holds that*

$$\sum_{k=0}^{\infty} \frac{s^k}{\phi(1) \cdots \phi(k)} < \infty. \tag{4.24}$$

Then the distribution of Z_∞ *is determined by its moments and* Z_∞ *has a moment-generating function, given by*

$$M_{Z_\infty}(s) = \sum_{k=0}^{\infty} \frac{s^k}{\phi(1) \cdots \phi(k)}, \quad \text{for } |s| < \delta.$$

Proof. Assume t is such that (4.24) holds. Then, using Theorem 4.16,

$$\sum_{k=0}^{\infty} \frac{s^k}{\phi(1) \cdots \phi(k)} = \sum_{k=0}^{\infty} \frac{s^k}{k!} \frac{k!}{\phi(1) \cdots \phi(k)} = \sum_{k=0}^{\infty} \frac{s^k \mathbf{E}\left(Z_\infty^k\right)}{k!} = M_{Z_\infty}(s).$$

The moment-generating function of Z_∞ is thus defined for $|s| < \delta$. The fact that the distribution is determined by its moments follows immediately from Theorem 2.2. □

The proposition also has a simple and useful corollary; a slight modification of a result in [5].

Corollary 4.18. *Assume that* $\phi(n) > 0$ *for all* $n \in \mathbb{N}$ *and that* $\phi(\infty) := \lim_{n\to\infty} \phi(n) \in \mathbb{R}_+ \cup \{+\infty\}$. *Then the distribution of* Z_∞ *is determined by its moments and* Z_∞ *has a moment-generating function* $M_{Z_\infty}(s)$ *defined for* $|s| < \phi(\infty)$.

Proof. By the ratio test for series convergence,

$$\lim_{n\to\infty}\left|\frac{\frac{s^{n+1}}{\phi(1)\cdots\phi(n+1)}}{\frac{s^n}{\phi(1)\cdots\phi(n)}}\right| = \lim_{n\to\infty}\left|\frac{s}{\phi(n+1)}\right| = \frac{|s|}{\phi(\infty)},$$

which is smaller than 1 if and only if $|s| < \phi(\infty)$. Then the series in Proposition 4.17 is convergent for such s, and the assertions follow. □

Negative moments

We shall see that in certain cases the negative moments of the functional

$$Z_\infty = \int_0^\infty e^{-X_s} ds$$

can be easily calculated and in fact determine its distribution. Note that the assumptions on X made here shall differ from those in the prior section. To this end, let X be a Lévy process with a well-defined function φ determined by

$$\mathbf{E}\left(e^{\lambda X_t}\right) = e^{t\varphi(\lambda)} < \infty, \quad \text{for all } t \geq 0, \lambda \geq 0. \tag{4.25}$$

If Ψ is the characteristic exponent of X, then $\varphi(\lambda) = -\Psi(-i\lambda)$ is an analytical extension of the characteristic exponent (see [2] for details). We shall also assume that X is such that $Z_\infty < \infty$ almost surely, i.e. by Proposition 4.6 that $\lim_{t\to\infty} X_t = +\infty$. This entails that

$$m = \mathbf{E}\left(X_1\right) = \varphi'(0+) \in (0, \infty). \tag{4.26}$$

Proposition 4.19 (Bertoin and Yor [2]). *Assume conditions* (4.25) *and* (4.26) *hold. Then, for all* $k \in \mathbb{N}$ *and* $t \in (0, +\infty]$, *we have* $\mathbf{E}\left(Z_t^{-k}\right) < \infty$. *Furthermore,*

$$\mathbf{E}\left(Z_\infty^{-k}\right) = m\frac{\varphi(1)\cdots\varphi(k-1)}{(k-1)!}. \tag{4.27}$$

If X *has no positive jumps, then the distribution of* Z_∞ *is determined by its negative integer moments.*

Proof. Let $a, \lambda > 0$. Then it follows from the Markov inequality that

$$\mathbf{P}(X_1 \geq a) = \mathbf{P}\left(e^{\lambda X_1} \geq e^{\lambda a}\right) \leq \frac{\mathbf{E}\left(e^{\lambda X_1}\right)}{e^{\lambda a}} = \exp\{\varphi(\lambda) - \lambda a\}.$$

Moreover, by the assumption that $\lim_{t\to\infty} X_t = +\infty$, there exists $a > 0$ sufficiently large that

$$\mathbf{P}\left(\inf_{0\leq t\leq 1} X_t < -a\right) \leq 1/2.$$

Thus, if we define $\tau_y := \inf\{t \in \mathbb{R}_+ : X_t > y\}$,

$$\begin{aligned}
\mathbf{P}\left(\sup_{0\leq t\leq 1} X_t \geq 2a\right) &= \mathbf{P}\left(\sup_{0\leq t\leq 1} X_t \geq 2a, X_1 < a\right) + \mathbf{P}\left(\sup_{0\leq t\leq 1} X_t \geq 2a, X_1 \geq a\right) \\
&\leq \mathbf{P}\left(\tau_{2a} < 1, X_1 - X_{\tau_{2a}} < -a\right) + \mathbf{P}\left(\sup_{0\leq t\leq 1} X_t \geq a, X_1 \geq a\right) \\
&\leq \mathbf{P}\left(\sup_{0\leq t\leq 1} X_t \geq 2a\right) \cdot \mathbf{P}\left(\inf_{0\leq t\leq 1} X_t < -a\right) + \mathbf{P}(X_1 \geq a) \\
&\leq \frac{1}{2}\mathbf{P}\left(\sup_{0\leq t\leq 1} X_t \geq 2a\right) + \mathbf{P}(X_1 \geq a),
\end{aligned}$$

where on the second line the Strong Markov Property was used. The resulting inequality implies

$$\mathbf{P}\left(\sup_{0\leq t\leq 1} X_t \geq 2a\right) \leq 2\mathbf{P}(X_1 \geq a) \leq 2\exp\{\varphi(\lambda) - \lambda a\}.$$

This further implies that for all $\lambda > 0$ and $0 < \epsilon < 1$,

$$\begin{aligned}
\mathbf{P}\left(Z_1 < \epsilon^2\right) &\leq \mathbf{P}\left(\int_0^1 \exp\{-\sup_{0\leq t\leq 1} X_t\} ds < \epsilon^2\right) \\
&= \mathbf{P}\left(\sup_{0\leq t\leq 1} X_t > 2\log(\epsilon^{-1})\right) \\
&\leq 2e^{\varphi(\lambda)}\epsilon^{\lambda}.
\end{aligned}$$

We note that $\mathbf{P}\left(Z_1^{-k} > \epsilon^{-2k}\right) = \mathbf{P}(Z_1 < \epsilon^2)$ and see that for $k \in \mathbb{N}$,

$$\begin{aligned}
\mathbf{E}\left(Z_1^{-k}\right) &= \int_0^\infty \mathbf{P}\left(Z_1^{-k} > x\right) dx = 2k\int_0^\infty y^{-2k-1}\mathbf{P}\left(Z_1^{-k} > y^{-2k}\right) dy \\
&\leq 2k\int_0^1 y^{-2k-1} 2e^{\varphi(\lambda)} y^{\lambda} dy + 2k\int_1^\infty y^{-2k-1} dy,
\end{aligned}$$

and as we can choose $\lambda = 2k+1$, we get

$$\int_0^1 y^{-2k-1} 2e^{\varphi(\lambda)} y^{\lambda} dy = 2\int_0^1 e^{\varphi(2k+1)} dy < \infty,$$

while clearly also

$$\int_1^\infty y^{-2k-1}dy < \infty$$

holds. As such,

$$\mathbf{E}\left(Z_1^{-k}\right) < \infty, \quad \forall k \in \mathbb{N}.$$

If $0 < t < 1$, an analogous argument based on the fact that

$$\mathbf{P}\left(\sup_{0\le s\le t} X_s \ge 2a\right) \le \mathbf{P}\left(\sup_{0\le s\le 1} X_s \ge 2a\right) \le \exp\{\varphi(\lambda) - \lambda a\},$$

leads to also $\mathbf{E}\left(Z_t^{-k}\right)$ being finite. On the other hand, for $t > 1$, $Z_t > Z_1$ implies finiteness. Thus, for all $t > 0$,

$$\mathbf{E}\left(Z_t^{-k}\right) < \infty, \quad \forall k \in \mathbb{N}.$$

Note that this includes the case when $t = +\infty$, so the first part of the proposition has been proven.

Now, we use the definition

$$Z_{s,\infty} = \int_s^\infty e^{-X_u}du.$$

As a function of s, $Z_{s,\infty}$ is monotonous and continuous, and so for $r > 0$ there is the identity (cf. the proof of Theorem 4.12)

$$Z_{t,\infty}^{-r} - Z_{0,\infty}^{-r} = r\int_0^t e^{-X_s} Z_{s,\infty}^{-(r+1)}ds. \tag{4.28}$$

We also note that the properties of the Lévy process implies that

$$Z_{s,\infty} = e^{-X_s} Z'_\infty,$$

where $Z'_\infty \overset{d}{=} Z_\infty$ is independent of X_s. We insert this into (4.28) and take the expected value, yielding

$$\mathbf{E}\left(Z_\infty^{-r}\right)\left(e^{t\varphi(r)} - 1\right) = r\int_0^t e^{s\varphi(r)}\mathbf{E}\left(Z_\infty^{-(r+1)}\right)ds.$$

From this the recursive relation

$$\mathbf{E}\left(Z_\infty^{-(r+1)}\right) = \frac{\varphi(r)}{r}\mathbf{E}\left(Z_\infty^{-r}\right) \tag{4.29}$$

is derived. Keep in mind that for any $r \in [0,1)$, $Z_\infty^{-r} < Z_\infty^{-1}$, and so the dominated convergence theorem can be used since it has already been established that $\mathbf{E}\left(Z_\infty^{-1}\right) < \infty$. As such, letting $r \to 0+$,

$$\mathbf{E}\left(Z_\infty^{-1}\right) = \lim_{r\to 0+} \frac{\varphi(r)}{r} = \varphi'(0+) = m.$$

Formula (4.27) is then acquired through induction.

