DISSERTATIO ACADEMICA

SUMMAS SERIERUM EX COËFFICIENTIBUS BINOMIALIBUS PECULIARI LEGE COMPOSITARUM COLLIGENDI METHODUM EXHIBENS;

QUAM

VENIA AMPL. FAC. PHILOS. IN IMP. ACAD. AB.

PRÆSIDE

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PRO GRADU PHILOSOPHICO

PUBLICÆ CENSURÆ MODESTE SUBJICIT

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h. a. m. c.

ABOÆ, Typis FRENCKELLIANIS.

§. 1.

Ponatur $A_{n.o} = 1$ & $A_{n.r} = \frac{n.\overline{n-1}\ldots\overline{n-r+1}}{1.2\ldots r} A_{n.o}$ sumto pro r numero quovis integro positivo, Quo pacto, si pro r substituantur numeri: $1, 2, 3$ &c. collectis terminis ea ratione conficiendis notissima hæc exsurgit series coëfficientium binomialium: $A_{n.o}$, $A_{n.1}$, $A_{n.2}$, $A_{n.3}$ &c. cui si subscribatur series ex terminis æquidifferentibus composita: a, $a+b$, $a+2b$, $a+3b$, &c., ductis in se invicem terminis correlatis utriusque seriei, nova obtinebitur series hæcce:

$a A_{n.o}, (a+b) A_{n.1}, (a+2b) A_{n.2}, (a+3b) A_{n.3}$, &c.,

cujus terminus generalis est

$$(a+rb) A_{n.r} = \frac{(a+rb).n.\overline{n-1}\ldots\overline{n-r+1}}{1.2\ldots r} A_{n.o}.$$

 Quod

Quod si in prima & ultima nostra serie signum: + præfixum sibi habeant termini valoribus indicis r hisce: α_1, α_2, α_3 &c. debiti, signum vero — termini, in quibus est $r = \beta_1$, β_2, β_3 &c., reliqui denique, si qui sunt termini, omnes expungantur, sequentes exstant series:

$$F_n = [A_{n.\alpha_1} + A_{n.\alpha_2} + A_{n.\alpha_3} \text{ \&c.}] - [A_{n.\beta_1} + A_{n.\beta_2} \; A_{n.\beta_3} \text{ \&c.}]$$

$$F'_n = [(a + \alpha_1 b) A_{n.\alpha_1} + (a + \alpha_2 b) A_{n.\alpha_2} \text{ \&c.}] - [(a + \beta_1 b) A_{n.\beta_1} + (a + \beta_2 b) A_{n.\beta_2} \text{ \&c.}].$$

Quia vero tum iste inter signa + — eligendi optio tum quoque terminorum expungendorum respectus impeditam reddunt generalem series hasce summandi rationem; hoc submoturi incommodum novum adhibuimus transformandi artificium.

Sit nimirum $f(r)$ istiusmodi functio ipsius r, ut posito r vel $= \alpha_1$ vel $= \alpha_2$ vel $= \alpha_3$ &c., fiat $f(r) = 1$, nec non posito r vel $= \beta_1$ vel $= \beta_2$ vel β_3 &c. $f(r) = -1$, denique pro reliquis quibusvis numeris integris positivis γ_1, γ_2, γ_3 &c. in locum ipsius r suffectis $f(r) = 0$. Quo pacto erit:

$$\begin{aligned} F_n = & f(\alpha_1) A_{n.\alpha_1} + f(\alpha_2) A_{n.\alpha_2} + f(\alpha_3) A_{n.\alpha_3} \text{ \&c.} \\ & + f(\beta_1) A_{n.\beta_1} + f(\beta_2) A_{n.\beta_2} + f(\beta_3) A_{n.\beta_3} \text{ \&c.} \\ & + f(\gamma_1) A_{n.\gamma_1} + f(\gamma_2) A_{n.\gamma_2} + f(\gamma_3) A_{n.\gamma_3} \text{ \&c.} \\ = & f(0) A_{n.0} \pm f(1) A_{n.1} + f(2) A_{n.2} \text{ \&c.} \end{aligned}$$

nec

nec non

$F'_n = (a+\alpha_1 b).f(\alpha_1) A_{n.\alpha_1} + (a+\alpha_2 b) f(\alpha_2) A_{n.\alpha_2}$ &c.
$+ (a+\beta_1 b).f(\beta_1) A_{n.\beta_1} + (a+\beta_2 b) f(\beta_2) A_{n.\beta_2}$ &c.
$+ (a+\gamma_1 b).f(\gamma_1) A_{n.\gamma_1} + (a+\gamma_2 b) f(\gamma_2) A_{n.\gamma_2}$ &c.
$= a.f(0) A_{n.0} + (a+f(1) A_{n.1} + (a+2b) f(0) A_{n.2}$ &c.

