DISSERTATIO GRADUALIS,
THEOREMATA EXHIBENS *GENERALIA INVENIENDIS RESIDUIS, EX DIVISIONE NUMERORUM ORIUNDIS, INSERVIENTIA.*

QUAM,

CONS. AMPL. FACULT. PHILOS. ABOËNS.

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9.

In variis disquisitionibus Arithmeticis haud parum interest nosse: quodnam peracta divisione numeri integri N per integrum D emergat residuum R. Cujus quidem Problematis solutio exhibet criteria, ex quibus dijudicari poterit, utrum numerus datus N dati D sit dividuus nec ne. Per se enim patet, ubi fuerit $R = 0$, fore, ut numerum N metiatur numerus D. Praeterea ex cognita generali residua definiendi methodo id capiet commodi computator, ut cavendo semet a tentandis divisionibus, plerumque scilicet vel frustraneis vel supervacaneis, tempori non minus quam operae insigniter parcere queat. Quare maxime expediet in hoc quaestionum genere enodando & ad theoriam exigendo elaborare. Neque vero hoc a computandi artificibus prorsus est neglectum. Occurrunt nempe quaedam ab ipsis tradita huc spectantia praecepta, nimis licet specialia, utpote pro paucissimis tantum iisdemque minimis divisoribus adornata. Quae ipsa quoque cum sine ullis plerumque demonstrationibus exposita deprehendantur; placuit nobis, Specimen Academicum edituris, universae huic materiae accuratius pertractandae manum admovere, eandemque ad principia, quantum fieri potuit, generalia revocare. Quo in opere num quid nostra valuerit industria, perlectis hisce pagellis aequo cuique Lectori judicandi erit locus.

§ 1.

Denotante A numerum quemcunque integrum unitate majorem facillime patet, numerum quemvis integrum N exhiberi posse per seriem secundum potestates positivas integras ipsius A procedentem: $a_n A^n + a_{n-1} A^{n-1} \ldots + a_1 A + a_0 A^0$, ita videlicet comparatam, ut sint coëfficientes $a_n, a_{n-1}, \ldots a_1, a_0$ numeri integri ipso A minores.

Brevitatis gratia formulam hanc: $a_n A^n + a_{n-1} A^{n-1} \ldots + a_1 A + a_0 A^0$ in sequentibus expressisse juvabit per $(a_n, a_{n-1}, \ldots a_1, a_0)$.

Præterea, ne ullus ambiguitati relinquatur locus, monendi sunt Lectores, ne valores coëfficientium indeterminatorum $a_n, a_{n-1} \ldots a_1, a_0$ ab adjectis litteræ a indicibus n, $n-1$, &c. aliquatenus pendere putent, neve obliviscantur hos characteres eo tantum consilio adhiberi, ut, quis cuique coëfficientium in serie competat locus, ultro pateat.

§ 2.

Ex adoptata (§. præced.) designandi lege erit, ob identitatem formularum: $a_n A^n + a_{n-1} A^{n-1} \ldots + a_1 A + a_0 A^0$ atque $(a_n, a_{n-1}, \ldots a_1, a_0)$, $A\ (= 1\,A + 0\,A^0) = (1,0)$, A^2 $(= 1\,A^2 + 0\,A + 0\,A^0) = (1,0)^2 = (1,0,0)$, A^3 $(= 1\,A^3 + 0\,A^2 + 0\,A + 0\,A^0) = (1,0)^3 = (1,0,0,0)$ atque generatim $A^n = (1,0)^n = (1,0,0,\ldots)$, subsequentibus videlicet notam unitatis tot cyphris, quot continet index n unitates. Unde $a_1 A = a_1 (1,0) = (a_1, 0)$, $a_2 A^2 = a_2 (1,0)^2 = (a_2, 0, 0)$, $a_3 A^3 = a_3 (1,0)^3 = (a_3, 0, 0, 0)$ nec non $a_n A^n = a_n (1,0)^n$.