Finally, if X lacks positive jumps, the measure Π is such that

$$\int_{\mathbb{R}} \left(1 - e^{\theta x}\right) \Pi(dx) = \int_{(-\infty,0)} \left(1 - e^{\theta x}\right) \Pi(dx),$$

which implies that $\varphi(\lambda) \leq a\lambda + \frac{1}{2}\sigma^2\lambda^2$, for some $a, \sigma \in \mathbb{R}$. In other words, $\varphi(k) = \mathcal{O}(k^2)$ as $k \to \infty$. Then there exists a constant $c > 0$ such that for any $k \in \mathbb{N}$,

$$\mathbf{E}\left(Z_\infty^{-k}\right) = m\frac{\varphi(1)\cdots\varphi(k-1)}{(k-1)!} \leq c^k k!$$

Using this bound, take the series s from Theorem 2.2 and observe that

$$s\left(\frac{1}{2c}\right) = \sum_{k=0}^{\infty} \frac{\mathbf{E}\left(Z_\infty^{-k}\right)(1/2c)^k}{k!} \leq \sum_{k=0}^{\infty} \frac{1}{2^k} < \infty.$$

As such, by Theorem 2.2 the distribution of Z_∞^{-1} is determined by its integer moments. Consequently, Z_∞ is determined by its negative integer moments. □

4.3 The distribution of a perpetuity

This section contains a large number of examples of distributions of the random variable Z_∞. The high point is a short proof for Theorem 4.20 – the distribution discovered by Dufresne [12] – while most subsequent examples are taken from an article by Gjessing and Paulsen [15]. We complement their work by adding some examples of our own and by providing alternative derivations for several of their distributions.

The goal is to express the random value Z_∞ either such that an explicit density is known, or such that the value Z_∞ can easily be simulated by taking samples from other distributions. When these goals are achievable the value of the perpetuity can easily be calculated.

4.3.1 Dufresne's perpetuity

A search for explicit distributions of discounted perpetuities essentially started with Dufresne's 1990 paper [12], and several examples have been found since. The methods used for discovering new distributions vary; Dufresne originally used an argument involving weak convergence from a discrete setting to continuous time, others have used e.g. martingale theory or integro-differential equations. This section is dedicated to presenting a large number of examples.

First, Dufresne's main result is presented with the argument for part (b) due to Bertoin and Yor [2].

Theorem 4.20 (Dufresne's perpetuity). *Let Z be defined by $Z_t = \int_0^t e^{-\tilde{X}_s} ds$, where $\widetilde{X}_t = \gamma t + \sigma X_t$, with $\sigma \neq 0$ and X a standard Brownian motion. Then:*

(a) If $\gamma \leq 0$ then $Z_\infty = \infty$ a.s.

(b) If $\gamma > 0$ then $Z_\infty^{-1} \sim \Gamma(2\gamma/\sigma^2, 2/\sigma^2)$.

Proof. (a) Follows from the fact that $\widetilde{X}_t$ does not drift to $+\infty$ and Proposition 4.6.

(b) Check that

$$\mathbf{E}\left(e^{\lambda \tilde{X}_t}\right) = e^{\lambda\gamma t}\mathbf{E}\left(e^{\lambda\sigma X_t}\right) = e^{\lambda\gamma t + t\frac{\sigma^2\lambda^2}{2}} = e^{t\left(\gamma\lambda + \frac{\sigma^2\lambda^2}{2}\right)} < \infty \ \forall t, \lambda \in \mathbb{R}.$$

We define $\varphi(\lambda) = \gamma\lambda + \frac{\sigma^2\lambda^2}{2}$ and use Proposition 4.19 to compute negative integer moments for Z_∞. The negative integer moments are given by

$$\mathbf{E}\left(Z_\infty^{-k}\right) = \mathbf{E}\left(\tilde{X}_1\right)\frac{\varphi(1)\cdots\varphi(k-1)}{(k-1)!} = \gamma\left(\frac{\sigma^2}{2}\right)^{k-1}\prod_{j=1}^{k-1}\left(\frac{2\gamma}{\sigma^2} + j\right)$$

$$= \left(\frac{\sigma^2}{2}\right)^{k}\prod_{j=0}^{k-1}\left(\frac{2\gamma}{\sigma^2} + j\right) = \left(\frac{\sigma^2}{2}\right)^{k}\frac{\Gamma\left(k + \frac{2\gamma}{\sigma^2}\right)}{\Gamma\left(\frac{2\gamma}{\sigma^2}\right)}.$$

This is recognized as the k:th moment of a $\Gamma(2\gamma/\sigma^2, 2/\sigma^2)$-distribution. Since the negative moments determine the distribution according to Proposition 4.19 the assertion has been proven. □

Alternative proofs of the above theorem have been found by showing that the expression is equivalent to the random last exit time of a Bessel process by M. Yor in 1992 [34], and by means of martingale theory by Milevsky in 1997 [24]. There are also several other proofs which we do not refer to.

4.3.2 Other perpetuities without jumps

Dufresne's result excludes some variants of his perpetuity, such as one with Brownian payments instead of a constant continuous cash flow. In this section other possible settings that involve Brownian interest or payments, but no jump processes. Unless otherwise specified, X and Y are assumed to be standard Brownian motions.

An especially simple method of finding the distribution of a perpetuity is by deriving it from another a priori known perpetuity. The following lemma provides one such method.

Lemma 4.21 (Gjessing and Paulsen [15])**.** *Let* $Z_\infty = \int_0^\infty e^{-R_t} dW_t$*, with* W *a standard Brownian motion and let* R *be a Lévy process with* $\lim_{t\to\infty} R_t = +\infty$*, independent of* W*, and further let* $A = \int_0^\infty e^{-2R_t} dt$*. Then*

$$Z_\infty \overset{d}{=} U\sqrt{A},$$

where $U \sim N(0,1)$*,* $U \perp A$*.*

Proof. Conditioning on R and using the Itô isometry, we get

$$\mathbf{E}\left(Z_\infty^2 | \sigma(R)\right) = \int_0^\infty e^{-2R_t} dt = A.$$

By the assumptions on R we have that A is finite a.s. As such, $Z_\infty \sim N(0, A)$, and so $Z_\infty/\sqrt{A} \sim N(0,1)$. The random variable $U = Z_\infty/\sqrt{A}$ is therefore independent of R and so $U \perp A$. □

Proposition 4.22 (Gjessing and Paulsen [15])**.** *Let* $\widetilde{X}_t = \gamma t + \sigma X_t$*, with* $\gamma > 0$ *and* $\sigma \neq 0$*. Then*

$$\int_0^\infty e^{-\tilde{X}_t} dY_t \overset{d}{=} \frac{1}{\sqrt{2\gamma}} T,$$

where $T \sim T(2\gamma/\sigma^2)$*.*

Proof. Consider that Dufresne's perpetuity (Theorem 4.20) has the distribution

$$\int_0^\infty e^{-2\tilde{X}_t} dt \overset{d}{=} \frac{1}{G}, \quad G \sim \Gamma\left(\frac{\gamma}{\sigma^2}, \frac{1}{2\sigma^2}\right).$$

By using Lemma 4.21,

$$\int_0^\infty e^{-\tilde{X}_t} dY_t \stackrel{d}{=} \frac{U}{\sqrt{G}},\ U \sim N(0,1), U \perp G.$$

Recall that

$$\frac{1}{\sigma^2} G \sim \Gamma\left(\frac{\gamma}{\sigma^2}, \frac{1}{2}\right) = \chi^2\left(\frac{2\gamma}{\sigma^2}\right),$$

and that

$$U\sqrt{\frac{\nu}{V}} \sim T(\nu),$$

if $V \sim \chi^2(\nu)$. Then,

$$\frac{U}{\sqrt{G}} = \frac{U/\sqrt{\sigma^2}}{\sqrt{G/\sigma^2}} \stackrel{d}{=} \frac{U/\sqrt{\sigma^2}}{\sqrt{\chi^2(2\gamma/\sigma^2)}} = \frac{U}{\sqrt{\sigma^2}} \cdot \sqrt{\frac{\sigma^2}{2\gamma}} \cdot \sqrt{\frac{2\gamma/\sigma^2}{\chi^2(2\gamma/\sigma^2)}} \stackrel{d}{=} \frac{1}{\sqrt{2\gamma}} T.$$

□

Proposition 4.23 (Gjessing and Paulsen [15]). *Let $\gamma > 0$ and let $\widetilde{Y}_t = \mu t + \zeta Y_t$. Then,*

$$\int_0^\infty e^{-\gamma t} d\widetilde{Y}_t \sim N\left(\frac{\mu}{\gamma}, \frac{\zeta^2}{2\gamma}\right).$$

Proof. This simple proof is omitted. □

Proposition 4.24 (Gjessing and Paulsen [15]). *Let $\widetilde{X}_t = \gamma t + \sigma X_t, \widetilde{Y}_t = \mu t + \zeta Y_t$, with $\gamma > 0, \sigma \neq 0, \zeta \neq 0$. Then*

$$\int_0^\infty e^{-\tilde{X}_t} d\widetilde{Y}_t$$

is finite a.s. according to Proposition 4.7. It is a Pearson type IV distribution with density given by

$$f(z) = \frac{K}{(\sigma^2 z^2 + \zeta^2)^{1/2+\gamma/\sigma^2}} \exp\left\{\frac{2\gamma}{\sigma\zeta} \arctan\left(\frac{\sigma}{\zeta} z\right)\right\},$$

where K is a normalizing constant.

Proof. Proved in [27] for $\gamma > \sigma^2$ and in [26] when $\gamma > 0$. □

We have already exhausted every possibility without jump processes in $\widetilde{X}$ or $\widetilde{Y}$. Let us therefore turn to the case with jump processes.

4.3.3 Application of discrete perpetuities

Below, N_X, N_Y shall denote independent Poisson processes, both also independent of X and Y, with respective intensity parameters λ_X and λ_Y. Unless otherwise specified, the jump sizes shall be i.i.d. copies of $S_X \sim Exp(\alpha)$ and $S_Y \sim Exp(\beta)$.

First, original proofs for a few simple cases omitted from [15] shall be presented. The idea is to identify the continuous-time perpetuity with the Vervaat-class perpetuities of the discrete-time section.

