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Effective large-scale model of boson gas from microscopic theory

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Abstract

An effective large-scale model of interacting boson gas at low temperatures is constructed from first principles. The starting point is the generating function of time-dependent Green functions at finite temperature. The perturbation expansion is worked out for the generic case of finite time interval and grand-canonical density operator with the use of the S-matrix functional for the generating function. Apparent infrared divergences of the perturbation expansion are pointed out. Regularization via attenuation of propagators is proposed and the relation to physical dissipation is studied. Problems of functional-integral representation of Green functions are analyzed. The proposed large-scale model is explicitly renormalized at the leading order.

1. Introduction

Investigation of the dynamics of the superfluid transition is an important problem with long history. According to the seminal paper of Hohenberg and Halperin dynamics of the transition is described by a phenomenological stochastic model: either model E or model F in the standard

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classification [1]. In these models even the two-loop calculation does not allow to find an unambiguous stable fixed point and calculate the sign of the correction exponent ω exactly (the sign of ω heavily depends on the fixed point at which it is calculated) [2]. Results of four-loop calculations of the static critical index α [3] and the experimental result [4] are usually regarded as an argument for the infrared (IR) stability of the simpler model E. The two-loop results in model E by De Dominicis and Peliti [5,6] demonstrate two different fixed points, one of which can be IR stable. A numerical error in these works was found by Dohm [7], whose results were confirmed recently in [8]. A different approach to the investigation of the IR stability of the theory was proposed in [9]. It is based on the analysis of the influence of hydrodynamic modes. However, a reliable result for the physically relevant large value of the expansion parameter ε has not yet been achieved and the dynamic critical exponent of the superfluid transition is not known.

Contrary to the basic critical exponents, the correction exponents ω depend on the model: models of the same class of equivalence may even have different numbers of coupling constants and, consequently, exponents ω. Therefore, it is not unreasonable to analyze "the most exact model" and we suggest to use the model based on time-dependent Green functions at finite temperature (GF@FT) to this end. As was shown in [9,10] behaviour of this model is more similar to that of model F than of model E.

The formalism of nonequilibrium Green functions (NEGF) is a versatile tool for studying time evolution of many-particle quantum systems both near equilibrium, and in suitable nonequilibrium states and at arbitrary time scales (for reviews, see, e.g. [11–14]). In most cases, however, detailed calculations are carried out for time-dependent Green functions, which are the near-equilibrium variant of the nonequilibrium Green functions. This is the case in the present paper as well. A great deal of the literature on NEGF and GF@FT is devoted to the analysis of kinetic phenomena on the basis of the functional equations of motion for the generating function of NEGF and GF@FT (Schwinger equations and Dyson equations in the quantum-field theory parlance).

We intend to analyze the dynamics of a boson gas with the aid of the field-theoretic renormalization group. To this end the standard perturbation theory is the customary tool. In this setup the construction of NEGF was first put forward by Keldysh [15] for solutions independent of initial conditions. Here, we will arrive at the approximation of Keldysh from a generic treatment. In description of NEGF it is customary to use products of time-dependent operators ordered along a contour on an auxiliary complex time plane. This trick puts each time-ordered exponential of an evolution operator in the NEGF in a correspondence with a certain part of the contour [11–16]. Here, following the classic monograph [17], we propose to express each time-ordered exponential with the aid of a S-matrix functional of a c-number field and fuse the product of time-ordered products into a single normal product. The result is a generating function of GF@FT, in which different dummy fields correspond to different evolution operators, but the structure is completely generic in the sense of the universal notation of [17]. In particular, the perturbation expansion of the generating function is generated by the standard S-matrix functional. In this approach the origin of $n \times n$ propagators and contractions is self-evident (n is the number of time-ordered products in the definition of NEGF) and the functional equations of motion follow automatically according to the universal scheme [17].

This paper is organized as follows. In Sec. 2 perturbation theory for generic Green functions at finite temperature is constructed in a finite time interval with the use of the S-matrix functional obtained by standard Wick theorems and fusion rules in the functional–differential form for time-ordered products of operator functionals. Feynman rules are established for the model obtained with three auxiliary fields. Sec. 3 is devoted to the analysis of apparent IR divergences of the perturbation theory in the limit of whole time axis. In Sec. 4 a regularization by attenuation of propagators is put forward and the limit to the whole time axis with initial condition in the infinite past carried out. New field variables are introduced leading to propagator set of retarded and advanced Green functions together with the Keldysh functions. The origin of dissipation is discussed in Sec. 5. The effective large-scale model is put forward in Sec. 6. Renormalization of the effective model is analyzed in Sec. 7. The method of calculation of two-loop contributions to renormalization constants is presented in Appendix A.

2. Perturbation theory for Green functions at finite temperature

Grand-canonical expectation value of the time-ordered product of Heisenberg field operators is the definition used here for the time-dependent Green functions at finite temperature (GF@FT)

$$G_n(x_1, x_2, \dots, x_n) = \mathrm{Tr}\left\{ \frac{\exp\big[-\beta(\hat{H} - \mu\hat{N})\big]}{Z_G} \, T\left[\hat{\varphi}_H(x_1)\hat{\varphi}_H(x_2)\cdots\hat{\varphi}_H(x_n)\right] \right\}. \tag{1}$$

Here, Z_G is the partition function, the field operator has two components and consists of the annihilation and creation operators of a scalar bosonic field

$$\hat{\varphi}_H(x) := \begin{pmatrix} \hat{\psi}_H(t, \mathbf{x}) \\ \hat{\psi}^+_H(t, \mathbf{x}) \end{pmatrix}. \tag{2}$$

The time evolution of the Heisenberg operators in (1) is generated by the operator $\hat{H} - \mu\hat{N}$. For instance, the Heisenberg (annihilation) field operator is

$$\hat{\psi}_H(t, \mathbf{x}) = \exp\left[\frac{i(t - t_0)}{\hbar}(\hat{H} - \mu\hat{N})\right]\hat{\psi}(\mathbf{x})\exp\left[-\frac{i(t - t_0)}{\hbar}(\hat{H} - \mu\hat{N})\right],$$

where $\hat{\psi}(\mathbf{x})$ is the Schrödinger field operator. To analyze the effect of various auxiliary time instants introduced in the construction of the perturbation theory we have introduced explicitly a reference time instant for operator evolution t_0. In should be noted that the time-dependent Green functions at finite temperature (1) are independent of t_0.

Perturbation expansion is constructed with the use of the interaction representation and field operators in the interaction representation will be denoted without subscript e.g.

$$\hat{\psi}(t, \mathbf{x}) = \exp\left[\frac{i(t - t_0)}{\hbar}(\hat{H}_0 - \mu\hat{N})\right]\hat{\psi}(\mathbf{x})\exp\left[-\frac{i(t - t_0)}{\hbar}(\hat{H}_0 - \mu\hat{N})\right], \tag{3}$$

where $\hat{H} = \hat{H}_0 + \hat{V}$.

Time-dependent Green functions at finite temperature are a special case of non-equilibrium Green functions. In the latter the expectation value is calculated with the use of some non-equilibrium density operator instead of the grand-canonical density operator in (1). Non-equilibrium Green functions are often introduced and elaborated on the basis of the idea of Schwinger [18] of a contour-ordered product of operators in a complex time plane [13,14] instead of the usual time-ordered product used in (1). In this approach the main goal is not perturbation expansion but the Cauchy problem of Dyson equations (called Kadanoff–Baym equations in this context) for Green functions, from which, of course, perturbation expansion may be inferred.

Here, our aim is to construct the standard perturbation expansion on the whole time axis with the subsequent renormalization-group analysis of an effective large-scale model inferred from the perturbation expansion. For this setup of the problem we find it useful to resort to the construction of the perturbation theory on the basis of standard Wick's theorems [19]. The problem of representation the expected value of time-ordered product of Heisenberg operators with the aid of Wick's theorems is usually discussed on the traditional basis of pairing of operators [11,16]. Here, we apply the elegant functional–differential representation [17], which also allows a straightforward transformation to a functional integral in a way, which, in particular, elucidates the problems appearing in the construction of the functional integral for GF@FT.

The first step is presentation of the time-ordered product of the Heisenberg operators in (1) in the form of the time-ordered product of Dirac operators (see, e.g., [17]):

$$T\left[\hat{\varphi}_H(x_1)\hat{\varphi}_H(x_2)\cdots\hat{\varphi}_H(x_n)\right] = \hat{U}\left(t_0,t_f\right) T\left\{\hat{\varphi}(x_1)\hat{\varphi}(x_2)\cdots\hat{\varphi}(x_n)\exp\left[-\frac{i}{\hbar}\int\limits_{t_i}^{t_f} dt\, V_n(\hat{\varphi}(t))\right]\right\}\hat{U}\left(t_i,t_0\right)\,, \tag{4}$$

where $t_f > t_l >> t_i\ \forall l = 1,\ldots,n$. The evolution operator $\hat{U}$ in (4) is

$$\hat{U}(t,t') = \exp\left[\frac{i(t-t_0)}{\hbar}(\hat{H}_0-\mu\hat{N})\right]\exp\left[-\frac{i(t-t')}{\hbar}(\hat{H}-\mu\hat{N})\right] \times\exp\left[-\frac{i(t'-t_0)}{\hbar}(\hat{H}_0-\mu\hat{N})\right]$$

and $V_n(t)$ is the interaction operator functional in the normal form, i.e. $V_n(\hat{\varphi}) = N[V_n(\hat{\varphi})]$. This implies the amendment of the definition of the chronological product at coinciding times as the normal product. Note that – contrary to the Green functions – the chronological product (4) does depend on the reference time instant t_0.

Having in mind the eventual passing to the limit $t_f\to\infty$, $t_i\to-\infty$, we use the representation of both evolution operators in (4) in the form of the antichronological exponential:

$$\hat{U}(t',t) = \widetilde{T}\exp\left[\frac{i}{\hbar}\int\limits_{t'}^{t} V_n(\hat{\varphi}(u))\,du\right]\,,$$

where $\widetilde{T}$ stands for the antichronological product and $t' < t$ but the time-integral in the exponential is written in the natural ordering with the lesser time argument as the lower limit.

In perturbation theory is convenient to carry out the trace in (1) in the basis spanned by the eigenstates of the free operator $\hat{H}_0-\mu\hat{N}$. Factorization of the corresponding density operator transforms the exponential of the grand canonical density operator into the evolution operator of Euclidean field theory, which is convenient to express in the form of a chronological exponential in the Euclidean "time":

$$\hat{U}_E(t,0) = \exp\left[\frac{t}{\hbar}(\hat{H}_0-\mu\hat{N})\right]\exp\left[-\frac{t}{\hbar}(\hat{H}-\mu\hat{N})\right] = T\exp\left[-\frac{1}{\hbar}\int\limits_0^t V_n(\hat{\varphi}(u))\,du\right] \tag{5}$$

with the eventual substitution of $t=\beta\hbar$ for the upper limit. In the interaction picture evolution of the Euclidean field operators is given by the rule

$$\hat{\varphi}_E(t,\mathbf{x}) = \exp\left[\frac{t}{\hbar}(\hat{H}_0-\mu\hat{N})\right]\hat{\varphi}(\mathbf{x})\,\exp\left[-\frac{t}{\hbar}(\hat{H}_0-\mu\hat{N})\right],$$

where the subscript refers to the Euclidean field theory obtained by the substitution $(t - t_0) \to -it$. Due to the presence of the finite reference time t_0 we shall use different notation for the operators with Euclidean time evolution (without t_0) instead of using the Dirac operators (3) with imaginary time.