Sic *e. gr.* facto $n = 8$, habebitur
$N = (a_8, a_7, a_6, a_5, a_4, a_3, a_2, a_1, a_0)$

$= a_8$

$$
\left.
\begin{array}{rl}
= & a_8\,(1,0)^8 \\
+ & a_7\,(1,0)^7 \\
+ & a_6\,(1,0)^6 \\
+ & a_5\,(1,0)^5 \\
+ & a_4\,(1,0)^4 \\
+ & a_3\,(1,0)^3 \\
+ & a_2\,(1,0)^2 \\
+ & a_1\,(1,0) \\
+ & a_0
\end{array}
\right\}
\begin{array}{rr}
= & (a_8,0,0,0,0,0,0,0,0) \\
+ & (a_7,0,0,0,0,0,0,0) \\
+ & (a_6,0,0,0,0,0,0) \\
+ & (a_5,0,0,0,0,0) \\
+ & (a_4,0,0,0,0) \\
+ & (a_3,0,0,0) \\
+ & (a_2,0,0) \\
+ & (a_1,0) \\
+ & a_0.
\end{array}
$$

Ex hoc exemplo facile patet, ſub generali noſtro Schemate contineri vulgarem illam, quæ in recepto dudum computandi Syſtemate Decadico valet, numeros exhibendi rationem. Quippe in hoc Syſtemate numerus A, denarium denotans, exprimitur per 10, ejusque poteſtates A^2, A^3, &c. per 100, 1000 &c. reſpective, numerus vero N, ſi e. gr. ſumatur $= 5A^4 + 8A^3 + 7A^2 + 9A + 2 = (5,8,7,9,2)$, tranſit hoc pacto in 58792.

Nos quidem, ne ullum formulæ generalis, pro numero indefinito A adornatæ, cum ſpecialisſimis ejusdem valoribus, ex determinato A (vices nimirum denarii ſuſtinente) oriundis, confundendæ periculum ſuboriretur, inter proximos quosque characteres numericos ſignum hoc: , interjiciendum cenſuimus.

§. 3.

Si fuerint s & m numeri integri poſitivi & quidem sm non $> n$; vi receptæ a nobis deſignandi methodi obtinebitur

$$\begin{aligned} N &= (a_n, a_{n-1}, \ldots a_1, a_0) \\ &= (a_n (1,0)^{n-sm} + a_{n-1} (1,0)^{n-sm-1} \ldots + a_{sm}) (1,0)^{sm} \\ &+ (a_{sm-1} (1,0)^{m-1} + a_{sm-2} (1,0)^{m-2} \ldots + a_{(s-1)m}) (1,0)^{(s-1)m} \\ &+ \ldots\ldots\ldots\ldots \\ &+ (a_{2m-1} (1,0)^{m-1} + a_{2m-2} (1,0)^{m-2} \ldots + a_m) (1,0)^m \\ &+ (a_{m-1} (1,0)^{m-1} + a_{m-2} (1,0)^{m-2} \ldots + a_0) \\ &= (a_n, a_{n-1}, \ldots a_{sm}) (1,0)^{sm} \\ &+ (a_{sm-1}, a_{sm-2}, \ldots a_{(s-1)m}) (1,0)^{(s-1)m} \\ &+ \ldots\ldots\ldots\ldots \\ &+ (a_{2m-1}, a_{2m-2}, \ldots a_m) (1,0)^m \\ &+ (a_{m-1}, a_{m-2}, \ldots a_0) \quad (I). \end{aligned}$$

Sumto *e. gr.* $n = 8$, erit $N = (a_8, a_7, \ldots a_1, a_0)$ (pro $m = 1$ & $s = 8$, ut in §. 2)

$$\text{(pro } m=2 \text{ \& } s=4) = \begin{array}{r} a_8 (1,0)^8 \\ + (a_7, a_6)(1,0)^6 \\ + (a_5, a_4)(1,0)^4 \\ + (a_3, a_2)(1,0)^2 \\ + (a_1, a_0) \end{array} \left.\begin{array}{l} \\ \\ \\ \\ \\ \end{array}\right\} = \begin{array}{r} (a_8, 0, 0, 0, 0, 0, 0, 0, 0) \\ + (a_7, a_6, 0, 0, 0, 0, 0, 0) \\ + (a_5, a_4, 0, 0, 0, 0) \\ + (a_3, a_2, 0, 0) \\ + (a_1, a_0); \end{array}$$

$$\text{(pro } m=3 \text{ \& } s=2) = \begin{array}{r} (a_8, a_7, a_6)(1,0)^6 \\ + (a_5, a_4, a_3)(1,0)^3 \\ + (a_2, a_1, a_0) \end{array} \left.\begin{array}{l} \\ \\ \\ \end{array}\right\} = \begin{array}{r} (a_8, a_7, a_6, 0, 0, 0, 0, 0, 0) \\ + (a_5, a_4, a_3, 0, 0, 0) \\ + (a_2, a_1, a_0); \end{array}$$