Proposition 4.25. *Let* $\gamma > 0$ *and* $\widetilde{Y}_t = \sum_{i=1}^{N_Y(t)} S_Y(i)$. *Then,*

$$\int_0^\infty e^{-\gamma t} d\widetilde{Y}_t \sim \Gamma\left(\frac{\lambda_Y}{\gamma}, \beta\right).$$

Proof. We have

$$\int_0^\infty e^{-\gamma t} d\widetilde{Y}_t = \sum_{k=1}^{\infty} S_Y(k) \prod_{j=1}^{k} e^{-\gamma \tau_j},$$

where $(\tau_j)_{j\in\mathbb{N}}$ is the sequence of i.i.d. sojourn times, each $\tau_j \sim Exp(\lambda_Y)$. We note that $\gamma\tau_j \sim Exp(\lambda_Y/\gamma)$. The proposition is now identified with Example 3.12. □

Proposition 4.26. *Let* $\gamma > 0$ *and* $\widetilde{Y}_t = \sum_{i=1}^{N_Y(t)} S_Y(i)$, *with jump sizes* $S_Y \sim$ $Laplace(0, \beta)$. *Then,*

$$\int_0^\infty e^{-\gamma t} d\widetilde{Y}_t \sim VG\left(0, \beta, 0, \frac{\lambda_Y}{2\gamma}\right).$$

Proof. Analogous to the above case, relying instead on Example 3.13. □

Recall that the variance-gamma (VG) distribution was defined in Chapter 3 as a part of Example 3.13.

Proposition 4.27. *Let* $\gamma > 0$ *and* $\widetilde{Y}_t = \mu t + \zeta Y_t \pm \sum_{i=1}^{N_Y(t)} S_Y(i)$. *Then,*

$$\int_0^\infty e^{-\gamma t} d\widetilde{Y}_t \overset{d}{=} N \pm G,$$

where $N \sim N(\mu/\gamma, \zeta^2/2\gamma)$ *and* $G \sim \Gamma(\lambda_Y/\gamma, \beta)$ *are independent.*

Proof. Directly from Propositions 4.23 and 4.25. □

Moreover, if $S_Y \sim Laplace(0, \beta)$ in the above, then from Proposition 4.26 we get that

$$\int_0^\infty e^{-\gamma t} d\widetilde{Y}_t \stackrel{d}{=} N + V,$$

where $V \sim VG\left(0, \beta, 0, \frac{\lambda_Y}{2\gamma}\right)$ is independent of N.

The next distribution is one due to Gjessing and Paulsen which we shall derive, once again, by identifying it with a discrete perpetuity. The method used by Gjessing and Paulsen is different.

Proposition 4.28 (Gjessing and Paulsen [15]). *Let* $\widetilde{X}_t = \sum_{i=1}^{N_X(t)} S_X(i)$, *with* $\widetilde{X}_t = 0$ *if* $N_X(t) = 0$. *Then,*

$$\int_0^\infty e^{-\tilde{X}_t} dt \sim \Gamma(1 + \alpha, \lambda_X).$$

Proof. Observe that

$$\int_0^\infty e^{-\tilde{X}_t} dt = \int_0^{T_1} dt + \sum_{k=1}^\infty \int_{T_k}^{T_{k+1}} \exp\left\{-\sum_{j=1}^k S_X(j)\right\} dt,$$

where $T_k, k = 1, 2, \ldots$ is the sequence of jump times for N_X. Then, if we denote by τ_k the k:th sojourn time of N_X, it follows that

$$\int_0^\infty e^{-\tilde{X}_t} dt = \tau_0 + \sum_{k=1}^\infty \tau_k \prod_{j=1}^k e^{-S_X(j)},$$

in which the sum can be identified with the perpetuity in Example 3.12, since the sojourn times $(\tau_k)_{k=0}^\infty$ is an i.i.d. sequence of $Exp(\lambda_X)$-distributed variables. Thus,

$$\sum_{k=1}^\infty \tau_k \prod_{j=1}^k e^{-S_X(j)} \sim \Gamma(\alpha, \lambda_X).$$

Recalling that $\tau_0 \sim Exp(\lambda_X) = \Gamma(1, \lambda_X)$, we get the conclusion

$$\int_0^\infty e^{-\tilde{X}_t} dt \sim \Gamma(1 + \alpha, \lambda_X).$$

□

The distribution given in Proposition 4.30 relies on a preliminary result. It is due to Gjessing and Paulsen, but the presented method of proof is once again different from theirs.

Lemma 4.29. *Let $\tau \sim Exp(\lambda)$, $\mu, \sigma > 0$ and W be a standard Brownian motion. Assume $\tau \perp W$. Further, let*

$$l_1 = \frac{k-\mu}{\sigma^2}, \quad l_2 = \frac{k+\mu}{\sigma^2},$$

and $k = \sqrt{\mu^2 + 2\lambda\sigma^2}$. Then,

$$\mu\tau + \sigma W_\tau \stackrel{d}{=} XF - (1-X)U,$$

where $F \sim Exp(l_1)$, $U \sim Exp(l_2)$ and $X \sim Ber(l_2/(l_1+l_2))$ and X, F, U are independent.

Proof. Let $C = XF - (1-X)U$. It is easy to compute the moment-generating function of C,

$$M_C(s) = \frac{l_1 l_2}{l_1 l_2 + (l_1 - l_2)s - s^2} = \frac{k^2 - \mu^2}{k^2 - \mu^2 - 2\mu s\sigma^2 - s^2\sigma^4}$$

$$= \frac{\mu^2 - +2\lambda\sigma^2 - \mu^2}{\mu^2 + 2\lambda\sigma^2 - \mu^2 - 2\mu s\sigma^2 - s^2\sigma^4} = \frac{\lambda}{\lambda - \mu s - \frac{\sigma^2 s^2}{2}}, \quad s < \min(l_1, l_2).$$

We carry out the proof by computing the moment-generating function of $\mu\tau + \sigma W_\tau$ as well.

$$\mathbf{E}\left(e^{s(\mu\tau + \sigma W_\tau)}\right) = \int_0^\infty \lambda e^{-\lambda t} \mathbf{E}\left(e^{s(\mu t + \sigma W_t)}\right) dt$$

$$= \int_0^\infty \lambda e^{\left(s\mu + \frac{\sigma^2 s^2}{2} - \lambda\right)t} dt = \frac{\lambda}{\lambda - \mu s - \frac{\sigma^2 s^2}{2}}, \quad s < \lambda/\mu.$$

The two moment-generating functions are equal in a neighbourhood of zero, and so the variables are equal in distribution. □

Proposition 4.30 (Gjessing and Paulsen [15]). *Let $\widetilde{X}_t = \sum_{i=1}^{N_X(t)} S_X(i)$ and $\widetilde{Y}_t = \mu t + \zeta Y_t$, with $\zeta \neq 0$. Then,*

$$Z_\infty = \int_0^\infty e^{-\tilde{X}_t} d\widetilde{Y}_t \stackrel{d}{=} G_1 - G_2,$$

where $G_1 \sim \Gamma(a, l_1)$ and $G_2 \sim \Gamma(b, l_2)$ are independent and

$$a = 1 + \frac{\alpha}{2} + \frac{\alpha\mu}{2k}, \quad b = 1 + \frac{\alpha}{2} - \frac{\alpha\mu}{2k},$$

$$l_1 = \frac{k-\mu}{\zeta^2}, \quad l_2 = \frac{k+\mu}{\zeta^2},$$

and $k = \sqrt{\mu^2 + 2\gamma_X \zeta^2}$.

Proof. Observe that

$$Z_\infty = \int_0^\infty e^{-\tilde{X}_t} d\widetilde{Y}_t = \sum_{k=0}^{\infty} (\mu\tau_k + \zeta Y_{\tau_k}) \prod_{j=1}^{k} e^{-S_X(j)},$$

where the sequence $(\tau_j)_{j=0}^{\infty}$ are the i.i.d. $Exp(\lambda_X)$-distributed sojourn times of N_X. We write $V_j = e^{-S_X(j)}$ and $C_j = \mu\tau_j + \zeta Y_{\tau_j}$, for $j = 0, 1, 2, \ldots$. Then note that Z_∞ must satisfy

$$Z_\infty = V Z_\infty + C, \quad \text{where } V, C, Z_\infty \text{ all independent.}$$

The proposition is now clearly a consequence of Lemma 4.29 and Corollary 3.15. □

Remark 4.31 (Gjessing and Paulsen [15])**.** Proposition 4.30 with $\mu = 0$ gives a different distribution than Lemma 4.21 for $\int_0^\infty e^{-\tilde{X}_t} dY_t$. In fact, by applying both one arrives at the stochastic identity

$$G_1 - G_2 \stackrel{d}{=} U\sqrt{G},$$

where $G_1 \perp G_2$, $G_1, G_2 \sim \Gamma(1 + a, b)$, and $G \sim \Gamma(1 + a, b^2/2)$.

4.3.4 Identifying the distribution by its moments

Recall the method of proof used by Bertoin and Yor to prove Theorem 4.20. In this section we prove some further results using similar arguments. The first two examples are due to Gjessing and Paulsen, but their method of proof differs.

Proposition 4.32 (Gjessing and Paulsen [15])**.** *Let* $\widetilde{X}_t = \gamma t + \sum_{i=1}^{N_X(t)} S_X(i)$, *with* $\gamma > 0$. *Then,*

$$Z_\infty = \int_0^\infty e^{-\tilde{X}_t} dt \stackrel{d}{=} \frac{1}{\gamma} B,$$

where

$$B \sim \beta(1 + \alpha, \lambda_X/\gamma).$$

Proof. In this case

$$\mathbf{E}\left(e^{-\theta \tilde{X}_t}\right) = e^{-\theta\gamma t} \mathbf{E}\left(e^{-\theta \sum_{k=1}^{N_X(t)} S_X(k)}\right) = e^{-t\theta\gamma} e^{\lambda_X t\left(\frac{\alpha}{\alpha+\theta} - 1\right)}$$

$$= \exp\left\{-t\left(\gamma\theta + \lambda_X \frac{\theta}{\alpha+\theta}\right)\right\}.$$

As such, the Laplace exponent of X is

$$\phi(\theta) = \gamma\theta\left(1 + \frac{\lambda_X}{\gamma(\alpha+\theta)}\right).$$

It is easy to see that $\phi(n) > 0$ for all $n \in \mathbb{N}$. Moreover, $\phi(\infty) = +\infty$, so by Corollary 4.18 the distribution of the perpetuity is determined by its positive integer moments. That being the case, we use Formula 4.23 and see that

$$\mathbf{E}\left(Z_\infty^n\right) = n! \prod_{k=1}^{n} k^{-1}\gamma^{-1}\left(\frac{\gamma(\alpha+k)+\lambda_X}{\gamma(\alpha+k)}\right)^{-1}$$

$$= \gamma^{-n} \prod_{k=0}^{n-1} \frac{\alpha+1+k}{\alpha+1+\frac{\lambda_X}{\gamma}+k}.$$

The product is identified as the moments of a $\beta(\alpha+1, \lambda_X/\gamma)$-distribution, which settles the proof. □

The following is a simple consequence of the above.