Thus, substitution of representations (4) and (5) in (1) gives rise to the expectation value of a product of four time-ordered products of operators. With the choice $t_i < t_0$ we arrive at the representation

$$G_n(x_1, x_2, \dots, x_n) = \mathrm{Tr}\Bigg(\frac{\exp\big[-\beta(\hat{H}_0 - \mu\hat{N})\big]}{Z_G} T\exp\left[-\frac{1}{\hbar}\int_0^{\beta\hbar} V_n(\hat{\varphi}(t))\,dt\right] \times \widetilde{T}\exp\left[\frac{i}{\hbar}\int_{t_0}^{t_f} V_n(\hat{\varphi}(t))\,dt\right] T\left\{\hat{\varphi}(x_1)\hat{\varphi}(x_2)\cdots\hat{\varphi}(x_n)\exp\left[-\frac{i}{\hbar}\int_{t_i}^{t_f} V_n(\hat{\varphi}(t))\,dt\right]\right\} \times \widetilde{T}\exp\left[\frac{i}{\hbar}\int_{t_i}^{t_0} V_n(\hat{\varphi}(t))\,dt\right]\Bigg). \tag{6}$$

Here, Wick's theorems allow to fuse the product of time-ordered products to a single normal-ordered product [17] leading to the expression:

$$G_n(x_1, x_2, \dots, x_n) = \mathrm{Tr}\Bigg(\frac{\exp\big[-\beta(\hat{H}_0 - \mu\hat{N})\big]}{Z_G} \times N\Bigg\{\exp\left(\frac{1}{2}\sum_{l=1}^{4}\frac{\delta}{\delta\varphi_l}\Delta_{ll}\frac{\delta}{\delta\varphi_l} + \sum_{k<l}\frac{\delta}{\delta\varphi_k}n_{kl}\frac{\delta}{\delta\varphi_l}\right)\varphi_3(x_1)\cdots\varphi_3(x_n) \times \exp\Big[-\frac{1}{\hbar}\int_0^{\beta\hbar} V_n(\varphi_1)\,dt + \frac{i}{\hbar}\int_{t_0}^{t_f} V_n(\varphi_2)dt - \frac{i}{\hbar}\int_{t_i}^{t_f} V_n(\varphi_3)\,dt + \frac{i}{\hbar}\int_{t_i}^{t_0} V_n(\varphi_4)\,dt\Big]\Bigg\}\Bigg|_{\substack{\varphi_{2,3,4}=\hat{\varphi}\\ \varphi_1=\hat{\varphi}_E}}\Bigg), \tag{7}$$

where $V_n(t, \varphi)$ is the normal form of the interaction functional and we have denoted explicitly all four field arguments, whose labelling follows the order of factors in the operator product in (6). In the exponential differential reduction operator all integrals are implied and the contractions are defined in the standard manner (subscripts refer to enumeration of the fields.): Δ_{33} is the chronological contraction

$$\Delta(x, x') = T\left[\hat{\varphi}(x)\hat{\varphi}(x')\right] - N\left[\hat{\varphi}(x)\hat{\varphi}(x')\right], \tag{8}$$

$\Delta_{22} = \Delta_{44}$ is the antichronological contraction

$$\widetilde{\Delta}(x, x') = \tilde{T}\left[\hat{\varphi}(x)\hat{\varphi}(x')\right] - N\left[\hat{\varphi}(x)\hat{\varphi}(x')\right], \tag{9}$$

and Δ_{11} is the Euclidean chronological contraction

$$\Delta_E(x,x') = T\left[\hat\varphi_E(x)\hat\varphi_E(x')\right] - N\left[\hat\varphi_E(x)\hat\varphi_E(x')\right]. \tag{10}$$

For economy of space we do not list explicitly all six normal contractions in (7) but remind the rule of construction:

$$n(x,x') = \hat\varphi(x)\hat\varphi(x') - N\left[\hat\varphi(x)\hat\varphi(x')\right], \tag{11}$$

where the field operators are either Dirac operators or Euclidean free field operators depending on the label of the field argument in (7).

The grand-canonical expectation value of the normal product for arbitrary operator functional F is calculated with the aid of the relation

$$\frac{\mathrm{Tr}\left\{\exp\left[-\beta(\hat H_0-\mu\hat N)\right]N[F(\hat\varphi)]\right\}}{\mathrm{Tr}\,\exp\left[-\beta(\hat H_0-\mu\hat N)\right]} = \exp\left(\frac12\frac{\delta}{\delta\varphi}d\frac{\delta}{\delta\varphi}\right)F(\varphi)\Big|_{\varphi=0}, \tag{12}$$

where the kernel of the functional differential operator is the expected value of the normal product of fields (we shall call it the thermal contraction)

$$d(x,x') = \frac{\mathrm{Tr}\left\{\exp\left[-\beta(\hat H_0-\mu\hat N)\right]N\left[\hat\varphi(x)\hat\varphi(x')\right]\right\}}{\mathrm{Tr}\,\exp\left[-\beta(\hat H_0-\mu\hat N)\right]}. \tag{13}$$

Since in (7) there are fields with both Euclidean and Dirac evolution rules and – consequently – functions (13) differing by temporal behaviour, it is convenient to write the quadratic form of the reduction operator (12) as the completed square over the four fields of (7), which adds the function d to all matrix elements of the 4×4 propagator matrix in (7). Therefore

$$\begin{aligned}G_n(x_1,x_2,\ldots,x_n) = \frac{Z_0}{Z_G}\Bigg\{&\exp\Bigg[\frac12\sum_{l=1}^{4}\frac{\delta}{\delta\varphi_l}\Delta_{ll}\frac{\delta}{\delta\varphi_l} + \sum_{k<l}\frac{\delta}{\delta\varphi_k}n_{kl}\frac{\delta}{\delta\varphi_l}\\ &+\frac12\sum_{k,l=1}^{4}\frac{\delta}{\delta\varphi_k}d_{kl}\frac{\delta}{\delta\varphi_l}\Bigg]\varphi_3(x_1)\cdots\varphi_3(x_n)\,\exp\Bigg[-\frac1\hbar\int_0^{\beta\hbar}V_n(\varphi_1)\,dt\\ &+\frac{i}{\hbar}\int_{t_0}^{t_f}V_n(\varphi_2)dt - \frac{i}{\hbar}\int_{t_i}^{t_f}V_n(\varphi_3)\,dt + \frac{i}{\hbar}\int_{t_i}^{t_0}V_n(\varphi_4)\,dt\Bigg]\Bigg\}\Bigg|_{\varphi_i=0},\end{aligned} \tag{14}$$

where $Z_0 = \mathrm{Tr}\,\exp\left[-\beta(\hat H_0-\mu\hat N)\right]$. This is almost the final form for generating the perturbation expansion of the GF@FT. Apart from the product of fields in front of the interaction exponential, expression (14) is of the form of the S-matrix functional [17]. The four field variables and the corresponding time integrals may be put in correspondence with the contributions produced by different parts of the contour in the complex time plane in the contour-ordered approach.

Simplifications require an analysis of the dependence on the time parameters t_i, t_0 and t_f introduced in the construction of the perturbation expansion. The original GF@FT (1) are independent of all three and, in principle, their values may be chosen from the point of view of convenience of calculations. Before discussing this issue, let us specify the model analyzed further. We are investigating a non-relativistic gas of scalar bosons with a local repulsive density-density interaction near the critical point of condensation. Thus, the Hamilton operator is chosen in the form

$$\hat{H} = \int d\mathbf{x} \left[\hat{\psi}^+(t,\mathbf{x}) \left(-\frac{\hbar^2\nabla^2}{2m} - \mu \right) \hat{\psi}(t,\mathbf{x}) + \frac{g}{4} \hat{\psi}^{+2}(t,\mathbf{x}) \hat{\psi}^2(t,\mathbf{x}) \right] \tag{15}$$

which, following the tradition of quantum field theory, is written in the normal form, so that the interaction functional V_n is obtained by simply replacing the field operators by the corresponding functions, i.e. by omitting operator hats.

All contractions of two creation operators as well as two annihilation operators vanish. Therefore, the contractions of the two-component field (2) are 2×2 matrices with zero diagonal elements. In the time-wave-vector representation the chronological contractions are (plane-wave basis)

$$\Delta(t,t';\mathbf{k}) = iG^R(t-t',\mathbf{k}) = \theta(t-t') \exp\left[-i\omega(\mathbf{k})(t-t')\right] , \tag{16}$$

$$\widetilde{\Delta}(t,t';\mathbf{k}) = -iG^A(t-t',\mathbf{k}) = \theta(t'-t) \exp\left[-i\omega(\mathbf{k})(t-t')\right] , \tag{17}$$

$$\Delta_E(t,t';\mathbf{k}) = \theta(t-t') \exp\left[-\omega(\mathbf{k})(t-t')\right] , \tag{18}$$

where

$$\omega(\mathbf{k}) = \frac{\epsilon(\mathbf{k})}{\hbar} = \frac{1}{\hbar} \left(\frac{\hbar^2 \mathbf{k}^2}{2m} - \mu \right)$$

and G^R, G^A establish the connection between the usual retarded and advanced Green functions of nonrelativistic kinetic theory [20].

The thermal contractions (as well as the normal contractions) differ by the rule of time evolution of the Dirac operator and the Euclidean operator:

$$\begin{aligned} d_{DD}(t,t';\mathbf{k}) &= \exp\left[-i\omega(\mathbf{k})(t-t')\right] \overline{n}(\mathbf{k}) , \\ d_{EE}(t,t';\mathbf{k}) &= \exp\left[-\omega(\mathbf{k})(t-t')\right] \overline{n}(\mathbf{k}) , \\ d_{DE}(t,t';\mathbf{k}) &= \exp\left[\omega(\mathbf{k})(-i(t-t_0)+t')\right] \overline{n}(\mathbf{k}) , \\ d_{ED}(t,t';\mathbf{k}) &= \exp\left[\omega(\mathbf{k})(-t+i(t'-t_0))\right] \overline{n}(\mathbf{k}) , \end{aligned} \tag{19}$$

where the subscripts refer to the time evolution of the field operators in the averaged operator product $d(x,x') = \langle \hat{\psi}^+(t',\mathbf{x}') \hat{\psi}(t,\mathbf{x}) \rangle$ and $\overline{n}(\mathbf{k})$ is the mean occupation number of the state with $\mathbf{k}$ in the free boson gas:

$$\overline{n}(\mathbf{k}) = \frac{1}{\exp\left[\beta\epsilon(\mathbf{k})\right] - 1} .$$

All simple contractions generated by normal-ordered products of two field operators vanish due to definition (11). The rest have the same time dependence as the thermal contractions, but the mean occupation number is replaced by the unity:

$$\begin{aligned} n_{DD}(t,t';\mathbf{k}) &= \exp\left[-i\omega(\mathbf{k})(t-t')\right] , \\ n_{EE}(t,t';\mathbf{k}) &= \exp\left[-\omega(\mathbf{k})(t-t')\right] , \\ n_{DE}(t,t';\mathbf{k}) &= \exp\left[\omega(\mathbf{k})(-i(t-t_0)+t')\right] , \\ n_{ED}(t,t';\mathbf{k}) &= \exp\left[\omega(\mathbf{k})(-t+i(t'-t_0))\right] . \end{aligned} \tag{20}$$

It should be noted that the purely oscillating time-dependence combined with the absence of restrictions on the wave numbers in simple contractions with Dirac evolution brings about certain problems in calculations in perturbation theory, as will be demonstrated in the next section.