Unde liquet esse:

$T(F_n) = f(r) A_{n.r}$
$T(F'_n) = (a+rb).f(r) A_{n.r}$
terminos generales serierum F_n & F'_n.

Exempl. 1. $f(r) = 1^r = 1$. Quo in casu est
$T(F_n) = 1^r A_{n.r} = A_{n.r}$
$T(F'_n) = (a+rb).1^r. A_{n.r} = (a+rb) A_{n.r}$
$F_n = 1^0 A_{n.0} + 1^2 A_{n.2} + 1^3 A_{n.3}$ &c.
$= A_{n.0} + A_{n.1} + A_{n.2}$ &c.
$F'_n = a A_{n.0} + (a+b) A_{n.1} + (a+2b) A_{n.2}$ &c.

Exempl. 2. Sit $f(r) = (-1)^r$ ideoque
$T(F_n) = (-1)^r A_{n.r}$, $T(F'_n) = (a+rb).(-1)^r A_{n.r}$
Quo pacto, cum sit $(-1)^{2m} = 1$, nec non
$(-1)^{2m+1} = -1$, habebitur:

$F_n = (-1)^0 A_{n.0} + (-1)^1 A_{n.1} + (-1)^2 A_{n.0}$ &c.
$= A_{n.0} - A_{n.1} + A_{n.2}$ &c.
$F'_n = a A_{n.0} - (a+b) A_{n.1} + (a+2b) A_{n.2}$ &c.

 Exempl. 3.

Exempl. 3. $f(r) = \frac{1}{2}(1 + (-1)^r)$. Unde, cum sit $\frac{1}{2}(1 + (-1)^0) = \frac{1}{2}(1 + (-1)^2) = \frac{1}{2}(1 + (-1)^4) = ..$ $= \frac{1}{2}(1 + (-1)^{2m}) = \frac{1}{2}(1 + 1) = 1$, nec non $\frac{1}{2}(1 + (-1)^1)$ $= \frac{1}{2}(1 + (-1)^3) = .. = \frac{1}{2}(1 + (-1)^{2m+1}) = \frac{1}{2}(1 - 1) = 0$, obtinebitur:

$F_n = A_{n.0} + A_{n.2} + A_{n.4}$ &c.

$F'_n = a A_{n.0} + (a + 2b) A_{n.2} + (a + 4b) A_{n.4}$ &c.

Exempl. 4. Fiat $f(r) = \frac{1}{2}(1 - (-1)^r)$ in formulis $T(F_n)$ & $T(F'_n)$. Quo pacto liquet fore $f(0) = f(2) = f(4) = .. = f(2m) = 0$ nec non $f(1) = f(3) = f(5) = .. = f(2m + 1) = 1$.

Quamobrem erit:

$F_n = A_{n.1} + A_{n.3} + A_{n.5}$ &c.

$F'_n = (a + b) A_{n.1} + (a + 3b) A_{n.3} + (a + 5b) A_{n.5}$ &c.

Exempl. 5. Ponatur $f(r) = Cos(rq)$. Quo pacto, ob $Cos\, 4mq = 1$, $Cos\, \overline{4m + 2}.q = -1$, $Cos\, \overline{4m + 1}.q$ $= Cos\, \overline{4m + 3}\, q. = 0$, erit:

$T(F_n) = Cos(rq)\, A_{n.r}$,

$T(F'_n) = (a + rb)\, Cos(rq)\, A_{n.r}$. Unde:

$F_n = Cos(0.q)\, A_{n.0} + Cos\, q\, A_{n.1} + Cos\, 2q\, A_{n.2}$ &c.

$= A_{n.0} - A_{n.2} + A_{n.4}$ &c.

F'_n

$F'_n = a \, Cos(o.q) \, A_{n.o} + (a + b) \, Cos q \, A_{n.1} +$
$(a + 2b) \, Cos \, 2q \, A_{n.2} + (a + 3b) \, Cos \, 3q \, A_{n.3}$ &c.
$= a \, A_{n.o} - (a + 2b) \, A_{n.2} + (a + 4b) \, A_{n.4}$ &c.