Proposition 4.33 (Gjessing and Paulsen [15]). *Let* $\widetilde{X}_t = \gamma t + \sum_{i=1}^{N_X(t)} S_X(i)$, *with* $\gamma > 0$. *Then,*

$$\int_0^\infty e^{-\tilde{X}_t} dY_t \overset{d}{=} U\sqrt{\frac{B}{2\gamma}},$$

where

$$U \sim N(0,1), \quad B \sim \beta(1+\alpha, \lambda_X/2\gamma).$$

Proof. From Proposition 4.32 with a direct application of Lemma 4.21. □

The next two distributions are new. The idea is to take a similar case to Proposition 4.32, but with negative jumps. Consequently a lower bound for the drift factor is needed for the discounted perpetuity to be finite.

Proposition 4.34. *Let* $\widetilde{X}_t = \gamma t - \sum_{i=1}^{N_X(t)} S_X(i)$, *with* $\gamma > \lambda_X/\alpha$. *Then,*

$$Z_\infty = \int_0^\infty e^{-\tilde{X}_t} dt \overset{d}{=} \frac{1}{\gamma B},$$

where

$$B \sim \beta(\alpha - \lambda_X/\gamma, \lambda_X/\gamma).$$

Proof. The distribution will be identified by its negative integer moments. To this end, we verify that conditions (4.25) and (4.26) hold for $\widetilde{X}$. Observe that for $\theta \geq 0$,

$$\mathbf{E}\left(e^{\theta \tilde{X}_t}\right) = e^{\theta\gamma t}\mathbf{E}\left(e^{-\theta\sum_{j=1}^{N_X(t)} S_X(j)}\right) = e^{\theta\gamma t}e^{\lambda_X t\left(\frac{\alpha}{\alpha+\theta}-1\right)} = e^{\theta\gamma t-\lambda_X t\frac{\theta}{\alpha+\theta}}.$$

Thus, we have $\mathbf{E}\left(e^{\theta \tilde{X}_t}\right) = e^{t\varphi(\theta)}$, where φ is given by

$$\varphi(\theta) = \theta\left(\gamma - \frac{\lambda_X}{\alpha+\theta}\right),$$

and $\mathbf{E}\left(e^{\theta \tilde{X}_t}\right) = e^{t\varphi(\theta)} < \infty$ (conditions (4.25)) does indeed hold. Moreover, we have

$$\varphi'(\theta) = \gamma - \frac{\lambda_X}{\alpha+\theta} + \frac{\lambda_X}{(\alpha+\theta)^2}\theta.$$

As $\theta \downarrow 0$, we get $\varphi(\theta) \to \gamma - \lambda_X/\alpha > 0$. Thus, condition (4.26) holds and

$$m = \varphi'(0+) = \gamma - \lambda_X/\alpha.$$

We have now checked that the conditions in Proposition 4.19 are hold. Thus, Z_∞ has negative moments of all orders and since $\widetilde{X}$ lacks positive jumps, the distribution of Z_∞ is determined by the negative integer moments. By formula (4.27),

$$\mathbf{E}\left(Z_\infty^{-k}\right) = \left(\gamma - \frac{\lambda_X}{\alpha}\right)\frac{\prod_{m=1}^{k-1} m\left(\gamma - \frac{\lambda_X}{\alpha+m}\right)}{(k-1)!}$$
$$= \prod_{m=0}^{k-1}\left(\gamma - \frac{\lambda_X}{\alpha+m}\right) = \gamma^k\prod_{m=0}^{k-1}\frac{\alpha - \frac{\lambda_X}{\gamma} + m}{\alpha+m},$$

which is identified as the k:th moment of the random variable γB, where

$$B \sim \beta(\alpha - \lambda_X/\gamma, \lambda_X/\gamma).$$

It follows that $Z_\infty^{-1} \overset{d}{=} \gamma B$, which proves the assertion. □

Again, the next example is a direct consequence of the prior one.

Proposition 4.35. *Let $\widetilde{X}_t = \gamma t - \sum_{i=1}^{N_X(t)} S_X(i)$, with $\gamma > \lambda_X/\alpha$. Then,*

$$Z_\infty = \int_0^\infty e^{-\tilde{X}_t}dY_t \overset{d}{=} \frac{U}{\sqrt{2\gamma B}},$$

where

$$U \sim N(0,1),\ B \sim \beta\left(\frac{\alpha}{2} - \frac{\lambda_X}{2\gamma}, \frac{\lambda_X}{2\gamma}\right),\ U \perp B.$$

Proof. From Proposition 4.34 and Lemma 4.21. □

We have now provided alternative derivations for a large number of examples, yet the list in [15] is not yet exhausted. Subsequent distributions are not easily found by reliance on prior methods, but we include them in order to give the reader an idea of the scope of the current knowledge in the research literature.

4.3.5 The Gjessing-Paulsen method

In this section, let X, Y be independent Brownian motions as before, and let $\widetilde{X}$ and $\widetilde{Y}$ be given by

$$\widetilde{X}_t = \gamma t + \sigma X_t + \sum_{k=1}^{N_X(t)} S_X(k),$$
$$\widetilde{Y}_t = \mu t + \zeta Y_t - \sum_{k=1}^{N_Y(t)} S_Y(k),$$

where N_X, N_Y are independent Poisson processes, both also independent of X and Y. The sequences $S_X(k), S_Y(k), k = 1, 2, \ldots$, are i.i.d. random jump variables. We shall also make use of the Laplace exponent of $\widetilde{X}$, defined as the function H_L in $\mathbf{E}\left(e^{-\kappa \tilde{X}_t}\right) = e^{-tH_L(\kappa)}$, for $\kappa \geq 0$. We change notation here because we will consistently use its expression given below:

$$H_L(\kappa) = \gamma\kappa - \frac{\sigma^2\kappa^2}{2} + \lambda_X(1 - m_L(\kappa)), \tag{4.30}$$

where $m_L(\kappa) = \mathbf{E}\left(e^{-\kappa S_X}\right)$. We also need the characteristic exponent of $\widetilde{Y}$ at time $t = 1$, that is $\eta_C(u) = \log\left[\mathbf{E}\left(e^{iu\tilde{Y}_1}\right)\right]$. Thus,

$$\eta_C(u) = \mu u i - \frac{\zeta^2 u^2}{2} - \lambda_X(1 - \rho_C(-u)), \tag{4.31}$$

where $\rho_C(u) = \mathbf{E}\left(e^{iuS_Y}\right)$.

Finally, with

$$Z_\infty = \int_0^\infty e^{-\tilde{X}_t} d\widetilde{Y}_t,$$

we will denote the characteristic function of Z_∞ by Ψ_C, that is $\Psi_C(u) = \mathbf{E}\left(e^{iuZ_\infty}\right)$, and its Laplace transform by Ψ_L, so that $\Psi_L(u) = \mathbf{E}\left(e^{-uZ_\infty}\right)$. The method of

Gjessing and Paulsen consists of deriving differential equations satisfied by Ψ_C, or Ψ_L, and finding solutions to them. Their differential equations are given in the following two lemmas.

Lemma 4.36 (Gjessing and Paulsen [15]). *Let Z_∞ and Ψ_C be as above and let H_L and η_C be as in (4.30) and (4.31). Suppose that $H_L(2) > 0$ holds and $\mathbf{E}\left(|S_Y|^2\right) < \infty$. Then,*

(a) Ψ_C is twice continuously differentiable and satisfies the differential equation

$$\frac{1}{2}\sigma^2 u^2 \Psi_C''(u) - \left(\gamma - \frac{1}{2}\sigma^2\right) u\Psi_C'(u)$$

$$+ \eta_C(u)\Psi_C(u) + \lambda_X \int_{-\infty}^{+\infty} (\Psi_C(ue^{-s}) - \Psi_C(u)) dF_{S_X}(s) = 0, \tag{4.32}$$

with boundary conditions

$$\Psi_C(0) = 1, \quad |\Psi_C(u)| \leq 1, \ \forall u \in \mathbb{R}.$$

(b) Suppose further that $\mathbf{E}\left(|S_Y|^3\right) < \infty$ and that $S_X \sim Exp(\alpha)$. Then, if also $H_L(3) > 0$ is confirmed, Ψ_C is three times continuously differentiable and satisfies the differential equation

$$\begin{aligned}\frac{1}{2}\sigma^2 u^2 \Psi_C'''(u) &+ \left(\frac{1}{2}(\alpha+3)\sigma^2 - \gamma\right) u\Psi_C''(u) \\ &+ \left(\eta_C(u) - \left((1+\alpha)\left(\gamma - \frac{1}{2}\sigma^2\right) + \lambda_X\right)\right)\Psi_C'(u) \\ &+ \left(\alpha\frac{\eta_C(u)}{u} - \zeta^2 u - \lambda_Y \rho_C'(-u) + i\mu\right)\Psi_C(u) = 0.\end{aligned} \tag{4.33}$$

Proof. The proof is based on the theory of integro-differential equations, which is why the reader is instead referred to the source. See [15], but note that a partial result is proved in [27]. □

There is a second equation in the case when $\widetilde{Y}$ is non-decreasing in t, i.e. it is given by

$$\widetilde{Y}_t = \mu t + \sum_{k=1}^{N_Y(t)} S_Y(k), \tag{4.34}$$

where $S_Y \geq 0$ almost surely. In this case the equation also holds given slightly weaker conditions on S_Y.