In terms of fields ψ and ψ^+ corresponding to the field operators of the Hamiltonian (15), the functional representation (14) assumes the form

$$G_n(x_1, x_2, \ldots, x_n) = \frac{Z_0}{Z_G}\Bigg\{\exp\Bigg[\sum_{l=1}^{4}\frac{\delta}{\delta\psi_l}\Delta_{ll}\frac{\delta}{\delta\psi_l^+} + \sum_{k<l}\frac{\delta}{\delta\psi_k}n_{kl}\frac{\delta}{\delta\psi_l^+} + \sum_{k,l=1}^{4}\frac{\delta}{\delta\psi_k}d_{kl}\frac{\delta}{\delta\psi_l^+}\Bigg]$$

$$\times\, \psi_3(x_1)\cdots\psi_3(x_m)\psi_3^+(x_{m+1})\cdots\psi_3^+(x_n)\exp\Bigg[-\frac{g}{4\hbar}\int_0^{\beta\hbar}dt\int d\mathbf{x}\,\psi_1^{+2}(t,\mathbf{x})\psi_1^2(t,\mathbf{x})$$

$$+\frac{ig}{4\hbar}\int_{t_0}^{t_f}dt\int d\mathbf{x}\,\psi_2^{+2}(t,\mathbf{x})\psi_2^2(t,\mathbf{x}) - \frac{ig}{4\hbar}\int_{t_i}^{t_f}dt\int d\mathbf{x}\,\psi_3^{+2}(t,\mathbf{x})\psi_3^2(t,\mathbf{x})$$

$$+\frac{ig}{4\hbar}\int_{t_i}^{t_0}dt\int d\mathbf{x}\,\psi_4^{+2}(t,\mathbf{x})\psi_4^2(t,\mathbf{x})\Bigg]\Bigg\}\Bigg|_{\substack{\psi_i=0\\ \psi_i^+=0}}. \quad (21)$$

The complete 4×4 matrix of contractions contains different combinations of functions defined in (16), (17), (18), (19) and (20):

$$\underline{\Delta} = \begin{pmatrix} \Delta_E + d_{EE} & n_{ED} + d_{ED} & n_{ED} + d_{ED} & n_{ED} + d_{ED} \\ d_{DE} & \widetilde{\Delta} + d_{DD} & n_{DD} + d_{DD} & n_{DD} + d_{DD} \\ d_{DE} & d_{DD} & \Delta + d_{DD} & \underset{\sim}{n_{DD}} + d_{DD} \\ d_{DE} & d_{DD} & d_{DD} & \widetilde{\Delta} + d_{DD} \end{pmatrix} \quad (22)$$

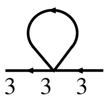
and each matrix element contains an oscillating part. It will be shown later that this is inconvenient in the analysis of perturbation theory and, moreover, is the source of unusual IR divergences.

It should be noted that the perturbation expansion (21), (22) refers to the most generic case of GF@FT: the time limits t_f and t_i as well the reference time t_0 are arbitrary (except for the convention of ordering) and the density operator is that of the interacting system. In principle, any Hermitian operator polynomial in field operators and with a free-field-like quadratic part could be used in $\hat{\rho}$ instead of $\hat{H} - \mu\hat{N}$ (with, of course, the corresponding changes in contractions following from definitions (8), (9), (10), (11) and (13)). In particular, (21), (22) are a generalization of Wagner's result with 3×3 propagator matrix [16].

3. Unusual and usual divergences of the perturbation theory

In Feynman diagrams of the perturbation expansion of Green functions in quantum field theory several divergences are met. In particle field theories the main concern are UV divergences in the Fourier space of wave vectors, which may be dealt with by the theory of renormalization to arrive at a meaningful physical theory. In massless models IR divergences are a major problem, to which there is no generic solution yet. In relativistic models of particle physics wave vectors and frequencies may be treated on equal footing due to the propagator structure of perturbation theory. In non-relativistic field theory, as in the present problem, wave-vector and frequency variables have to be treated separately. This is also the case in the field-theoretic approach to critical phenomena. In the latter, all divergences are brought about by the wave-vector space. It turns out that in non-relativistic quantum field theory divergences appear also in the time (frequency) integrations of Feynman diagrams due to the oscillatory behaviour of the propagators of quantum mechanics (in contrast to the exponential attenuation in stochastic field theory). Some of these divergences are due to peculiarities of representation and disappear order by order in perturbation theory, others are genuine UV of IR singularities related to the continuum or thermodynamic limit of the model. In this section we are mainly discussing apparent temporal divergences brought about by the structure of perturbation theory.

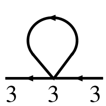
In the perturbation expansion of ordinary Green functions – calculated as expectation values of time-ordered products of Heisenberg operators in the ground state instead of the grand-canonical trace over all basis states – the limit $t_f \to \infty$, $t_i \to -\infty$ is customary to facilitate the use of the Fourier transform. In case of the usual Green functions this limit may be carried out in each Feynman graph of the perturbation expansion separately. This is not the case here, however. Consider, for instance, the simplest one-loop self-energy contribution of the physical fields to the two-point Green function $G_2(x_1, x_2) = \mathrm{Tr}\left\{\hat{\rho}_G T\left[\hat{\psi}_H(x_1)\hat{\psi}^+_H(x_2)\right]\right\}$ (the subscript refers to the number of the field variables in (21))

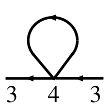


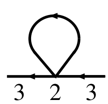
$$
\begin{aligned}
&= \int_{t_i}^{t_f} dt\ [\Delta(t_1 - t) + d_{DD}(t_1 - t)]\, \Sigma^{(1)}_{\psi_3^+\psi_3} \\
&\times [\Delta(t - t_2) + d_{DD}(t - t_2)] = -ig \int d\mathbf{p}\left[\theta(0) + \overline{n}(\mathbf{p})\right] \\
&\times \int_{t_i}^{t_f} dt\ \exp[-i\omega(\mathbf{k})(t_1 - t)]\,[\theta(t_1 - t) + \overline{n}(\mathbf{k})]\exp[-i\omega(\mathbf{k})(t - t_2)] \\
&\times [\theta(t - t_2) + \overline{n}(\mathbf{k})] = -ig \int d\mathbf{p}\,\overline{n}(\mathbf{p})\ \exp[-i\omega(\mathbf{k})(t_1 - t_2)] \\
&\times \left[(t_1 - t_2) + (t_1 - t_i + t_f - t_2)\overline{n}(\mathbf{k}) + (t_f - t_i)\overline{n}^2(\mathbf{k})\right].
\end{aligned}
$$

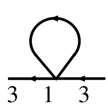
Here, the wave-vector integral of the propagator with coinciding time arguments is taken over the closed loop. The wave-vector integral of the mean occupation number $n(\mathbf{p})$ is convergent and gives rise to a finite factor. We use the convention in which the time-ordered product at coinciding time arguments is defined as the normal product, which amounts to that formally $\theta(0) = 0$. The time integral, on the contrary, gives rise to terms which diverge in the limit $t_f \to \infty$, $t_i \to -\infty$. It should be noted that if the model is analyzed in the frequency representation on the whole time axis (i.e. after passing to the limit $t_f \to \infty$, $t_i \to -\infty$) this singularity appears in the form of a pinch singularity (integration contour in the complex frequency plane is pinched between two poles of propagators or contractions) as a product of two frequency δ functions with coinciding arguments [14].

However, there are other similar graphs corresponding to interaction terms with fields φ_2, φ_3 and φ_4 in the generic notation (7). All one-loop contributions to G_2 (labels are numbers of fields) are

$G^{(1)}_2(x_1, x_2) =$ 

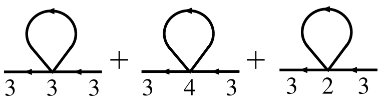
$+$ 

$+$ 

$+$ 

.

Divergences in the second and fourth term cancel those of the first, thus leading to a finite result:



$$
\begin{aligned}
&= \int_{t_i}^{t_f} dt\, [\Delta(t_1 - t) + d_{DD}(t_1 - t)]\, \Sigma^{(1)}_{\psi_3^+\psi_3}\, [\Delta(t - t_1) + d_{DD}(t - t_1)] \\
&\quad + \int_{t_i}^{t_0} dt\, [n_{DD}(t_1 - t) + d_{DD}(t_1 - t)]\, \Sigma^{(1)}_{\psi_4^+\psi_4} d_{DD}(t - t_2) \\
&\quad + \int_{t_0}^{t_f} dt\, [n_{DD}(t_1 - t) + d_{DD}(t_1 - t)]\, \Sigma^{(1)}_{\psi_2^+\psi_2} d_{DD}(t - t_2) \\
&= -ig \int d\mathbf{p}\, \overline{n}(\mathbf{p})\, \exp[-i\omega(\mathbf{k})(t_1 - t_2)]\, (t_1 - t_2)\left[\theta(t_1 - t_2) + \overline{n}(\mathbf{p})\right]. \qquad (23)
\end{aligned}
$$

The linear growth in time $t_1 - t_2$ of (23) obtained is not a divergence, but nevertheless it is unusual behaviour of the loop correction. The meaning of this linear growth is more transparent in the frequency space: the Fourier transformation of the right side of (23) with respect to $t_1 - t_2$ gives rise to the following expression

$$
\begin{aligned}
&-ig \int d\mathbf{p}\, \overline{n}(\mathbf{p}) \left[\frac{-1}{(\omega - \omega(\mathbf{k}) + i\delta)^2} - 2\pi i\, \delta'(\omega - \omega(\mathbf{k}))\, \overline{n}(\mathbf{k}) \right] \\
&\qquad = g \int d\mathbf{p}\, \overline{n}(\mathbf{p}) \frac{\partial}{\partial \omega} \left[\frac{i}{\omega - \omega(\mathbf{k}) + i\delta} + 2\pi\, \delta(\omega - \omega(\mathbf{k}))\, \overline{n}(\mathbf{k}) \right], \qquad (24)
\end{aligned}
$$

where δ' is the derivative of the δ function and the expression in square brackets on the right side is the Fourier transform of the propagator $\Delta + d_{DD}$. Thus, the expression (23) is simply the leading term of an expansion in a shift of the frequency. It should be noted that (24) cannot be obtained by direct substitution of propagators in frequency representation.