$$\text{(pro } m=4 \text{ \& } s=2) = \begin{array}{r} a_8 (1,0)^8 \\ + (a_7, a_6, a_5, a_4)(1,0)^4 \\ + (a_3, a_2, a_1, a_0) \end{array} \left.\begin{array}{l} \\ \\ \\ \end{array}\right\} = \begin{array}{r} (a_8, 0, 0, 0, 0, 0, 0, 0, 0) \\ + (a_7, a_6, a_5, a_4, 0, 0, 0, 0) \\ + (a_3, a_2, a_1, a_0); \end{array}$$

$$\text{(pro } m=5 \text{ \& } s=1) = \begin{array}{r} (a_8, a_7, a_6, a_5)(1,0)^5 \\ + (a_4, a_3, a_2, a_1, a_0) \end{array} \left.\begin{array}{l} \\ \\ \end{array}\right\} = \begin{array}{r} (a_8, a_7, a_6, a_5, 0, 0, 0, 0, 0) \\ + (a_4, a_3, a_2, a_1, a_0); \end{array}$$

& sic porro.

§. 4.

THEOREMA, Retento pro N valore per formulam (I) exhi-

exhibito fiat $N(m,r) = (a_n, a_{n-1}, \ldots a_{sm})\, r^s + (a_{sm-1}, a_{sm-2}, \ldots a_{(s-1)m})\, r^{s-1} \ldots + (a_{2m-1}, a_{2m-2}, \ldots a_m)\, r + (a_{m-1}, a_{m-2}, \ldots a_0)$ (II); dico fore, ut numerus $(1,0)^m - r$ metiatur numerum $N - N(m,r)$.

Veritas hujusce Theorematis sequenti evincitur ratiocinio. Posito $(1,0)^m - r = M$ (unde $(1,0)^m = M + r$), substituatur in formula (I) $M + r$ pro $(1,0)^m$. Unde, evolutis potestatibus binomii $(M + r)$ nec non collectis in unam summam, quæ dicatur Mq, terminis factore communi M gaudentibus, obtinebitur
$N = (a_n, a_{n-1}, \ldots a_{sm})\, r^s + (a_{sm-1}, a_{sm-2}, \ldots a_{(s-1)m})\, r^{s-1} + \ldots + (a_{2m-1}, a_{2m-2}, \ldots a_m)\, r + (a_{m-1}, a_{m-2}, \ldots a_0) + Mq = N(m,r) + Mq$. Quamobrem erit $N - N(m,r) = Mq$ dividuus numeri M, h. e. ipsius $(1,0)^m - r$.

§. 5.

Quod si ponatur $N(m,r) = N'$ & dehinc progrediaris ad novos terminos: $N'' = N'(m,r)$, $N''' = N''(m,r)$, &c. quorum quisque a proxime præcedenti eadem lege formetur ac N' a N, ponendo videlicet r pro $(1,0)^m$; ope Theorematis præced. facile conficitur, fore singulos hosce numeros: $N - N''$ ($= (N - N') + (N' - N'')$), $N - N'''$ ($= (N - N'') + (N'' - N''')$), &c. dividuos numeri $(1,0)^m - r$.

Observandum præterea est, hanc seriem non posse non convergere, quoties fuerit $N > (1,0)^m$ & $r < (1,0)^m$, & quidem eo citius, quo minores fuerint numeri m & r, ita tamen, ut continuari nequeat nisi usque dum incideris in terminum numero $(1,0)^m$ minorem.

§. 6.

THEOREMA. Sit D factor quicunque numeri $(1,0)^m - r$; nec non residuum, ex divisione numeri N per D oriundum $= R$, singuli termini seriei: N', N'', N''' &c., per eundem D divisi, idem exhibebunt residuum R.

Primo quidem est (Hyp.) $N = Dq + R$, sumto quoto ex divisione ipsius N per D prodeunte $= q$. Praeterea, cum sit numerus $N - N'$ ($= N - N(m, r)$). dividuus numeri $(1, 0)^m - r$ (§ 5) ideoque etiam ipsius D; licebit statuere $N - N' = Dq'$, adhibito pro q' numero integro. Quibus concessis liquet fore $N' = N - (N - N') = Dq + R - Dq'$ $= D(q - q') + R$.