In this case, we use the Laplace transform of Z_∞, defined as $\Psi_L(u) = \mathbf{E}\left(e^{-uZ_\infty}\right)$, for $u \geq 0$. We also use the Laplace exponent of $\widetilde{Y}$ at time 1, i.e. $\eta_L(u) = -\log \mathbf{E}\left(e^{-u\tilde{Y}_1}\right)$. Using expression (4.34), this gives

$$\eta_L(u) = \mu u + \lambda_Y(1 - \rho_L(u)), \tag{4.35}$$

where $\rho_L(u) = \mathbf{E}\left(e^{-uS_Y}\right)$. We give the Gjessing-Paulsen differential equation for Ψ_L in the lemma below.

Lemma 4.37 (Gjessing and Paulsen [15])**.** *Let $\widetilde{Y}$ be as in* (4.34)*. Suppose the conditions*

$$\mathbf{E}\left(\widetilde{X}_1\right) > 0, \quad \mathbf{E}\left(|S_Y|\right) < \infty, \quad \mathbf{E}\left(S_X^4\right) < \infty$$

hold. Then,

(a) Ψ_L is twice continuously differentiable on $(0, \infty)$ and satisfies the differential equation

$$\begin{aligned} &\frac{1}{2}\sigma^2 u^2 \Psi_L''(u) - \left(\gamma - \frac{1}{2}\sigma^2\right) u \Psi_L'(u) \\ &- \eta_L(u)\Psi_L(u) + \lambda_X \int_{-\infty}^{+\infty} (\Psi_L(ue^{-s}) - \Psi_L(u)) dF_{S_X}(s) = 0, \end{aligned} \tag{4.36}$$

with boundary conditions

$$\Psi_L(0) = 1, \quad \lim_{u\to\infty} \Psi_L(u) = 0.$$

(b) If also $S_X \sim Exp(\alpha)$, then Ψ_L is three times continuously differentiable and satisfies the differential equation

$$\begin{aligned} &\frac{1}{2}\sigma^2 u^2 \Psi_L'''(u) + \left(\frac{1}{2}(\alpha+3)\sigma^2 - \gamma\right) u \Psi_L''(u) \\ &- \left(\eta_L(u) + \left((1+\alpha)\left(\gamma - \frac{1}{2}\sigma^2\right) + \lambda_X\right)\right) \Psi_L'(u) \\ &- \left(\alpha \frac{\eta_L(u)}{u} - \lambda_Y \rho_L'(u) + \mu\right) \Psi_L(u) = 0. \end{aligned} \tag{4.37}$$

Proof. Like Lemma 4.36, see [15]. □

Finding valid solutions to the differential equations of Lemmas 4.36 and 4.37, with various assumptions on the parameters, directly leads to a characteristic function (or Laplace transform) of Z_∞. The following example is included in [15] but only proved in [25].

Proposition 4.38 (Nilsen and Paulsen [25]). *Let* $\widetilde{X}_t = \gamma t + \sigma X_t$*, and* $\widetilde{Y}_t = \sum_{i=1}^{N_Y(t)} S_Y(i)$*. Then*

$$\int_0^\infty e^{-\tilde{X}_t} d\widetilde{Y}_t \overset{d}{=} \frac{G}{B},$$

where $G \sim \Gamma(b, \beta)$ *and* $B \sim \beta(a, 1+b)$ *are independent and*

$$a = \frac{2\gamma}{\sigma^2}, \quad b = \frac{1}{\sigma^2}\left(\sqrt{\gamma^2 + 2\lambda_Y \sigma^2} - \gamma\right).$$

Proof. In this case Equation (4.36) is applicable. Since $1 - \rho_L(u) = \frac{u}{\beta+u}$, we have $\eta_L(u) = \frac{\lambda_Y u}{\beta+u}$ and the equation has the form

$$\frac{1}{2}\sigma^2 u \Psi_L''(u) - (\gamma - \frac{1}{2}\sigma^2)\Psi_L'(u) - \frac{\lambda_Y u}{\beta + u}\Psi(u) = 0.$$

Some manipulation and a variable transform of $v = -u/\beta$, with $f(v) = \Psi_L(u)$, gives the equation the form

$$v(1-v)f''(v) + \left(1 - \frac{2\gamma}{\sigma^2} - v\left(1 - \frac{2\gamma}{\sigma^2}\right)\right) f'(v) + \frac{2\lambda_Y \beta}{\sigma^2} f(v) = 0,$$

with boundary conditions of $f(0) = 1$ and $\lim_{v \to -\infty} f(v) = 0$. This is a *hypergeometric differential equation*, for which a general solution is known. The derivations are somewhat technical, but one arrives at the solution (transformed back into terms of u, Ψ_L)

$$\Psi_L(u) = K \int_0^1 y^{a+b-1}(1-y)^b(\beta y + u)^{-b} dy,$$

where K is an arbitrary constant. See [25] for the details.

Furthermore, $(\beta y + u)^{-b}$ is the Laplace transform of a $\Gamma(\beta y, b)$-distributed random variable, which is why

$$(\beta y + u)^{-b} = \int_0^\infty \frac{1}{\Gamma(b)} z^{b-1} e^{-\beta y z} e^{-uz} dz.$$

By applying this and changing the order of integration, one gets

$$\Psi_L(u) = \int_0^\infty K \left[\int_0^1 y^{a+b-1}(1-y)^b e^{-\beta y z} dy\right] z^{b-1} e^{-uz} dz,$$

in which the density f_{Z_∞} can be identified because of the uniqueness of the Laplace transform. We get

$$f_{Z_\infty}(z) = K z^{b-1} \int_0^1 y^{a+b-1}(1-y)^b e^{-\beta y z} dy.$$

To see that this is the desired density, calculate the density f_Z, where $Z = G/B$. By the law of total probability,

$$f_Z(z) = \int_0^1 y f_G(yz) f_B(y) dy = \frac{\Gamma(a+b+1)}{\Gamma(a)\Gamma(b+1)} \frac{\beta^b}{\Gamma(b)} z^{b-1} \int_0^1 y^{a+b-1}(1-y)^b e^{-\beta z y} dy.$$

This is indeed the correct form, and allows identification of the constant K. □

The prior proof involved some technical details that were omitted for the sake of brevity. Finding the next distribution similarly involves solving a hypergeometric differential equation, and similarly we omit the details.

Proposition 4.39 (Gjessing and Paulsen [15]). *Let* $\widetilde{X}_t = \gamma t + \sum_{i=1}^{N_X(t)} S_X(i)$, *with* $-\lambda_X/\alpha < \gamma < 0$. *Then*

$$\int_0^\infty e^{-\tilde{X}_t} dt \overset{d}{=} -\frac{1}{\gamma} B_2,$$

where

$$B_2 \sim \beta_2 \left(1+\alpha, -\frac{\alpha}{\gamma}\left(\gamma + \frac{\lambda_X}{\alpha}\right)\right).$$

Proof. In this case $\eta_L(u) = u$ and $\sigma = 0$, so (4.37) becomes

$$\gamma u \Psi_L''(u) + (u + (1+\alpha)\gamma + \lambda_X)\Psi_L'(u) + (1+\alpha)\Psi_L(u) = 0.$$

With slight manipulation and a variable transform of $v = -u/\gamma$ and $g(v) = \Psi_L(u)$, this changes into the *confluent hypergeometric differential equation*

$$vg''(v) + (1 + \alpha + \frac{\lambda_X}{\gamma} - v)g'(v) - (1+\alpha)g(v) = 0.$$

This differential equation has a solution satisfying the boundary conditions (yet see [15] for further details)

$$g(v) = K \int_0^\infty y^\alpha (1+y)^{\frac{\lambda_X}{\gamma}-1} e^{-vy} dy,$$

where K is a constant. Now it is a simple matter to transform back, yielding

$$\Psi_L(u) = \int_0^\infty K y^\alpha (1+y)^{\frac{\lambda_X}{\gamma}-1} e^{\frac{u}{\gamma}y} dy = \mathbf{E}\left(e^{-u(-\frac{1}{\gamma}B_2)}\right),$$

if

$$K = \frac{\Gamma(1-\lambda_X/\gamma)}{\Gamma(1+\alpha)\Gamma(-\frac{\alpha}{\gamma}(\gamma + \lambda_X/\alpha))}.$$

The uniqueness of the Laplace transform guarantees the result. □

Proposition 4.40 (Gjessing and Paulsen [15]). *Let $\widetilde{X}_t = \gamma t + \sum_{i=1}^{N_X(t)} S_X(i)$, with $-\lambda_X/\alpha < \gamma < 0$. Then,*

$$\int_0^\infty e^{-\widetilde{X}_t} dY_t \stackrel{d}{=} U\sqrt{-\frac{1}{2\gamma}B_2},$$

where

$$U \sim N(0,1) \quad \text{and} \quad B_2 \sim \beta_2\left(1+\frac{\alpha}{2}, \frac{\lambda_X}{2\gamma}\right)$$

are independent.

Proof. Application of Lemma 4.21 to Proposition 4.39. □

The final two are mixture distributions between Γ-variables.

Proposition 4.41 (Gjessing and Paulsen [15]). *Let $\widetilde{X}_t = \sum_{i=1}^{N_X(t)} S_X(i)$ and $\widetilde{Y}_t = \sum_{j=1}^{N_Y(t)} S_Y(t)$. Then,*

$$\int_0^\infty e^{-\tilde{X}_t} d\widetilde{Y}_t \sim (1-k)\Gamma((1+k)\alpha+1, k\beta) + k\Gamma((1-k)\alpha, k\beta),$$

i.e. a mixture distribution between two gamma-variables, with proportion $k = \frac{\lambda_X}{\lambda_X+\lambda_Y}$.