It is not difficult to see in the frequency space that the term (23) corresponds to a shift (renormalization) of the chemical potential by the amount $g \int d\mathbf{p}\, n(\mathbf{p})$. More conveniently this fact may be seen by the use of the reduced vertex [17] with respect to the generation of thermal contractions, i.e. (generic notation)

$$
V_{\mathrm{red}}(\varphi) = \exp\left(\frac{1}{2} \frac{\delta}{\delta\varphi} d \frac{\delta}{\delta\varphi} \right) V(\varphi),
$$

which in the case of interaction in (15) amounts to (integrals in functionals are implied)

$$
\begin{aligned}
V_{\mathrm{red}}(\psi, \psi^+) &= \exp\left(\frac{\delta}{\delta\psi^+} d \frac{\delta}{\delta\psi} \right) \left(\frac{g}{4} \psi^{+2} \psi^2 \right) \\
&= \frac{g}{4} \psi^{+2} \psi^2 + g \left(\int d\mathbf{p}\, \overline{n}(\mathbf{p}) \right) \psi^+ \psi + \frac{g}{2} \left(\int d\mathbf{p}\, \overline{n}(\mathbf{p}) \right)^2 .
\end{aligned}
$$

The last term on the right side is an unimportant constant, but the second directly yields a fluctuation correction to the chemical potential. We recall that when reduced vertices are used, there are no more closed loops of single propagators in the Feynman graphs of the model. Thus, the apparent divergences produced by closed loops of single propagators may be dealt with explicitly once and for all.

However, oscillating time dependence occurs in other graphs as well with the subsequent generation of apparent divergences. It is utterly annoying to collect all the relevant terms prior passing to the limit of whole time axis. Therefore, we find it convenient to regularize these IR divergences so that each separate graph has a finite limit, when $t_f \to \infty$, $t_i \to -\infty$.

Traces of the unusual divergences appear in following sections in the form of poles in the temporal regulator introduced in Sec. 4.

4. Temporal regularization of the perturbation theory

To deal with finite quantities in each separate Feynman diagram of the perturbation expansion we introduce attenuation with respect to time in both time directions from the present instant by defining the regularized propagators as follows

$$\begin{aligned}\Delta_{\rm reg}(t,t';\mathbf{k}) &= \theta(t-t')\exp\left[-i\omega(\mathbf{k})(t-t')-\gamma(t-t')\right],\\ \widetilde{\Delta}_{\rm reg}(t,t';\mathbf{k}) &= \theta(t'-t)\exp\left[-i\omega(\mathbf{k})(t-t')+\gamma(t-t')\right],\end{aligned} \tag{25}$$

$$\begin{aligned}d_{DD\,\rm reg}(t,t';\mathbf{k}) &= \exp\left[-i\omega(\mathbf{k})(t-t')-\gamma|t-t'|\right]\overline{n}(\mathbf{k}),\\ d_{DE\,\rm reg}(t,t';\mathbf{k}) &= \exp\left[\omega(\mathbf{k})(-i(t-t_0)+t'-\gamma|t-t_0|)\right]\overline{n}(\mathbf{k}),\\ d_{ED\,\rm reg}(t,t';\mathbf{k}) &= \exp\left[\omega(\mathbf{k})(-t+i(t'-t_0)-\gamma|t'-t_0|)\right]\overline{n}(\mathbf{k}),\end{aligned} \tag{26}$$

$$\begin{aligned}n_{DD\,\rm reg}(t,t';\mathbf{k}) &= \exp\left[-i\omega(\mathbf{k})(t-t')-\gamma|t-t'|\right],\\ n_{DE\,\rm reg}(t,t';\mathbf{k}) &= \exp\left[\omega(\mathbf{k})(-i(t-t_0)+t'-\gamma|t-t_0|)\right],\\ n_{ED\,\rm reg}(t,t';\mathbf{k}) &= \exp\left[\omega(\mathbf{k})(-t+i(t'-t_0)-\gamma|t'-t_0|)\right].\end{aligned} \tag{27}$$

The dependence of the attenuation coefficient $\gamma > 0$ on k will be specified later. Physically, this temporal regularization corresponds to introduction of energy dissipation to a Hamiltonian quantum system. This attenuation is a feature brought about by loop corrections in the perturbation theory anyway [21] and, as will be demonstrated in the following section, is a crucial feature in the renormalization of the model.

With the regularized propagators (25), (26) and (27) we may safely pass to the limit $t_i \to -\infty$ and $t_f \to \infty$. Having done this there is a choice for the value of the reference time instant t_0. If we put $t_0 = 0$ (or choose any other finite value), then the vertices of the density operator field remain connected to others and we are left with the full perturbation theory (21), (22) with 16 contractions and propagators. For problems with time scales of the order of relaxation time (to equilibrium or steady state) this is the appropriate choice. Here, we are interested in the situation at times much larger than the relaxation time and in this case it is reasonable to send the reference time t_0 to $-\infty$, which makes the correlation functions $d_{DE\,\rm reg}$, $d_{ED\,\rm reg}$, $n_{DE\,\rm reg}$ and $n_{ED\,\rm reg}$ vanish and also removes the contribution of the fields ψ_4 and ψ_4^+ in (21). It should be borne in mind, however, that in this argument the density operator is time-independent. There are other options to deal with the statistical averaging of the time-ordered product of Heisenberg operators [13,14], which we do not discuss here.

Thus, we are left with the set of three pairs of fields and the propagator matrix

$$\underline{\Delta}_{\rm reg} = \begin{pmatrix} \Delta_E + d_{EE} & 0 & 0 \\ 0 & \widetilde{\Delta}_{\rm reg} + d_{DD\,\rm reg} & n_{DD\,\rm reg} + d_{DD\,\rm reg} \\ 0 & d_{DD\,\rm reg} & \Delta_{\rm reg} + d_{DD\,\rm reg} \end{pmatrix} .$$

The structure of the propagator matrix reveals that the pair of fields ψ_1^+ and ψ_1 is decoupled from the rest in this approach and the only temperature dependence remains in the propagators. Little reflection shows that the functional in ψ_1^+ and ψ_1 gives rise to Z_G/Z_0, therefore the functional representation (21) is replaced by

$$G_n(x_1, x_2, \ldots, x_n) = \left\{ \exp\left(\sum_{l=2}^{3} \frac{\delta}{\delta\psi_l} \underline{\Delta}_{{\rm reg}\,ll} \frac{\delta}{\delta\psi_l^+} \right) \psi_3(x_1)\cdots\psi_3^+(x_n) \right.$$
$$\left. \left. \times \exp\left[\frac{ig}{4\hbar} \int_{-\infty}^{\infty} dt \int d\mathbf{x}\, \psi_2^{+2}\psi_2^2 - \frac{ig}{4\hbar} \int_{-\infty}^{\infty} dt \int d\mathbf{x}\, \psi_3^{+2}\psi_3^2 \right] \right\} \right|_{\substack{\psi_i=0 \\ \psi_i^+=0}}, \quad (28)$$

with the regularized propagator

$$\underline{\Delta}_{\rm reg} = \begin{pmatrix} \widetilde{\Delta}_{\rm reg} + d_{DD\,\rm reg} & n_{DD\,\rm reg} + d_{DD\,\rm reg} \\ d_{DD\,\rm reg} & \Delta_{\rm reg} + d_{DD\,\rm reg} \end{pmatrix} .$$

This gives rise the perturbation theory for GF@FT proposed by Keldysh [15]. From this line of argument it is clear that the limit $t_0 \to -\infty$ in the present approach is tantamount to neglecting all correlations in the initial distribution of particles.

In construction of the perturbation theory as well as in the analysis of divergences it is convenient to use a different set of Green functions [15]. To this end, let us introduce a new set of fields (written in terms of functional variables, this corresponds to the change of Green function matrices in the original paper of Keldysh [15])

$$\begin{pmatrix} \eta \\ \eta^+ \\ \xi \\ \xi^+ \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \psi_3 - \psi_2 \\ \psi_3^+ - \psi_2^+ \\ \psi_2 + \psi_3 \\ \psi_2^+ + \psi_3^+ \end{pmatrix} . \quad (29)$$

In terms of these Retarded–Advanced–Keldysh (RAK) fields the reduction operator of the perturbation expansion is

$$P = \exp\left[-\frac{\delta}{\delta\eta} \widetilde{\Delta}_{\rm reg} \frac{\delta}{\delta\xi^+} + \frac{\delta}{\delta\xi} \Delta_{\rm reg} \frac{\delta}{\delta\eta^+} + \frac{\delta}{\delta\xi} \Delta^K_{\rm reg} \frac{\delta}{\delta\xi^+} \right], \quad (30)$$

where $\Delta^K_{\rm reg}$ is the (regularized) Keldysh function

$$\Delta^K_{\rm reg}(t, t'; \mathbf{k}) = \exp\left[\omega(\mathbf{k})(-i(t - t') - \gamma|t - t'|) \right] [1 + 2\overline{n}(\mathbf{k})] . \quad (31)$$

The interaction functional becomes

$$S_I(\eta, \eta^+, \xi, \xi^+) = \frac{g}{4} \int_{-\infty}^{\infty} dt \int d\mathbf{x} \left(\psi_2^{+2}\psi_2^2 - \psi_3^{+2}\psi_3^2 \right)$$

$$= -\frac{g}{4}\int_{-\infty}^{\infty} dt \int d\mathbf{x}\left(\eta^+\xi^+\xi^2 + \xi^{+2}\xi\eta + \xi^+\eta^+\eta^2 + \eta^{+2}\eta\xi\right)$$
$$= V_1 + V_2 + V_3 + V_4\,, \quad (32)$$

where the shorthand notation on the right side will be used further to identify the different field structures of the interaction terms. For purposes of renormalization the common coupling constant g will also be replaced later by a set of coupling constants with indices following the labelling introduced in (32).