Simili probabitur ratiocinio, ex reliquis quoque terminis N'', N''' &c. peracta per numerum D divisione, idem emergere residuum R.

§. 7.

Sicubi evenerit, ut in residua negativa incideris, id quidem non offendat. Quia enim est $Dq - R = D(q - 1)$ $+ D - R = D(q - 1) + R'$; facillimus est a residuo negativo $-R$ ad positivum R' transitus. Quippe erunt haec residua, nisi signi diversitatem respexeris, alterum alterius complementum ad ipsum divisorem D, ob $R' = D - R$.

Sic *e. gr.* sumto $D = 5$, facile liquet, residuorum habito respectu hasce exsurgere formas: $5q + 1$, $5q - 4$; $5q + 2$, $5q - 3$; $5q + 3$, $5q - 2$; $5q + 4$, $5q - 1$; quarum binae quaeque omnino reciprocantur.

§. 8.

§. 8.

Exhibito numero quocunque N per seriem secundum potestates denarii procedentem, ut in calculis numericis vulgo solet, erit, vi Theorematis §. 6 traditi, numerus quivis D, qui metitur numerum $10^m - r$, ita comparatus, ut ex divisis per D numeris: N, N', N'', N''' &c. idem exoriatur residuum R.

Ex. I. Sit $N = 4836985$ atque $10^m - r = 99 = 10^2 - 1$. Unde $m = 2$ & $r = 1$. Quibus adhibitis valoribus obtinebitur $N(2,1) = 4 + 83 + 69 + 85 = 241 = N'$, nec non $N'(2,1) = 2 + 41 = 43 = N''$. Erunt vero numeri jam eruti 241, 43 ejus indolis, ut peracta divisione per factores singulos ipsius 99, qui sunt: 3, 9, 11, 33, 99, eadem exhibeant residua ac ipse $N = 4836985$ Quare sufficiet residua ex minimo horum, puta 43, oriunda investigasse. At cum facillime pateat, hunc numerum comprehendi sub formis: $3q + 1$, $9q + 7$, $11q + 10$, $33q + 10$, $99q + 43$; idem quoque de ceteris est asserendum.

Ex. 2. Sumto $N = 9103297$ fiat $m = 2$ & $r = -2$, ita ut habeatur $10^m - r = 10^2 + 2 = 102 = 2.3.17$ Quo pacto erit $N(2,-2) = -9.8 + 10.4 - 32.2 + 97 = -72 + 40 - 64 + 97 = 1$. Quia vero unitas, tentata per quemcunque divisorem divisione, relinquit residuum $= 1$; hinc liquet, esse numerum propositum 9103297 ejus naturae, ut divisus per quemvis factorem ipsius 102, (unum scilicet ex hisce numeris: 2, 3, 6, 17, 34, 51, 102) exhibeat residuum $R = 1$.

Ex. 3. Fiat $N = 24263$ & $10^m - r = 10^2 - 5 = 95 = 5.19$. Quo in casu est $m = 2$ & $r = 5$. Hinc vero emergit

$N(2,5)$

$$N(2,5) = \left.\begin{matrix} 2.25 \\ +\,42.5 \\ +\,63 \end{matrix}\right\} = \left.\begin{matrix} 50 \\ +\,210 \\ +\,63 \end{matrix}\right\} = 323;\quad N'(2,5) = \left.\begin{matrix} 3.5 \\ +\,23 \end{matrix}\right\} = \begin{matrix} 15 \\ +\,23 \\ =38. \end{matrix}$$

Est jam $N'' = 38$ sub formis: $5q + 3$, $19q$, nec non $95q + 38$. Unde colligitur ad easdem pertinere formas $N' = 323$ atque $N = 24263$.

§. 9.

Exhibeat numerum N series quæcunque secundum potestates sexagenarii integras positivas adornata. Quo pacto (§. 6) liquet posse formari seriem N', N'', N''' &c. ejus naturæ, ut termini singuli, divisi per factorem quemcunque D numeri $60^m - r$, idem relinquant residuum R.