Proof. We have $\gamma = \mu = \sigma = \zeta = 0$ and $\eta_L(u) = \lambda_Y u/(\beta+u)$. With this, all of the higher order terms are zero and (4.37) becomes

$$\left(\frac{\lambda_Y u}{\beta+u} + \lambda_X\right)\Psi_L'(u) + \left(\frac{\lambda_Y \alpha}{\beta+u} + \frac{\lambda_Y \beta}{(\beta+u)^2}\right)\Psi_L(u) = 0.$$

Rearranging,

$$\frac{\Psi_L'(u)}{\Psi_L(u)} = -\frac{\lambda_Y\left(\alpha+\frac{\beta}{\beta+u}\right)}{(\lambda_Y+\lambda_X)\left(u+\frac{\lambda_X}{\lambda_X+\lambda_Y}\beta\right)} = -(1-k)\frac{\alpha\beta+\beta+\alpha u}{(\beta+u)(k\beta+u)}.$$

Integrating, one arrives at

$$\log\Psi_L(u) = -(1-k)\int_0^u \frac{\alpha\beta+\beta+\alpha s}{(\beta+s)(k\beta+s)} ds.$$

By means of partial fraction decomposition, this gets simplified into

$$\log\Psi_L(u) = -(1-k)\int_0^u \left[\frac{\alpha\beta+\beta+\alpha s}{-(1-k)\beta(\beta+s)} + \frac{\alpha\beta+\beta+\alpha s}{(1-k)\beta(k\beta+s)}\right] ds$$

$$= \log\left(\frac{\beta+u}{\beta}\right) + (1+(1-k)\alpha)\log\left(\frac{k\beta}{k\beta+u}\right).$$

Thus,

$$\Psi_L(u) = \left(\frac{\beta+u}{\beta}\right)\left(\frac{k\beta}{k\beta+u}\right)^{1+\alpha(1-k)}$$

$$= \left(\frac{k\beta}{k\beta+u}\right)^{1+(1-k)\alpha} + \frac{u}{\beta}\left(\frac{k\beta}{k\beta+u}\right)\left(\frac{k\beta}{k\beta+u}\right)^{(1-k)\alpha},$$

and since $\frac{u}{k\beta+u} = 1 - \frac{k\beta}{k\beta+u}$, we get

$$\Psi_L(u) = (1-k)\left(\frac{k\beta}{k\beta+u}\right)^{1+(1-k)\alpha} + k\left(\frac{k\beta}{k\beta+u}\right)^{(1-k)\alpha}.$$

Noting that a $\Gamma((1+k)\alpha+1, k\beta)$-distributed variable has the Laplace transform

$$\left(\frac{k\beta}{k\beta+u}\right)^{1+(1-k)\alpha},\ u \geq 0$$

completes the proof. □

Proposition 4.42 (Gjessing and Paulsen [15]). *Let* $\widetilde{X}_t = \sum_{i=1}^{N_X(t)} S_X(i)$ *and* $\widetilde{Y}_t = \mu t - \sum_{j=1}^{N_Y(t)} S_Y(t)$, *with* $\mu \neq 0$. *Then,*

$$\int_0^\infty e^{-\tilde{X}_t} d\widetilde{Y}_t \overset{d}{=} G_1 - G_2,$$

where $G_1 \sim \Gamma(a, l_1)$ *and* G_2 *has the mixture distribution*

$$G_2 \sim (1-k)\Gamma(b, l_2) + k\Gamma(b-1, l_2),$$

with

$$a = \frac{1}{2}\alpha(1+c)+1,\ b = \frac{1}{2}\alpha(1-c)+1,\ l_1 = \frac{1}{2\mu}(R+\lambda_X+\lambda_Y-\mu\beta),$$

$$l_2 = \frac{1}{2\mu}(R-\lambda_X-\lambda_Y+\mu\alpha_Y),\ k = l_2/\beta,\ c = (\lambda_X+\mu\beta-\lambda_Y)/R$$

and

$$R = \sqrt{(\lambda_X+\lambda_Y-\mu\beta)^2 + 4\mu\lambda_X\beta}.$$

Proof. With $\gamma = \sigma = \zeta = 0$ and $\eta_C(u) = i\mu u - \frac{i\lambda_Y u}{\beta+iu}$, (4.33) takes the form

$$\left(i\mu u - \frac{i\lambda_Y u}{\beta+iu} - \lambda_X\right)\Psi_C'(u) + i\left((1+\alpha)\mu - \frac{\alpha\lambda_Y}{\beta+iu} - \frac{\beta\lambda_Y}{(\beta+iu)^2}\right)\Psi_C(u) = 0.$$

Similarly to the proof of Proposition 4.41, a solution involving partial fraction decomposition can be found. We omit the details, but one gets

$$\Psi_C(u) = \left(\frac{l_1}{l_1 - iu}\right)^a \left(\frac{l_2}{l_2 + iu}\right)^b \frac{\mu\beta + iu}{\mu\beta},$$

which similarly to the proof of Proposition 4.41 can be rewritten as

$$\Psi_C(u) = \left(\frac{l_1}{l_1 - iu}\right)^a \left[(1-k)\left(\frac{l_2}{l_2 + iu}\right)^b + k\left(\frac{l_2}{l_2 + iu}\right)^{b-1}\right].$$

This is easily identified as the desired characteristic function of $G_1 - G_2$. □

Chapter 5

Applications

In this chapter we briefly overview some applications of the concepts contained within this thesis. We do not discuss applications that have already been treated in the prior chapters (e.g. applications to stochastic equations or the application of discrete perpetuities to finding the distribution of a continuous-time discounted perpetuity); instead, this final chapter is included for the sake of discussion and review. Most proofs are omitted and we shall often refer the reader to other papers for arguments and examples.

5.1 Stock valuation

In this section I propose a new *dividend discount model* (DDM) for the price of a dividend-paying stock. The model, which can essentially be described as a stochastic DDM with a known probability distribution for the price of the stock, is based on Example 3.16 and the work on DDMs [18] due to Gordon and Shapiro. We shall also see that the model can be used to statistically estimate the cost of equity of a company.

Whereas DDM approaches to stock valuation are less popular than the *capital asset pricing model* (CAPM), the discovery of a stochastic DDM with a known density function for the stock price is interesting from an academic point of view. Nearly all of the drawbacks of the DDM still apply to this stochastic DDM; the only improvement is allowing the parameters of the model to be stochastic.

5.1.1 Introduction to dividend discount models

This section explains the preliminary financial concepts necessary to understand the proposed DDM. As such, this exposition relies heavily on the papers of Gordon and Shapiro [18, 17].

Generally, when a firm requires capital it can do two things: 1) take on debt in form of bonds or loans, or 2) sell shares of its equity. The *cost of capital* is rate of return required for the investors or lenders in order to provide capital to the firm (cf. risk-free interest rates). This return is provided in the form of interest in the case of lending, while investors expect dividends or a growth in value of the assets of the firm. The cost of capital when issuing shares is called the *cost of equity capital.* While the cost depends on the market, it cannot be directly observed and instead has to be estimated e.g. by using the CAPM equations.

Dividend discount models are based on the assumption that the price of the stock is equal to the present value of all future dividend payments, discounted by the cost of equity capital. Since dividends are typically paid out monthly or yearly, a DDM always has a discrete-time setting.

The simplest DDM is the case where the cost of equity capital, r, and the dividend growth rate, g, are both constant. This model, also known as the *Gordon growth model*, was introduced by Gordon in his 1959 paper [17]. In this case the dividend at time k is $D_k = D_0(1+g)^k$, where D_0 is the dividend payment of the current time period. It then follows that the price P of the stock is given by

$$P = \sum_{k=1}^{\infty} D_0 \frac{(1+g)^k}{(1+r)^k}, \tag{5.1}$$

where $r, g > -1$. Provided that $g < r$, a formula for the price can be calculated with the power series formula, yielding

$$P = \frac{D_0(1+g)}{r-g}. \tag{5.2}$$

Formula (5.2) can be used to put a price on a dividend-paying stock, but also has a second use. If one assumes that the known market price is P, the cost of capital r can be solved for, yielding an estimate

$$r = g + \frac{D_0(1+g)}{P}. \tag{5.3}$$

Formula (5.3) is in particular useful since the cost of capital is unobservable.

5.1.2 Stochastic DDMs

Let $G_1, G_2, \ldots,$ be i.i.d. random positive growth factors for the dividend payments and $R_1, R_2, \ldots,$ be i.i.d. random positive growth factors for the cost of equity capital. We assume that the processes $(G_k)_{k\in\mathbb{N}}$ and $(R_k)_{k\in\mathbb{N}}$ are independent. Formula (5.1) now takes the form

$$P = \sum_{k=1}^{\infty} D_0 \frac{G_1 G_2 \cdots G_k}{R_1 R_2 \cdots R_k}, \tag{5.4}$$

which can be rewritten as

$$P = \sum_{k=1}^{\infty} D_0 V_1 V_2 \cdots V_k, \tag{5.5}$$

where $V_j := G_j/R_j, j = 1, 2, \ldots.$

It is easy to see that Equation (5.5) corresponds to a perpetuity with constant cost D_0 and i.i.d. discount factors $V_1, V_2, \ldots.$ One needs only check that

$$\mu := \mathbf{E}(\log V_1)$$

exists and that $\mu < 0$, in order to see by Theorem 3.2, that the series (5.5) converges a.s.

The next proposition presents the new model, which is based on an example from Chapter 3.

Proposition 5.1. *Let $G_i \sim \Gamma(g, 1)$ and $R_i \sim \Gamma(r, 1), i = 1, 2, \ldots,$ where $r > g > 0$, be independent i.i.d. random variables. Further, let $V_i := G_i/R_i, i = 1, 2, \ldots.$ Then,*

$$P = \sum_{k=1}^{\infty} V_1 V_2 \cdots V_k \sim \beta_2(g, r-g).$$

Proof. Since for each $i = 1, 2, \ldots,$ $G_i \perp R_i$, it holds that $V_i \sim \beta_2(g, r)$ for all $i \in \mathbb{N}$. Then Example 3.16 yields the result. □

The model of Proposition 5.1 is unlikely to have any practical significance. The random variable P can be used to calculate the expected gain from holding a dividend-paying stock, but the assumptions of identical and independent gamma-distributed growth factors are very unrealistic. A trader willing to use such an approximation would, most likely, also be willing to use the simpler, deterministic model and Formula (5.2).

Furthermore, the assumptions on the cost of equity capital can not be justified by statistical methods, as the random variables $R_1, R_2, \dots$ are unobservable. A potential work-around would be to calculate the CAPM estimate of the costs of capital and treat the estimate for month k as an observation of the random variable R_k, although that would introduce even larger uncertainty into the model.