In the regularized model a functional-integral representation may be constructed with the use of the standard trick [17], which here assumes the form

$$\exp\left[-\frac{\delta}{\delta\eta}\widetilde{\Delta}_{\text{reg}}\frac{\delta}{\delta\xi^+} + \frac{\delta}{\delta\xi}\Delta_{\text{reg}}\frac{\delta}{\delta\eta^+} + \frac{\delta}{\delta\xi}\Delta^K_{\text{reg}}\frac{\delta}{\delta\xi^+}\right]$$
$$= \frac{1}{C}\int\mathcal{D}E\int\mathcal{D}E^+\int\mathcal{D}X\int\mathcal{D}X^+$$
$$\times\exp\left(-X^+\left[\frac{\partial}{\partial t} + i\omega(\mathbf{k}) - \gamma\right]E - E^+\left[\frac{\partial}{\partial t} + i\omega(\mathbf{k}) + \gamma\right]X\right.$$
$$\left. - 2E^+\gamma\left[1 + 2\overline{n}(\mathbf{k})\right]E + E\frac{\delta}{\delta\eta} + E^+\frac{\delta}{\delta\eta^+} + X\frac{\delta}{\delta\xi} + X^+\frac{\delta}{\delta\xi^+}\right),$$

where the normalization factor is

$$C^{-1} = \det\left|\frac{1}{2\pi}\left[\frac{\partial}{\partial t} + i\omega(\mathbf{k}) - \gamma\right]\frac{1}{2\pi}\left[\frac{\partial}{\partial t} + i\omega(\mathbf{k}) + \gamma\right]\right|$$

Linear exponentials of derivatives produce shifts in the integration variables. Therefore, the functional–differential representation (28) of Green functions gives rise to the functional integral representation of Green functions of RAK fields in the form

$$G(A, A^+, B, B^+) = \exp\left[-\frac{\delta}{\delta\eta}\widetilde{\Delta}_{\text{reg}}\frac{\delta}{\delta\xi^+} + \frac{\delta}{\delta\xi}\Delta_{\text{reg}}\frac{\delta}{\delta\eta^+} + \frac{\delta}{\delta\xi}\Delta^K_{\text{reg}}\frac{\delta}{\delta\xi^+}\right]$$
$$\times\exp\left[\frac{i}{\hbar}S_I(\eta^\pm,\xi^\pm) + A^+\xi + +B^+\eta + A\xi^+ + B\eta^+\right]\Bigg|_{\eta^\pm=\xi^\pm=0}$$
$$= \int\mathcal{D}\eta\int\mathcal{D}\eta^+\int\mathcal{D}\xi\int\mathcal{D}\xi^+\exp\left(-\xi^+\left[\frac{\partial}{\partial t} + i\omega(\mathbf{k}) - \gamma\right]\eta\right.$$
$$- \eta^+\left[\frac{\partial}{\partial t} + i\omega(\mathbf{k}) + \gamma\right]\xi - \eta^+\left\{2\gamma\left[1 + 2\overline{n}(\mathbf{k})\right]\right\}\eta + \frac{i}{\hbar}S_I(\eta^\pm,\xi^\pm)$$
$$\left. + A^+\xi + +B^+\eta + A\xi^+ + B\eta^+\right). \quad (33)$$

It should be noted that this connection is unambiguous in the regularized model only, when $\gamma > 0$ and Δ^K_{reg} is not a solution of the free-field equation of motion.

In representation (33) of the generating function of Green functions in RAK variables of the regularized model the propagator matrix is given unambiguously as the inverse of the differential operator in the following quadratic form of the action functional

$$S_0(\eta,\eta^+,\xi,\xi^+) = \int_{-\infty}^{\infty} dt \int d\mathbf{k}\, \eta^+(t,-\mathbf{k}) \left(i\hbar\frac{\partial}{\partial t} - \frac{\hbar^2 k^2}{2m} + \mu + i\hbar\gamma \right) \xi(t,\mathbf{k})$$
$$+ \int_{-\infty}^{\infty} dt \int d\mathbf{k}\, \xi^+(t,-\mathbf{k}) \left(i\hbar\frac{\partial}{\partial t} - \frac{\hbar^2 k^2}{2m} + \mu - i\hbar\gamma \right) \eta(t,\mathbf{k})$$
$$+ 2i \int_{-\infty}^{\infty} dt \int d\mathbf{k}\, \eta^+(t,-\mathbf{k}) \left[(1+2\overline{n})\, \hbar\gamma \right] \eta(t,\mathbf{k})\,.$$

Separation of advanced and retarded propagators allows to prove a loop theorem similar to that in stochastic field theory [3]. The idea is based on two properties of the propagators in the reduction operator (30). First, only retarded and advanced propagators are attached to field arguments η and η^+ in the interaction functional. Second, in the retarded and advanced propagators the lesser time argument is always that of η or η^+. Therefore, in a time-ordered propagator starting from the η or η^+ field of a vertex, the time argument grows from that of the (time-local) vertex.

In the vertices brought about by the interaction functional (32), there is always at least one field η or η^+ which thus gives rise to at least one time-ordered propagator with time direction from the vertex. In interaction vertices there is always at least one field ξ or ξ^+ as well, which makes it possible (but not necessary, in general) to attach a time-ordered propagator to that vertex with the time direction to the vertex. Due to these properties, if in a connected graph there is a time-ordered propagator between two vertices, then in this graph there is necessarily a chain of time-ordered propagators with the same direction of time in all of them. There are now two possibilities: the end points of the chain of time-ordered propagators correspond to external field arguments of a connected Green function or the chain forms a closed loop, in which case the value of the whole graph is equal to zero.

This theorem has an important consequence: one-irreducible graphs with external ξ or ξ^+ arguments only vanish identically, because they are bound to have closed loops of successive step functions in time. This property preserves the field structure of the propagator matrix (30) to all orders in perturbation theory. Of course, this must be so, because the change of variables (29) is based on a transformation of the full propagator matrix in Dyson equations of the model. However, there is more here. The Dyson equation allows to make conclusions about two-point functions. From the loop theorem it follows in particular, that the four-point one-irreducible function $\Gamma_{\xi\xi\xi^+\xi^+} = 0$ so that there are limitations on the generation of vertex structures in the perturbation expansion.

5. The origin of dissipation

It turns out that the attenuation introduced above may be regarded not only as regularization, but has a physical meaning as well. It describes the physical dissipation produced directly by the loop corrections in the present model.

To discuss the origin of dissipation we use the Dyson equation

$$D^{-1} = \underline{\Delta}^{-1} - \Sigma\,.$$

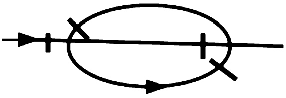
Here, D is the matrix of full propagators, $\underline{\Delta}$ the matrix of bare propagators in the unregularized model

$$\underline{\Delta} = \begin{pmatrix} 0 & -\widetilde{\Delta} \\ \Delta & \Delta^K \end{pmatrix},$$

and Σ contains self-energy graphs.

The one-loop contribution to Σ is not interesting. Due to the presence of $\overline{n}$ these graphs do not have UV divergences and lead to a redefinition of the chemical potential only. It is necessary to consider two-loop graphs. According to the loop theorem of previous Section, the Σ matrix contains the elements $\Sigma_{\eta^+\eta}$, $\Sigma_{\eta^+\xi}$ and $\Sigma_{\xi^+\eta}$.

Let us introduce a momentum cutoff Λ and consider $\Sigma_{\eta^+\xi}$ in two-loop approximation. It consists of diagrams:



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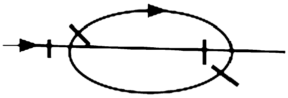

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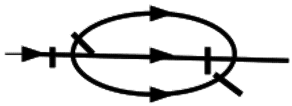


.

Several similar terms in different graphs cancel, whereafter we arrive at the relation (in the wavevector-frequency representation)

$$\Sigma_{\eta^+\xi} = \frac{-g^2}{(2\pi)^6} \int d^3k \int d^3q \int_0^\infty dt \exp\left[-i\left(\frac{\hbar k^2}{2m} - \frac{\hbar q^2}{2m} + \frac{\hbar(\mathbf{k}-\mathbf{q})^2}{2m} - \frac{\mu}{\hbar}\right)t\right]$$
$$\left\{n\left[\epsilon(\mathbf{q})\right] - \frac{1}{2} n\left[\epsilon(\mathbf{k})\right] n\left[\epsilon(\mathbf{k}-\mathbf{q})\right] + n\left[\epsilon(\mathbf{k})\right] n\left[\epsilon(\mathbf{q})\right]\right\}. \quad (34)$$

The expression is presented at zero external frequency and wave vector for simplicity. The existence of dissipation is ensured by the nonvanishing real part of this expression. Using the reference formula

$$\int_0^\infty dt \exp\left\{-i\left[\frac{\hbar k^2}{2m} - \frac{\hbar q^2}{2m} + \frac{\hbar(\mathbf{k}-\mathbf{q})^2}{2m} - \frac{\mu}{\hbar}\right]t\right\}$$
$$= -i\left[\frac{\hbar k^2}{2m} - \frac{\hbar q^2}{2m} + \frac{\hbar(\mathbf{k}-\mathbf{q})^2}{2m} - \frac{\mu}{\hbar}\right]^{-1}$$
$$+ \pi\delta\left(\frac{\hbar k^2}{2m} - \frac{\hbar q^2}{2m} + \frac{\hbar(\mathbf{k}-\mathbf{q})^2}{2m} - \frac{\mu}{\hbar}\right) \quad (35)$$

it may be concluded that the first term of the right side of (35) leads to the divergent in Λ imaginary part of (34) but the second term in the right side of (35) yields the positive convergent real part of $\Sigma_{\eta^+\xi}$.

The matrix of D^{-1} is triangular, therefore

$$\det D^{-1} = -\det D^{-1}_{\eta^+\xi} \det D^{-1}_{\xi^+\eta}.$$

The zeros of this determinant yield the quasi-particle spectrum of the system. $\Sigma_{\xi^+\eta}$ is complex conjugated to $\Sigma_{\eta^+\xi}$ that ensures attenuation of all two-point Green functions in time.

Due to presence of $\overline{n}$, analytic calculations in the present model are rather cumbersome. Therefore, a more detailed analysis of the dissipation problem will be made in Section 6 on the example of infra-red (IR) effective theory with the asymptotic expression for $\overline{n}$.

6. Effective IR theory

Our main task announced above is an investigation of the critical behaviour with the aid of the renormalization group. For the renormalization group analysis it is necessary to construct an effective large-scale model with definite canonical (engineering) dimensions of the fields and the subsequent dimensions of parameters. Prescription of canonical dimensions implies that all propagators of perturbation theory possess the property of generalized homogeneity. The choice of the parameters of the latter then determines the region of dynamic variables in which the asymptotic behaviour is sought. The average occupation number $\overline{n}(\mathbf{p})$ factor is singular in the critical region $p^2 \sim \mu \to 0$ leading to the inequality $\overline{n} \gg 1$. In the original variables determination of canonical dimensions is obscured by the structure of the dynamic propagators $\Delta + d_{DD}$ and $\widetilde{\Delta} + d_{DD}$, in which the thermal contraction d_{DD} yields the dimensionally leading contribution in the IR limit and the dynamics of the model is lost. The situation is quite different in the RAK variables, in which the propagators (16), (17) possess the usual generalized homogeneity of a non-relativistic theory at the outset and approximations are needed in the Keldysh function (31) only.