Ex. 1. Proposito Arcu $81^s\ 8^\circ\ 13'\ 12''$ fiat $(1,0)^m - r = 60^2$. Unde erit in formula (II) $m = 2$ & $r = 0$. Jam vero est, in systemate Sexagenorum (sumto minuto secundo $= 1$), $(1,0) = 1'$, $(1,0)^2 = 1^\circ$, $(1,0)^3 = 60^\circ$ $(= 2.30^\circ) = 2^s$. Quare, ob 81^s $(= 40.2^s + 30^\circ) = 40(1,0)^3 + 30(1,0)^2$, erit in hoc exemplo $a_3 = 40$, a_2 $(= 30 + 8) = 38$, $a_1 = 13$, $a_0 = 12$; ideoque $N = (40, 38, 13, 12)$. Quibus substitutis valoribus eruitur $N(2,0) = (13, 12'') = 13'\ 12''$. Quo igitur constet, quænam emergant residua ex diviso Arcu proposito per singulos factores numeri $(1,0)^2 = 3600$, peragenda restat divisio Arcus $13'\ 12'' = 792''$ per eosdem divisores. Sunt vero hi (ob $60^2 = (2^2 \cdot 3 \cdot 5)^2$): 2, 3, 4, 5, 6, 8, 9, 10, 12, 15, 16, 18, 20, 24, 25, 30, 36, 40, 45, 48, 50, 60, 72, 75, 80, 90, 100, 120, 144, 150, 180, 200, 225, 240, 300, 360, 400, 450, 600, 720, 900, 1200, 1800, 3600. Quia autem est $\frac{792}{72} = 11$ numerus integer; constat hinc esse numerum N in hoc casu dividuum numeri 72, ideoque etiam omnium numerorum in-

inferiorum hunc metientium: 2, 3, 4, 6, 8, 9, 12, 18, 24, 36. Pro divisoribus 900, 1200, 1800, 3600, numerum 792 excedentibus, habetur $R = 792$. Pro reliquis vero, dividendo numerum 792, residua facile determinantur.

Ex. 2. Sumto pro N tempore $15^d\ 7^h\ 5'\ 11''$, sit D factor numeri 63 in calculo sexagenario exprimendi per $(1,0) + 3 = (1,3)$. Erit ergo in formula (II) $m = 1$ nec non $r = -3$. Quo pacto, ob $5^d (= 5 . 24^h) = 2 . 60^h = 2 (1,0)^3$, erit $N = 6 (1,0)^3 + 7 (1,0)^2 + 5 (1,0) + 11'' = (6,7,5,11'')$. Unde $N(1,-3) = -27 . 6'' + 9 . 7'' - 3 . 5'' + 11'' = -(2,42'') + (1,3'') - 15'' + 11'' = -(1,43'') = N'$, atque $N'' = N'(1,-3) = 3 . 1'' - 43'' = -40''$ sive, transeundo a valore negativo ad positivum (cfr. §. 7), $N'' = (1,3'') - 40'' = 23''$. Qui numerus cum sit sub formis: $3q + 2$, $7q + 2$, $9q + 5$, $21q + 2$, $63q + 23$; exhibebit quoque $N = 15^d\ 7^h .\ 5' .\ 11''$, peracta divisione per numeros hosce: 3, 7, 9, 21, 63 (factores videlicet numeri 63), hæc residua: 2, 2, 5, 2, 23 respective.

§. 10.

Allata jam exempla vim Theorematis supra (§. 6) exhibiti cognitam satis atque perspectam reddunt. Quumque ex formula $(1,0)^m - r$, apte determinando m & r, obtineatur numerus dividuus divisoris cujuscunque D; sequitur hinc generalem omnino esse expositam heic residua determinandi methodum, minimeque ad certas numerorum classes restringi. Ubi vero id agitur, ut formulæ nostræ in praxin transferantur, permultum interest, ut ii præ ceteris eligantur valores numerorum m & r, qui efficiant seriem N', N'', N''' &c. maxime convergentem. Cujus quidem delectus adhibendi ratio haud parum difficultatis persæpe habet. Specialiora vero huc spectantia præcepta angusti dissertatiunculæ nostræ non capiunt limites.

Corrigenda.

§. 1. lin.	5 & 6.	pro: a,	erit: a_1
— —	12.	pro: a^1	erit: a_1
§. 2. —	4.	pro: (1,0,0	erit: (1,0,0)
— —	6.	pro: (1,0^n	erit: $(1,0)^n$
§. 9. —	5.	post: idem inseratur:	ac ipse N