5.1.3 Estimating the cost of equity capital

A potential use for the model of Proposition 5.1 would be for using statistical inference to calculate an estimator of r, yielding a probability distribution for the monthly cost of equity capital, $R_k, k = 1, 2, \dots$.

If one decides on using data from N months back, then monthly dividend payments $d_1, d_2, \dots, d_N$ can be used as data and an estimator $\hat{g}$ can be calculated. If the monthly growth factors are assumed to be i.i.d. and independent, the monthly prices $p_1, p_2, \dots, p_N$ can also be treated as a sample from P. This allows one to compute a statistical estimator for r.

This method is crude and relies on strong assumptions. For this reason it remains unclear whether or not it yields better results than the Gordon growth model, although having a probabilistic model of the cost of equity capital is highly desirable for companies looking to sell shares. This is certainly the case for companies that only wish to sell shares if the cost of equity capital is below some threshold with a certain probability.

5.2 Approximating discrete-time models

In some special cases continuous-time models can provide for especially simple analytic derivations, while in other cases a discrete-time formulation can be more advantageous for analytic or numerical solutions. For this reason it can be useful for many applications, including finance, biology, etc., to be able to pick a suitable continuous approximation for a model with discrete time (or vice versa). A significant fraction of D. Dufresne's work [13, 12] is devoted to this kind of approximation. In this section, we review Dufresne's approximation procedure applied to cash flows.

5.2.1 Dufresne's procedure

The idea is to approximate a discrete cash flow with a continuous one without changing model statistics such as expected values and variances. The classical example in finance is how continuously compounded interest is introduced as a partitioning of discretely compounded interest into successively smaller time-intervals, where the continuous interest function converges to an exponential function.

Here, $(S_t)_{t\in\mathbb{R}}$ and $(Z_t)_{t\in\mathbb{R}}$ shall denote a continuous cash flow and its discounted value process, respectively, while $S_n(t)$ and $Z_n(t)$ denote discrete-time counterparts where a unit of time has been partitioned n times. The goal is to construct these processes such that the discrete-time processes converge to their continuous counterparts as $n \to \infty$. Dufresne [12] presents the construction in the following way.

For each $n \in \mathbb{N}$, define

$$S_n(t) := \sum_{j=0}^{[nt]-1} C_{n,j} U_{n,j+1} \cdots U_{n,[nt]} \tag{5.6}$$

$$Z_n(t) := \sum_{j=1}^{[nt]} C_{n,j} V_{n,1} \cdots V_{n,j} \tag{5.7}$$

where $V_{n,j} = U_{n,j}^{-1}$ and the following conditions are assumed for the random variables $C_{n,j}, U_{n,j}$:

a) For each n, $(C_{n,j})_{j\geq 0}$ and $(U_{n,j})_{j\geq 1}$ are mutually independent i.i.d. sequences.

b) $C_{n,1} \overset{d}{=} n^{-1}\mathbf{E}\left(C_{1,1}\right) + n^{-1/2}(C_{1,1} - \mathbf{E}\left(C_{1,1}\right))$, and $\mathbf{Var}\left(C_{1,1}\right) < \infty$.

c) The factors $U_{n,j}$ have

$$\mathbf{P}\left(U_{1,1} > 0\right) = 1, \mathbf{Var}\left(U_{1,1}\right) < \infty, \mathbf{Var}\left(\log U_{1,1}\right) < \infty,$$

and their distribution is given by either

i) $U_{n,1} \overset{d}{=} 1 + n^{-1}\mathbf{E}\left(U_{1,1} - 1\right) + n^{-1/2}(U_{1,1} - \mathbf{E}\left(U_{1,1}\right))$, or

ii) $\log(U_{n,1}) \overset{d}{=} n^{-1}\mathbf{E}\left(\log(U_{1,1})\right) + n^{-1/2}(\log(U_{1,1}) - \mathbf{E}\left(\log(U_{1,1})\right))$.

In the process (5.6), the payments $C_{n,j}$ are assumed to arrive in the beginning of time period $\left[\frac{j}{n}, \frac{j+1}{n}\right)$, while in (5.7) they arrive at the end of the period. These processes are constructed such that with $n = 1$ they are equal to the processes discussed in chapter 3 and for higher values of n they represent successive refinements of the processes.

The assumptions are made to assure that the successive partitioning of the time periods does not impact the fit of the model. Note in particular that the expected value and variance of the payments during one unit of time remain the same. This can be seen by observing that

$$\mathbf{E}\left(C_{n,1}\right) = \frac{\mathbf{E}\left(C_{1,1}\right)}{n}, \ \mathbf{Var}\left(C_{n,1}\right) = \frac{\mathbf{Var}\left(C_{1,1}\right)}{n},$$

and so, due to independence,

$$\mathbf{E}\left(\sum_{j=1}^{n} C_{n,j}\right) = \mathbf{E}\left(C_{1,1}\right), \ \mathbf{Var}\left(\sum_{j=1}^{n} C_{n,j}\right) = \mathbf{Var}\left(C_{1,1}\right).$$

One might say that they undergo a mean- and variance-preserving transformation. For the growth factors $U_{n,j}$ the same is done either such that the rates of return $R_{n,j} = U_{n,j} - 1$ or the geometric rates of return $\log(U_{n,j})$ have their first two moments preserved in a unit of time.

Next we present Dufresne's example. Keep in mind that Dufresne's work relies on the concept of *weak convergence*, which has not been treated in this thesis.

Proposition 5.2 (Dufresne [12]). *Let X and Y be independent standard Brownian motions and let*

$$\begin{aligned} \widetilde{X}_t &:= \gamma t + \sigma X_t, \\ \widetilde{Y}_t &:= \mu t + \zeta Y_t, \end{aligned} \tag{5.8}$$

where $\mu = \mathbf{E}\left(C_{1,1}\right), \zeta^2 = \mathbf{Var}\left(C_{1,1}\right)$, and either

i) $\gamma = \mathbf{E}\left(U_{1,1} - 1\right) - \frac{1}{2}\mathbf{Var}\left(U_{1,1}\right), \ \sigma^2 = \mathbf{Var}\left(U_{1,1}\right)$, *or* *ii)* $\gamma = \mathbf{E}(\log U_{1,1})$, $\sigma^2 = \mathbf{Var}(\log U_{1,1})$.

Then the sequence of processes $(S_n)_{n\geq 1}$ *converges weakly to a process* S *satisfying*

$$S_t = \int_0^t e^{\tilde{X}_t - \tilde{X}_s} d\widetilde{Y}_s. \tag{5.9}$$

Proposition 5.3 (Dufresne [12])**.** *With the same notation and conditions, the processes* $(Z_n)_{n\geq 1}$ *converge weakly to a process* Z *satisfying*

$$Z_t = \int_0^t e^{-\tilde{X}_s} d\widetilde{Y}_s. \tag{5.10}$$

Proof of propositions. See the appendix to [12]. □

Propositions 5.2-5.3 in fact show that discrete cash flows with i.i.d. growth (or discounting) factors and i.i.d. payments can be approximated by continuous cash flows (or vice versa). Recall from Proposition 4.1 that the process $(S_t)_{t\in\mathbb{R}}$ of (5.9) is a diffusion, satisfying the SDE

$$dS_t = (\alpha S_t + \mu)dt + \sigma S_t dX_t + \zeta dY_t,$$

where $\alpha = \gamma + \sigma^2/2$. Hence we have a method for approximating a discrete cash flow with a diffusion.

5.2.2 Some examples

Example 5.4 (Moments of an annuity)**.** Suppose an annuity pays a fixed amount $c > 0$ for $t \in \mathbb{N}$ time periods. Let $V_1, V_2, \ldots, V_t$ be i.i.d. discounting factors for the respective time periods and assume $\mathbf{E}(V_1) < 1$. The present value of the annuity is then

$$A(t) = \sum_{k=1}^{t} c \prod_{j=1}^{k} V_j.$$

The moments of this present value can obviously be calculated by using the formulas of chapter 3. By formula (3.12) the m:th moment has the form

$$\mathbf{E}(A(t)^m) = c^m \left(\sum_{j=0}^{m} d_{mj} \left[\mathbf{E}\left(V^j\right)\right]^t \right),$$

where V is an independent copy of the variables $V_1, V_2, \dots, V_t$ and the constants d_{mj} have to be recursively calculated by formulas (3.13).

Another possibility is to approximate the annuity with a continuous stream of cash. If the time periods are sufficiently short, by using Proposition 5.3, the present value is well approximated by the random variable

$$Z(t) = c \int_0^t e^{-\tilde{X}_s} ds,$$

where $\widetilde{X}_s = \gamma s + \sigma X_s$ and X is a standard Brownian motion, and

$$\gamma = -1 + \mathbf{E}\left(V_1^{-1}\right) - \frac{1}{2}\mathbf{Var}\left(V_1^{-1}\right), \ \sigma^2 = \mathbf{Var}\left(V_1^{-1}\right). \tag{5.11}$$

Knowing that $\widetilde{X}$ is a Brownian motion with drift, and so has a Laplace exponent $\phi(\lambda) = -\gamma\lambda - \frac{\sigma^2\lambda^2}{2}$, if $\gamma \neq 0$ then we can apply formula (4.22) and calculate the m:th moment

$$\mathbf{E}\left(Z(t)^m\right) = c^m m! \sum_{k=0}^{m-1} \frac{e^{tk(\gamma - \frac{\sigma^2}{2}k)} - e^{tm(\gamma - \frac{\sigma^2}{2}m)}}{\prod_{\substack{i=0 \\ i\neq k}}^{m}((k-i)(\gamma + \frac{\sigma^2}{2}(k+i)))}. \tag{5.12}$$

This approximation may not always perform well, but the moment calculations avoid the use of recursive formulas.

In the case when $t = \infty$, i.e. A is a perpetuity, the recursive moment formula (3.14) can be used. If one desires to avoid the use of a recursive formula, the same approximation can still be used and the formula (4.23) is applicable.

Example 5.5. In [12] D. Dufresne uses a discrete-time approximation of the process in (5.10) as a tool in carrying out the original proof of Theorem 4.20.