In IR limit the unregularized Keldysh function

$$\Delta^K_{\mathrm{IR}}(\omega\,, \mathbf{k}) = \frac{4\pi T_C\, \delta\left[\omega - \omega(\mathbf{k})\right]}{\hbar\omega(\mathbf{k})}\,,$$

where T_C is the critical temperature in energy units, is a generalized homogeneous function of the frequency and wave number with the canonical dimension -4 under scaling $\omega \to \lambda^2\omega$, $k \to \lambda k$. Therefore, it will be used to determine canonical dimensions. The renormalization theory will be applied here to the analysis of the large-scale behaviour of the model in a manner similar to that in the theory of critical phenomena and stochastic dynamics [3]. This approach is based on the intimate connection between the UV and IR divergences at the critical dimension (logarithmic theory).

Since there is no action in the unregularized model, canonical dimensions are calculated through the UV exponent (degree of divergence) d_γ of a one-irreducible graph γ of the model [3]

$$d_\gamma = d + 2 + (d-4)V_1 + (d-4)V_2 + dV_3 + dV_4 - \left(\frac{d+2}{2}\right)N_{\eta^+} - \left(\frac{d+2}{2}\right)N_\eta - \left(\frac{d-2}{2}\right)N_{\xi^+} - \left(\frac{d-2}{2}\right)N_\xi\,, \quad (36)$$

where V_i is the number of vertices i (labelled according to (32)) in the one-irreducible graph and N_ζ is the number of external arguments corresponding to the field ζ. From (36) it is immediately seen that the canonical dimensions of the fields are $d_\xi = d_{\xi^+} = d/2 - 1$, $d_\eta = d_{\eta^+} = d/2 + 1$ and the critical dimension of the model is $d_c = 4$. Since the coefficient of the interaction vertices V_3 and V_4 is always positive, these interaction structures are IR irrelevant and are omitted in the effective IR model.

However, the connection between the UV and IR divergences is less straightforward in the regularized model. To see this, consider the leading fluctuation contribution to the one-irreducible function $\Gamma_{\eta^+\eta}$ assuming constant attenuation factor γ. This is a two-loop graph (the fourth item in Table 2 and the only two-loop contribution to $\Gamma_{\eta^+\eta}$) containing three regularized Keldysh functions

$$\Delta^K_{\mathrm{IR\ reg}}(\omega, \mathbf{k}) = \frac{4\gamma T_C}{\hbar\omega(\mathbf{k})\left\{[\omega - \omega(\mathbf{k})]^2 + \gamma^2\right\}}.$$

Omitting irrelevant coefficients we have (here and henceforth $\gamma^{(i)}$ denotes a graph γ with vanishing external frequency and wave vector; the superscript refers to the number of loops)

$$\gamma^{(2)}_{\eta^+\eta} \propto \gamma \int \frac{d\mathbf{q}}{(2\pi)^d} \int \frac{d\mathbf{k}}{(2\pi)^d} \frac{1}{\omega(\mathbf{k})\omega(\mathbf{q})\omega(\mathbf{k}+\mathbf{q})} \frac{1}{\left\{\left[\omega(\mathbf{k}) + \omega(\mathbf{q}) - \omega(\mathbf{k}+\mathbf{q})\right]^2 + 9\gamma^2\right\}}. \tag{37}$$

According to the formal degree of divergence, the Fourier integral contributing to $\Gamma_{\eta^+\eta}$ has the wave–number dimension equal to $d_\gamma = 2(d-4)$. However, in the regularized model things are different. In the critical region we adopt $\omega(\mathbf{k}) \sim \hbar k^2/2m$. In this case the integral (37) is UV convergent at $d < 5$ and below five dimensions the dependence on the attenuation coefficient γ may be obtained by scaling it out to yield

$$\gamma_{\eta^+\eta} \propto \gamma^{d-4} \int \frac{d\mathbf{q}}{(2\pi)^d} \int \frac{d\mathbf{k}}{(2\pi)^d} \frac{1}{\omega(\mathbf{k})\omega(\mathbf{q})\omega(\mathbf{k}+\mathbf{q})} \frac{1}{\left\{\left[\omega(\mathbf{k}) + \omega(\mathbf{q}) - \omega(\mathbf{k}+\mathbf{q})\right]^2 + 9\right\}}. \tag{38}$$

Expression on the right side of (38) shows that its limit $\gamma \to 0$ diverges below four dimensions, which signals that there are IR divergences in the model below four dimensions as expected. The UV behaviour of the integral, however, is quite different from that of the unregularized model. To fix this we use the freedom in the choice of the attenuation parameter γ, which does not need to be an independent of the wave number constant.

In the critical region the temporal behaviour of the system is characterized by critical slowing down, i.e. by vanishing of time derivatives with wave number. Therefore, in order to include this property and preserve in the critical region the IR regularization due to attenuation we should have the attenuation factor vanishing with the wave number. Moreover, in order to be able to use the standard machinery of the RG for the large-scale analysis of the model, we choose the wave–number dependent attenuation factor of the form $\gamma \propto k^2$. From the technical point of view this choice allows to use the theory of UV renormalization and the RG for the analysis of the large-scale behaviour, because it leads to a model in which the IR divergences appear just below the dimension, at which the model is logarithmic. Indeed, if we choose $\gamma = a\omega(\mathbf{k}) \propto k^2$, where a is a positive number, we obtain

$$\gamma^{(2)}_{\eta^+\eta} \propto \int \frac{d\mathbf{q}}{(2\pi)^d} \int \frac{d\mathbf{k}}{(2\pi)^d} \frac{a}{\omega(\mathbf{k})\omega(\mathbf{q})\omega(\mathbf{k}+\mathbf{q})} \times \frac{\left[\omega(\mathbf{k}) + \omega(\mathbf{q}) + \omega(\mathbf{k}+\mathbf{q})\right]}{\left\{\left[\omega(\mathbf{k}) + \omega(\mathbf{q}) - \omega(\mathbf{k}+\mathbf{q})\right]^2 + \left[\omega(\mathbf{k}) + \omega(\mathbf{q}) + \omega(\mathbf{k}+\mathbf{q})\right]^2 a^2\right\}},$$

instead of (37). Now the wave–number dimension of this integral is $2d-8$ as in the unregularized model. This superficial degree of divergence corresponds to the usual UV divergence at four dimensions, to which dimensional regularization is applied under the renormalization.

This expression possesses a finite limit, when $a \to 0$ (see Appendix A) and produces a term to the effective action which completely changes the structure of the perturbation theory in the functional representation. The point is that with the set of original propagators there is no well-defined free part of the action in the functional-integral representation (because the Keldysh function is a solution of the homogeneous free-field equation of motion). However, with the account of the two-loop correction the Keldysh function retains the form of the regularized function, in which a in the numerator is replaced by a finite quantity of the order g^2. In this case the set of propagators may be inverted unambiguously giving rise to a well-defined action in the functional integral.

It should be noted that without the attenuation term the action of the model is fully symmetric with respect to the change of fields $\eta, \eta^+ \longleftrightarrow \xi, \xi^+$, since there is no term $\propto \eta^+\eta$ in that case. Introduction of attenuation breaks this symmetry and allows to introduce different canonical dimensions in accordance with the power-counting expression (36) for the fields directly in the action.

For simplicity, we will use the massless scheme to investigate the critical phenomena, in the model considered it implies $\mu = 0$, so that $\overline{n}(\mathbf{k}) \approx 2mT_C/k^2$. The extra parameters $\hbar$ and T_C can be eliminated from the propagators by the scaling of coordinates, time and fields. This leads to propagators of the effective IR theory in dimensionless variables in the form

$$\begin{aligned}
&\langle \xi(t,\mathbf{k})\xi^+(t',-\mathbf{k})\rangle_0 = \Delta^K_{\mathrm{IR\,reg}}(t,\mathbf{k}) = \frac{2}{k^2} e^{-iuk^2(t-t')-\alpha k^2|t-t'|},\\
&\langle \eta(t,\mathbf{k})\eta^+(t',-\mathbf{k})\rangle_0 = 0,\\
&\langle \xi(t,\mathbf{k})\eta^+(t',-\mathbf{k})\rangle_0 = \Delta_{reg}(t,\mathbf{k}) = \theta(t-t') e^{-iuk^2(t-t')-\alpha k^2|t-t'|},\\
&\langle \eta(t,\mathbf{k})\xi^+(t',-\mathbf{k})\rangle_0 = -\widetilde{\Delta}_{reg}(t,\mathbf{k}) = -\theta(t'-t) e^{-iuk^2(t-t')-\alpha k^2|t-t'|},
\end{aligned} \tag{39}$$

with the dimensionless parameters α and u.

Note that propagators (39) include dissipative factors. According to the analysis of Sec. 5, in the unregularized model dissipation appears as a result of two-loop corrections and it regularizes the apparent IR divergences in diagrams. Therefore, the formally small ($\sim g^2$) parameter occurs in the denominators of loop contributions. This is inconvenient from the point of construction of the perturbation theory. To avoid these problems the dissipative factors were introduced into the regularized propagators at the outset. The dissipation parameter α can be considered an additional regulator, whose smallness is not connected with the parameter of expansion g. With the aid of this regulator it is possible to construct the perturbation expansion in a straightforward manner. The value $\alpha = 0$ corresponds to the initial unregularized model, but it will be shown in the following sections that nonzero α is generated during the renormalization of the theory. Therefore, from the technical point of view we are dealing with "generation terms" [3] added to the basic action of the renormalized model at the outset in order to achieve multiplicative renormalizability. Recall that the dissipative term is chosen in the form αk^2 because we consider massless theory and require both the canonical dimensions of the effective IR model and the connection between IR and UV divergences to hold in the regularized model.

As a result of this analysis we arrive at the effective IR model with the basic action in the scaled variables (integrals implied here and henceforth)

$$\begin{aligned}
S = -4\alpha\eta^+\eta + \eta^+\left[-\frac{\partial}{\partial t} + \nabla^2(ui+\alpha)\right]\xi + \xi^+\left[-\frac{\partial}{\partial t} + \nabla^2(ui-\alpha)\right]\eta \\
- \frac{ig_1}{2}\eta^+\xi^+\xi^2 - \frac{ig_2}{2}\xi^{+2}\eta\xi
\end{aligned} \tag{40}$$

Table 1
Canonical dimensions of the fields and parameters of the effective IR model.

	ξ, ξ^+	η, η^+	α	g
d_Q^ω	$-\frac{1}{2}$	$\frac{1}{2}$	0	2
d_Q^k	$\frac{d}{2}$	$\frac{d}{2}$	0	$-d$
d_Q	$\frac{d}{2}-1$	$\frac{d}{2}+1$	0	$4-d$

with

$$g_1 = g_2 = g\,\frac{\hbar}{T_C}\left(\frac{\sqrt{2\pi}}{\lambda_T}\right)^d ,$$

where $\lambda_T = \sqrt{2\pi\hbar^2/mT_C}$ is the thermal de Broglie wavelength at the critical temperature T_C. In (40) the coupling constants are in fact equal, but the labels are introduced to anticipate the different renormalization of the vertices.

Model (40) is proposed as an alternative to critical dynamics, therefore the notation implies the change of the weight e^{iS} in the functional integral, which is usual in quantum mechanics, to e^S adopted in statistical physics [3,17].

In dynamic models it is convenient to use separate scaling dimensions with respect to temporal and spatial variables with the convention [3] $d_k^k = 1$, $d_\omega^\omega = 1$, $d_\omega^k = d_k^\omega = 0$ and define the full canonical dimension according to the scaling of the parabolic differential operator in action (40): $d_Q = d_Q^k + 2d_Q^\omega$. Canonical dimensions of fields and parameters inferred from the effective IR action are listed in Table 1.

7. Renormalization of the theory. Dissipation

For the purpose of the renormalization group analysis we shall consider model (40) in $4-\varepsilon$ space dimensions. The superficial UV divergences of one-irreducible Green functions Γ of model (40) are determined by the UV exponent (36) at the critical dimension $d_c = 4$, i.e.

$$d_\Gamma^* = 6 - 3N_{\eta^+} - 3N_\eta - N_{\xi^+} - N_\xi \,.$$

The structure of action (40) is such that $N_{\eta^+} + N_{\xi^+} = N_\eta + N_\xi$ (conservation of “charge”). According to the loop theorem of Sec. 4 there are no one-irreducible functions (and, consequently, counterterms) without fields η, η^+, such as $\Gamma_{\xi^+\xi}$, $\Gamma_{\xi^+\xi^+\xi\xi}$ and $\Gamma_{\xi^+\xi^+\xi^+\xi\xi\xi}$. Therefore, the divergent graphs correspond to one-irreducible functions $\Gamma_{\xi^+\eta}$, $\Gamma_{\eta^+\xi}$, $\Gamma_{\eta^+\eta}$, $\Gamma_{\eta^+\xi^+\xi\xi}$ and $\Gamma_{\xi^+\xi^+\xi\xi\eta}$. Thus, Green functions of model (40) can be renormalized by power counting and there are no counterterms of structure different from (40). The renormalized action can be written in the form

$$S = -Z_0\eta\eta^+ + \eta^+\left(-Z_1\frac{\partial}{\partial t} + Z_2\nabla^2\right)\xi + \xi^+\left(-Z_3\frac{\partial}{\partial t} + Z_4\nabla^2\right)\eta$$
$$- Z_5\eta^+\xi^+\xi^2 - Z_6\xi^{+2}\eta\xi \tag{41}$$

with complex renormalization constants $Z_1, ..., Z_6$. Because Z_5 and Z_6 can be different we have introduced two different charges g_1 and g_2 in the action (40).

Under complex conjugation, basic action (40) obeys the symmetries (α is considered a real number)

$$S(\eta^+, \eta, \xi^+, \xi, g_1, g_2) = S^*(-\eta^+, -\eta, \xi^+, \xi, g_2^*, g_1^*) = S^*(\eta^+, \eta, -\xi^+, -\xi, g_2^*, g_1^*).$$

Integration by parts and vanishing of surface terms is implied here. This symmetry leads to the constraints on the renormalization constants

$$Z_0 = Z_0^*, \quad Z_1 = -Z_3^*, \quad Z_2 = -Z_4^*, \quad Z_5 = -Z_6^* \tag{42}$$

where the change $(g_1, g_2) \to (g_2^*, g_1^*)$ on the r.h.s. of equalities is implied.

Calculation of the two-loop counterterm to Z_0 in the minimal subtraction scheme yields a result finite in the limit $\alpha \to 0$ and leads to

$$Z_0 = 4\alpha - \frac{g_1^2}{64\pi^4 \left(u^2+\alpha^2\right)\varepsilon}\left\{\pi u + 2u \arctan\left(\frac{u}{4\alpha} - \frac{3\alpha}{4u}\right)\right.$$
$$\left. + \alpha \log\left[\frac{4096\,\alpha^8}{\left(u^2+\alpha^2\right)\left(u^2+9\alpha^2\right)^3}\right]\right\},$$

where $g_2 = g_1$ is implied. Details of calculation can be found in Appendix A. This expression shows that Z_0 tends to a finite value in the limit $\alpha \to 0$. This is a demonstration of the property that the parameter α – introduced as a regulator – is generated in the process of renormalization.

The common hypothesis about the connection between dissipation in the hydrodynamic limit and the influence of hard modes to soft modes is here substantiated by the fact that the attenuation parameter α is generated due to UV renormalization. It may be said that the UV renormalization is a method to take into account the influence of hard modes on the soft in the sense of the Kadanoff transformation.

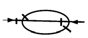
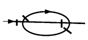
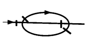
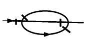
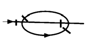
We have introduced different coupling constants g_1, g_2 for different vertices with the unrenormalized real-number values $g_1 = g_2$. To use the symmetry (42), we restrict our analysis to a hypersurface $g_1 = g_2^*$ for the time being. From relations (42) it follows that the action can be multiplicatively renormalized by the renormalization constants Z_η, Z_{η^+}, Z_ξ, Z_{ξ^+}, Z_{g1}, Z_α, Z_u. Moreover, the renormalization constants Z_α and Z_u are real. These constants are connected with the counterterms by equations

$$4\alpha Z_\alpha Z_{\eta^+} Z_\eta = Z_0,$$
$$Z_{\eta^+} Z_\xi = Z_1, \quad Z_\eta Z_{\xi^+} = Z_1^*,$$
$$Z_\alpha Z_{\eta^+} Z_\xi \alpha + i Z_{\eta^+} Z_\xi Z_u u = Z_2,$$
$$\frac{i g_1}{2} Z_{\eta^+} Z_\xi^2 Z_{\xi^+} Z_{g_1} = Z_5.$$

These equations can be resolved as

$$Z_{\eta^+} = Z_\eta^*, \quad Z_{\xi+} = Z_\xi^*, \quad Z_{\eta^+} Z_\xi = Z_1,$$
$$Z_\alpha = \frac{1}{\alpha}\mathrm{Re}(Z_2 Z_1^{-1}), \quad Z_{\eta^+} Z_\eta = \frac{1}{4\alpha} Z_0 Z_\alpha^{-1}$$
$$Z_u = \frac{1}{u}\mathrm{Im}(Z_2 Z_1^{-1}), \quad Z_{g_1} = -\frac{i}{2 g_1 \alpha} Z_5 Z_0 Z_1^{-2} (Z_1^*)^{-1} Z_\alpha^{-1}.$$

Table 2
Leading-order contributions to renormalization in the MS scheme. The diagrams are depicted in the first column, in the second the symmetry coefficients (S.C.) are quoted. The third column indicates the renormalization constant to which the diagram contributes. In the fourth column values of the pole parts of the diagrams are quoted in the normalization of propagators and vertices corresponding to the basic action (40). The product of the value in the fourth column and the symmetry coefficient of the second column yields the contribution of the graph to the counterterm of the renormalized action (41). Contributions to Z_1 are the coefficients of $i\omega$ and contributions to Z_2 are the coefficients of k^2 in the Maclaurin expansion of diagrams as functions of external frequency and wave number. Expressions for M_0, M_2 and M_3 too lengthy to fit the table are quoted in (43), (44) and (45), respectively.

Diagram	S.C.	Z_i	Value
	$\frac{1}{2}$	Z_5	$-\frac{g_1^2}{(iu+\alpha)8\pi^2\varepsilon}$
	1	Z_5	$-\frac{g_1^2}{8\alpha\pi^2\varepsilon}$
	1	Z_5	$\frac{g_1 g_1^*}{8\alpha\pi^2\varepsilon}$
	$\frac{1}{2}$	Z_0	$-\frac{2g_1 g_1^*}{(8\pi^2)^2\varepsilon}\frac{M_0}{8}$
	$\frac{1}{2}$	Z_1	$-\frac{g_1 g_1^*}{(8\pi^2)^2\varepsilon}\frac{M_2}{4}$
	$\frac{1}{2}$	Z_2	$-\frac{g_1 g_1^*}{(8\pi^2)^2\alpha\epsilon}\left[\frac{(u^2+3\alpha^2-2\alpha ui)}{2(u^2+9\alpha^2)}\right]$
	1	Z_1	$\frac{g_1^2}{(8\pi^2)^2\varepsilon}\frac{M_3}{4}$
	1	Z_2	$\frac{g_1^2}{(8\pi^2)^2\alpha\varepsilon}\left[\frac{(u^2+6\alpha^2-i\alpha u)}{4(u^2+9\alpha^2)}\right]$

There is an obvious symmetry in the initial action (40). Arbitrariness in the phases of renormalization constants of fields is a remnant of this symmetry.

We consider the renormalization of model (40) in space dimension $4-\varepsilon$ using dimensional regularization and the minimal subtraction (MS) scheme. Renormalization constants then have a form (1+ poles in ε).

The tadpole graphs are equal to zero in the framework of dimensional regularization and massless theory. Contributions of diagrams to the leading order renormalization (i.e. counterterms) in the MS scheme are presented in Table 2. Counterterms brought about by the watermelon graphs are coefficients of $i\omega$ and k^2 in the Mclaurin expansion of the expressions in the Fourier space.

After integration over the time variable divergences of diagrams were calculated with zero external wave number and a sharp IR cutoff in the last wave-vector integral. Due to the presence of both real and imaginary parts in the exponentials of propagators (39), the calculation of wave-vector integrals cannot be carried out in the fashion used in stochastic dynamics. Examples of calculation are briefly explained in Appendix A.

Explicit expressions too lengthy to fit Table 2 are the following:

$$M_0 = \frac{8}{\left(u^2+\alpha^2\right)}\left[\pi\, u + 2\, u\ \arctan\left(\frac{u}{4\alpha} - \frac{3\alpha}{4\, u}\right) + \alpha\ \log\frac{4096\alpha^8}{\left(u^2+\alpha^2\right)\left(u^2+9\alpha^2\right)^3}\right]. \quad (43)$$

$$\begin{aligned} M_2 = \frac{1}{\left(u^2+\alpha^2\right)^2}\Bigg(&2\alpha u\left[-\pi + 2\ \arctan\left(\frac{u}{2\alpha}+\frac{3\alpha}{2u}\right)\right] \\ &+ \left(u^2-\alpha^2\right)\log\frac{16\alpha^4}{\left(u^2+\alpha^2\right)\left(u^2+9\alpha^2\right)} \\ &- i\Bigg\{(u^2-\alpha^2)\left[\pi - 2\ \arctan\left(\frac{u}{2\alpha}+\frac{3\alpha}{2u}\right)\right] \\ &+ 2\alpha u\ \log\frac{16\alpha^4}{\left(u^2+\alpha^2\right)\left(u^2+9\alpha^2\right)}\Bigg\}\Bigg). \quad (44) \end{aligned}$$

$$\begin{aligned} M_3 = \frac{2}{\left(u^2+\alpha^2\right)^2}\Bigg\{&4\alpha u\ \arctan\frac{u}{3\alpha} - \left(u^2-\alpha^2\right)\log\frac{16\alpha^2}{u^2+9\alpha^2} \\ &- 2i\left[(u^2-\alpha^2)\ \arctan\frac{u}{3\alpha} + \alpha u\ \log\frac{16\alpha^2}{u^2+9\alpha^2}\right]\Bigg\}. \quad (45) \end{aligned}$$

8. Conclusion

In conclusion, we have analyzed the problem of first-principles construction of an effective large-scale model for weakly interacting boson gas in the vicinity of the critical point. We have used time-dependent Green functions at finite temperature to describe the dynamics of the microscopic model near equilibrium. The construction of perturbation theory has been carried out with the use of the functional form of Wick's theorems without any resort to a complex time variable, although resulting expressions may be interpreted in this framework as well. We have pointed out apparent IR divergences in individual Feynman graphs of the perturbation expansion, introduced a regularization in the form of temporal attenuation of propagators to deal with a finite model and analyzed the effect of these divergences at leading orders of both the original model and the proposed effective large-scale model.