For more applications of the theory of weak convergence, see Dufresne's earlier article [13]. In [12, ch. 5] Dufresne also applies the concepts to risk theory as a justification for an approximation of discrete risk processes with a diffusion. There are also entire textbooks that discuss applications to mathematical finance or simulation methods.

5.3 Risk theory

There is a connection between ruin probabilities and the distribution of a perpetuity, as noted by R. Norberg, J. Paulsen, and H. Gjessing in their respective papers [26, 27, 15]. In particular, it may be possible to calculate certain ruin probabilities from a-priori-known distributions of continuous perpetuities. We summarize the idea, largely following the exposition of [15].

In the following, let the insurance company's assets at time t be denoted by U_t, starting from $U_0 = u$. Let X, Y be standard Brownian motions and N_X, N_Y Poisson processes having intensity parameters λ_X, λ_Y, all independent. Let the process generating insurance profit and losses for the company be

$$\widetilde{Y}_t = \mu t + \zeta Y_t - \sum_{k=1}^{N_Y(t)} S_Y(k), \quad t \geq 0, \tag{5.13}$$

where $\mu, \zeta \in \mathbb{R}$ are constants and $S_Y(1), S_Y(2), \ldots$ are i.i.d. jump variables.

The company invests its assets and gets return given by the return-on-investment generating process

$$\widehat{X}_t = \left(\gamma + \frac{1}{2}\sigma^2\right) t + \sigma X_t + \sum_{k=1}^{N_X(t)} \hat{S}_X(k), \quad t \geq 0, \tag{5.14}$$

where $\gamma, \sigma \in \mathbb{R}$ are constants and the jump variables are given by

$$\hat{S}_X(k) = e^{S_X(k)} - 1, \; k = 1, 2, \ldots,$$

where $S_X(k), k = 1, 2, \ldots,$ are i.i.d. variables. For example, if $S_X(k)$ has an exponential distribution, then $\hat{S}_X(k)$ has a Pareto distribution.

We will also soon need the related process

$$\widetilde{X}_t = \gamma t + \sigma X_t + \sum_{k=1}^{N_X(t)} S_X(k), \quad t \geq 0, \tag{5.15}$$

where all parameters and variables are as above.

At time t, the company has assets given by

$$U_t = u + \widetilde{Y}_t + \int_0^t U_{s-} d\hat{X}_s. \tag{5.16}$$

Paulsen [27] has solved equation (5.16), yielding the solution

$$U_t = e^{\tilde{X}_t} \left(u + Z_t\right), \tag{5.17}$$

where

$$Z_t = \int_0^t e^{-\tilde{X}_s} d\widetilde{Y}_s. \tag{5.18}$$

It follows that the company is ruined at time t if and only if $Z_t < -u$, since $e^{\tilde{X}_t}$ is always positive. If we denote the time of ruin (when the initial assets are $U_0 = u$) by T_u, then T_u is a stopping time and

$$T_u \coloneqq \inf\{t : U_t < 0\} = \inf\{t : Z_t < -u\}. \tag{5.19}$$

Paulsen [27] proves the following theorem.

Theorem 5.6 (Paulsen [27])**.** *Assume $Z_\infty < \infty$ a.s. and let F be the distribution function of Z_∞. Then F is continuous and the probability of eventual ruin is*

$$\mathbf{P}\left(T_u < \infty\right) = \frac{F(-u)}{\mathbf{E}\left(F(-U_{T_u}) \mid T_u < \infty\right)}. \tag{5.20}$$

Proof. See [27]. □

Equation (5.20) would be most useful if an explicit expression for F were known, which is unfortunately not often the case. Further approximations or numerical methods may prove necessary in applications. Moreover, it is necessary to compute the expected liabilities at time of ruin, U_{T_u}, in order to use (5.20), which may require further assumptions on the jump sizes S_Y.

There is much more in [27] and in the later article by Gjessing and Paulsen [15] as well as Norberg's more general diffusion setting [26]. Dufresne [12] also treats both discrete-time and continuous-time risk processes.

Bibliography

[1] Tom M. Apostol. *Mathematical Analysis.* 2nd ed. Reading: Addison Wesley Longman, 1974.

[2] Jean Bertoin and Marc Yor. “Exponential functionals of Lévy processes.” In: *Probability Surveys* 2 (2005), pp. 192–212.

[3] Patrick Billingsley. *Probability and Measure.* 3rd ed. New York: John Wiley & Sons, 1995.

[4] Andreas Brandt. “The Stochastic Equation $Y_{n+1} = A_n Y_n + B_n$ with Stationary Coefficients.” In: *Advances in Applied Probability* 18.1 (1986), pp. 211–220.

[5] Philippe Carmona, Frédérique Petit, and Marc Yor. “On the distribution and asymptotic results for exponential functionals of Lévy processes”. In: *Exponential functionals and principal values related to Brownian motion.* Ed. by Marc Yor. Biblioteca de la Revista Matemática Iberoamericana, 1997, pp. 73–121.

[6] K.L. Chung and R.J. Williams. *Introduction to Stochastic Integration.* Boston: Birkhäuser, 1990.

[7] Rama Cont and Peter Tankov. *Financial Modelling with Jump Processes.* Boca Raton: Chapman and Hall/CRC, 2004.

[8] Leif Döring and Andreas E. Kyprianou. “Perpetual Integrals for Lévy Processes.” In: *Journal of Theoretical Probability* 29 (2016), pp. 1192–1198.

[9] Daniel Dufresne. “Algebraic properties of beta and gamma distributions, and applications.” In: *Advances in Applied Mathematics* 20 (1998), pp. 285–299.

[10] Daniel Dufresne. "G distributions and the beta-gamma algebra." In: *Electronic Journal of Probability* 15 (2010), pp. 2163–2199.

[11] Daniel Dufresne. "On the stochastic equation $L(X) = L[B(X + C)]$ and a property of gamma distributions." In: *Bernoulli* 2 (1996), pp. 287–291.

[12] Daniel Dufresne. "The distribution of a perpetuity, with applications to risk theory and pension funding." In: *Scandinavian Actuarial Journal* 1990.1 (1990), pp. 39–79.

[13] Daniel Dufresne. "Weak convergence of random growth processes with applications to insurance." In: *Insurance: Mathematics and Economics* 8.3 (1989), pp. 187–201.

[14] K. Bruce Erickson and Ross A. Maller. "Generalised Ornstein–Uhlenbeck Processes and the Convergence of Lévy Integrals." In: *Séminaire de Probabilités XXXVIII*. Ed. by Michel Émery, Marc Yor, and Michel Ledoux. Vol. 1857. Lecture Notes in Mathematics. Berlin Heidelberg New York: Springer, 2004, pp. 70–94.

[15] Håkon K. Gjessing and Jostein Paulsen. "Present value distributions with applications to ruin theory and stochastic equations." In: *Stochastic Processes and their Applications* 71.1 (1997), pp. 123–144.

[16] Charles M. Goldie and Ross A. Maller. "Stability of Perpetuities." In: *The Annals of Probability* 28.3 (2000), pp. 1195–1218.

[17] Myron J. Gordon. "Dividends, Earnings and Stock Prices." In: *Review of Economics and Statistics* 41.2 (1959), pp. 99–105.

[18] Myron J. Gordon and Eli Shapiro. "Capital Equipment Analysis: The Required Rate of Profit." In: *Management Science* 3.1 (1956), pp. 102–110.

[19] Kiyosi Itô. *Stochastic Processes*. Ed. by Ole E. Barndoff-Nielsen and Ken iti Sato. Berlin: Springer-Verlag, 2004.

[20] Ioannis Karatzas and Steven Shreve. *Brownian Motion and Stochastic Calculus*. 2nd ed. New York: Springer-Verlag, 1998.

[21] Samuel Karlin and Howard M. Taylor. *A First Course in Stochastic Processes*. 2nd ed. New York: Academic Press, Inc., 1975.

[22] Martin Kolb and Mladen Savov. "A characterization for the finiteness of perpetual integrals of Lévy processes." unpublished. 2019.

[23] Damien Lamberton and Bernard Lapeyre. *Introduction to Stochastic Calculus Applied to Finance*. 2nd ed. Boca Raton: Chapman and Hall/CRC, 2008.

[24] Moshe Arye Milevsky. “The present value of a stochastic perpetuity and the Gamma distribution.” In: *Insurance: Mathematics and Economics* 20.3 (1997), pp. 243–250.

[25] Trygve Nilsen and Jostein Paulsen. “On the distribution of a randomly discounted compound Poisson process.” In: *Stochastic Processes and their Applications* 61.2 (1996), pp. 305–310.

[26] Ragnar Norberg. “Ruin problems with assets and liabilities of diffusion type.” In: *Stochastic Processes and their Applications* 81.2 (1999), pp. 255–269.

[27] Jostein Paulsen. “Risk theory in a stochastic economic environment.” In: *Stochastic Processes and their Applications* 46.2 (1993), pp. 327–361.

[28] Nicolas Privault. *Notes on Stochastic Finance*. 2019.

[29] Paavo Salminen and Lioudmila Vostrikova. “On moments of integral exponential functionals of additive processes.” In: *Statistics and Probability Letters* 146 (2019), pp. 139–146.

[30] Paavo Salminen and Marc Yor. “Properties of perpetual integral functionals of Brownian motion with drift.” In: *Annales de l’Institut Henri Poincaré, Probabilités et Statistiques* 41.3 (2005), pp. 335–347.

[31] Ken-Iti Sato. *Lévy Processes and Infinitely Divisible Distributions*. Cambridge: Cambridge University Press, 1999.

[32] Kazimierz Urbanik. “Functionals on transient stochastic processes with independent increments.” In: *Studia Mathematica* 103.3 (1992), pp. 299–315.

[33] Wim Vervaat. “On a Stochastic Difference Equation and a Representation of Non-Negative Infinitely Divisible Random Variables.” In: *Advances in Applied Probability* 11.4 (1979), pp. 750–783.

[34] Marc Yor. “On Certain Exponential Functionals of Real-Valued Brownian Motion”. In: *Exponential Functionals of Brownian Motion and Related Processes*. Ed. by Marc Yor. Berlin: Springer-Verlag, 2001. Chap. 1, pp. 14–22.