It has been shown that the dissipative terms corresponding to this regularization are brought about by loop corrections of the perturbation expansion. These terms provide the time symmetry breaking leading to definite asymptotic behaviour at large scales. This produces the unambiguous representation of the renormalized perturbation theory in terms of a standard functional integral with fields vanishing at infinity, which is not possible in the original model.

As a result of this analysis, we have put forward a model which allows to use the quantum-field renormalization group to investigate the large-scale behaviour of the system. The renormalization-group problem is more difficult than usually due to the presence of a non-perturbative charge u. Therefore, the fixed points of the RG equations cannot be found analytically. Moreover, a similar analysis in models E and F of critical dynamics shows that the leading-order approximation is not sufficient to determine an IR stable fixed point. We are working on the RG analysis of the problem and hope to present detailed analysis in the near future.

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Appendix A. Calculation of Feynman diagrams

Consider the two-loop graph giving rise to a contribution in Z_0 (fourth item in Table 2). Omitting coupling constants and with the use of propagators (39) from the integral over the time variable we obtain

$$\gamma_{\eta^+\eta} = \int\limits_{q>m} \frac{d\mathbf{q}}{(2\pi)^d} \int \frac{d\mathbf{k}}{(2\pi)^d} \frac{1}{k^2 q^2 (\mathbf{k}-\mathbf{q})^2} \times \frac{16\left[k^2+(\mathbf{k}-\mathbf{q})^2+q^2\right]\alpha}{\left[k^2-q^2+(\mathbf{k}-\mathbf{q})^2\right]^2 u^2 + \left[k^2+(\mathbf{k}-\mathbf{q})^2+q^2\right]^2 \alpha^2},$$

where the inner $\mathbf{k}$ integral is taken over the whole wave-vector space. Due to symmetry of the integrand, in the decomposition

$$\frac{\left[k^2+(\mathbf{k}-\mathbf{q})^2+q^2\right]}{k^2 q^2 (\mathbf{k}-\mathbf{q})^2} = \frac{1}{k^2(\mathbf{k}-\mathbf{q})^2} + \frac{1}{(\mathbf{k}-\mathbf{q})^2 q^2} + \frac{1}{k^2 q^2}$$

the last two terms give rise to equal integrals and effectively we may replace

$$\frac{\left[k^2+(\mathbf{k}-\mathbf{q})^2+q^2\right]}{k^2 q^2 (\mathbf{k}-\mathbf{q})^2} \to \frac{1}{k^2(\mathbf{k}-\mathbf{q})^2} + \frac{2}{k^2 q^2}.$$

It is convenient to arrange the dependence on the scalar product $\mathbf{p}\cdot\mathbf{q} = pq\cos\theta$ in the simplest form by making the shift $\mathbf{k} \to \mathbf{k} + \mathbf{q}/2$. The result is

$$\gamma_{\eta^+\eta} = 64\,\alpha \int\limits_{q>m} \frac{d\mathbf{q}}{(2\pi)^d} \int \frac{d\mathbf{k}}{(2\pi)^d} \left[\frac{1}{(\mathbf{k}+\mathbf{q}/2)^2\,(\mathbf{k}-\mathbf{q}/2)^2} + \frac{2}{(\mathbf{k}-\mathbf{q}/2)^2 q^2}\right] \times \frac{4}{16k^4\left(u^2+\alpha^2\right) + 8k^2q^2\left(-u^2+3\alpha^2\right) + q^4\left(u^2+9\alpha^2\right)}. \quad \text{(A.1)}$$

Having pulled out the q dependence by the scaling $\mathbf{k} \to (q/2)\mathbf{k}$ we readily calculate the q integral:

$$\int\limits_{q>m} \frac{d\mathbf{q}}{(2\pi)^d} \frac{1}{q^{8-d}} = \frac{S_d}{(2\pi)^d} \frac{m^{2d-8}}{8-2d} \approx \frac{1}{16\pi^2 \varepsilon},$$

where S_d is the surface area of a sphere of unit radius in d dimensional space and $\varepsilon = 4-d$.

The remaining $\mathbf{k}$ integral has a finite limit, when $d \to 4$. Therefore, at the leading order of the $\varepsilon = 4-d$ expansion it is sufficient to calculate it in four dimensions. In this case in the angular part of the $\mathbf{k}$ integral the integral over the angle between $\mathbf{k}$ and $\mathbf{q}$ is taken by the reference formula [22]

$$\int\limits_0^\pi \frac{\sin^2\theta\, d\theta}{p^2+1-2p\cos\theta} = \begin{cases} \dfrac{\pi^2}{2}, & p<1, \\ \dfrac{\pi^2}{2p^2}, & p>1. \end{cases}$$

The integrand is independent of the other two angles, thus the angular integral over them yields 4π.

After the change of variables $k^2 = t$ we arrive at the sum of two rational integrals, which may be taken with the use of reference literature [22] and/or some (computer) algebra:

$$
\begin{aligned}
\gamma_{\eta^+\eta} = \frac{2\alpha}{(8\pi^2)^2\varepsilon}\Bigg[\int_0^1 \left(1+\frac{2}{1+t}\right)\frac{t^2\,dt}{u^2+9\alpha^2+t^2\left(u^2+\alpha^2\right)+t\left(-2u^2+6\alpha^2\right)} \\
+\int_1^\infty \left(1+\frac{2}{1+t}\right)\frac{t\,dt}{u^2+9\alpha^2+t^2\left(u^2+\alpha^2\right)+t\left(-2u^2+6\alpha^2\right)}\Bigg] + O(1) \\
= \frac{1}{32\pi^4\left(u^2+\alpha^2\right)\varepsilon}\Bigg\{\pi u + 2u\,\arctan\left(\frac{u}{4\alpha}-\frac{3\alpha}{4u}\right) \\
+\alpha\,\log\frac{4096\,\alpha^8}{\left(u^2+\alpha^2\right)\left(u^2+9\alpha^2\right)^3}\Bigg\} + O(1)\,.
\end{aligned}
$$

In a similar fashion the integral giving the contribution of the fifth item in Table 2 to the renormalization constant Z_1 may be calculated as

$$
\begin{aligned}
-i\frac{\partial\gamma^A_{\eta^+\xi}}{\partial\omega}\bigg|_{\omega=0} = \int_{q>m}\frac{d\mathbf{q}}{(2\pi)^d}\int\frac{d\mathbf{k}}{(2\pi)^d}\frac{1}{k^2q^2} \\
\times\frac{4}{\left[k^2(\alpha+iu)+q^2(\alpha+iu)+(\mathbf{k}-\mathbf{q})^2(\alpha-iu)\right]^2} \\
= -\frac{1}{128\pi^4\left(u^2+\alpha^2\right)^2\varepsilon}\Bigg\{2\alpha u\left[-\pi+2\,\arctan\left(\frac{u}{2\alpha}+\frac{3\alpha}{2u}\right)\right] \\
+\left(u^2-\alpha^2\right)\log\left[\frac{16\alpha^4}{\left(u^2+\alpha^2\right)\left(u^2+9\alpha^2\right)}\right]\Bigg\} \\
+\frac{i}{128\pi^4\left(u^2+\alpha^2\right)^2\varepsilon}\Bigg\{(u^2-\alpha^2)\left[\pi-2\,\arctan\left(\frac{u}{2\alpha}+\frac{3\alpha}{2u}\right)\right] \\
+2\alpha u\,\log\left[\frac{16\alpha^4}{\left(u^2+\alpha^2\right)\left(u^2+9\alpha^2\right)}\right]\Bigg\} + O(1). \qquad \text{(A.2)}
\end{aligned}
$$

The integral giving the contribution of the seventh item in Table 2 to the renormalization constant Z_1 is

$$
\begin{aligned}
-i\frac{\partial\gamma^R_{\eta^+\xi}}{\partial\omega}\bigg|_{\omega=0} = -\int_{q>m}\frac{d\mathbf{q}}{(2\pi)^d}\int\frac{d\mathbf{k}}{(2\pi)^d}\frac{1}{k^2q^2} \\
\times\frac{4}{\left[k^2(\alpha-iu)+q^2(\alpha+iu)+(\mathbf{k}-\mathbf{q})^2(\alpha+iu)\right]^2}
\end{aligned}
$$

$$= -\frac{1}{128\,\pi^4 \left(u^2+\alpha^2\right)^2 \varepsilon}\left\{4\,\alpha u \,\arctan\frac{u}{3\,\alpha} - \left(u^2-\alpha^2\right)\log\left(\frac{16\,\alpha^2}{u^2+9\,\alpha^2}\right)\right\}$$
$$+\frac{i}{64\,\pi^4\,(u^2+\alpha^2)^2\,\varepsilon}\left\{(u^2-\alpha^2)\arctan\frac{u}{3\,\alpha}+\alpha u\,\log\left(\frac{16\,\alpha^2}{u^2+9\,\alpha^2}\right)\right\}+O(1) \quad \text{(A.3)}$$

It should be noted that if we put $\alpha = 0$ in the denominators of the wave-vector integrands in (A.1), (A.3) and (A.2), we arrive at expressions of the type

$$\left[k^2(\alpha - iu) + q^2(\alpha - iu) + (\mathbf{k}-\mathbf{q})^2(\alpha + iu)\right]^2\Big|_{\alpha=0} = -4\,(\mathbf{k}\cdot\mathbf{q})^2\,u^2$$

which change the UV behaviour of the integrand in such way that – apart from the logarithmic UV divergence of the twofold wave-vector integral – the wave vector integrals over $\mathbf{k}$ and $\mathbf{q}$ separately become logarithmically divergent at four dimensions. In dimensional regularization this means that instead of the first-order pole in ε the leading singularity in these integrals is a second-order pole in ε. This is quite unusual, because the divergent $\mathbf{k}$ and $\mathbf{q}$ integrals do not correspond to any divergent subgraphs of the superficially divergent two-loop graphs. Therefore, it seems that in the perturbation theory of original model (i.e. without the attenuation of propagators) the standard diagrammatic approach to construction of UV renormalization does not work. In relativistic-invariant particle field theories this kind of problem does not appear.

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