Exempl. 6. $f(r) = Sin(rq)$ ideoque
$T(F_n) = Sin(rq) \, A_{n.r}$, $T(F_n') = (a + rb) \, Sin(rq) \, A_{n.r}$
Unde, cum sit $Sin \, 4mq = Sin \, 4m + 2q = o$ nec non
$Sin \, 4m + 1.q = 1$, $Sin \, 4m + 3.q = -1$, obtinetur:
$F_n = Sin(o.q) \, A_{n.o} + Sin \, q \, A_{n.r} + Sin \, 2q \, A_{n.2}$ &c.
$= A_{n.1} - A_{n.3} + A_{n.5}$ &c.
$F'_n = a \, Sin(o.q) . A_{n.o} + (a + b) . Sin \, q . A_{n.1} +$
$(a + 2b) . Sin \, 2q . A_{n.2} + (a + 3.b) \, Sin \, 3q . A_{n.3}$
$+ (a + 4b) . Sin \, 4q . A_{n.4}$ &c.
$= (a + b) A_{n.1} - (a + 3b) A_{n.3} + (a + 5b) A_{n.5}$ &c.

Exempl. 7. $f(r) = Cos(rq) + Sin(rq)$. Quo in casu habebitur:
$T(F_n) = (Cos(rq) + Sin(rq)) \, A_{n.r}$
$T(F'_n) = (a + rb) . (Cos(rq) + Sin(rq)) . A_{n.r}$.
Unde, cum sit $f(o) = f(4) = f(8) \, .. = f(4m)$
$= f(1) = f(5) = f(9) \, .. = f(4m+1) = 1$, nec non
$f(2) = f(6) = f(10) \, .. = f(4m+2) = f(3) = f(7)$
$= f(11) \, .. = f(4m+3) = -1$, eruitur:
$F_n = A_{n.o} + A_{n.1} - A_{n.2} - A_{n.3}$ &c.
$F'_n = a \, A_{n.o} + (a + b) . A_{n.1} - (a + 2b) . A_{n.2}$
$+ (a + 3b)$ &c.

Exempl.

Exempl. 8. Fiat denique $f(r) = Cos(rq) - Sin(rq)$. Quo pacto facile colligitur fore:

$T(F_n) = Cos(rq) - Sin(rq) A_{n.r}$

$T(F'_n) = (a + rb) . (Cos(rq) - Sin(rq)) . A_{n.r}$

$F_n = A_{n.0} - A_{n.1} - A_{n.2} + A_{n.3}$ &c.

$F'_n = a\ A_{n.0} - (a + b) A_{n.1} - (a + 2b) A_{n.2} + (a + 3b) A_{n.3}$ &c.

Schol. 1. In præcedentibus adhibita functionis $f(r)$ exempla id habent inter se commune, quod sit $f(r)$ vel $= + f(r + s) = + f(r + 2s) = .. = f(r + ms)$. vel $= - f(r + s) = + f(r + 2s) = - f(r + 3s) = .. = f(r + ms) . (-1)^m$, quodque $f(0), f(1), f(2), .. f(s-1)$ nullos admittant valores præter hosce: $+1, -1, 0$. Sic posito $s = 1$, obtinetur $f(r)$ vel ($= 1^r$ in *Exempl.* 1) $= f(r + 1) = f(r + 2) = .. = f(r + m)$, vel ($= (-1)^r$ in *Exempl.* 2) $= - f(r + 1) = + f(r + 2) = .. = f(r + m) . (-1)^m = f(r + 2m)$, nec non $f(0) = 1^0$ vel $= (-1)^0 = 1$; posito $s = 2$, $f(r) = f(r + 2) = f(r + 4) = .. = f(r + 2m)$ in *Exempll.* 1, 2, 3, 4 nec non $f(0) = 1$, $f(1) = 0$ in *Exempl.* 3, $f(0) = 0$, $f(1) = 1$ in *Exempl.* 4, vel $f(r) = - f(r + 2) = f(r + 4) = .. = f(r + 2m) . (-1)^m = f(r + 4m)$ in *Exemplis* 5, 6, 7, 8, prætereaque $f(0) = 1$, $f(1) = 0$ in *Ex.* 5, $f(0) = 0$, $f(1) = 1$ in *Ex.* 6., $f(0) = f(1) = 1$ in *Ex.* 7, $f(0) = 1$, $f(1) = -1$ in *Ex.* 8.

Schol.

Schol. 2. Exempla octo formulæ F_n supra exhibita Analystis jam dudum fuere notissima. Quorum si singula peculiari præfixo secundum ordinem exemplorum indice distinguantur, constat esse:

${}^1F_n = (1+1)^n = 2^n$; ${}^2F_n = (1-1)^n = 0^n (=0)$, excepto ${}^2F_0 = 1$); ${}^3F_n = \frac{1}{2}({}^1F_n + {}^2F_n) = 2^{n-1} + \frac{1}{2} . 0^n$ ($= 2^{n-1}$, excepto ${}^3F_0 = 1$); ${}^4F_n = \frac{1}{2}({}^1F_n - {}^2F_n) = 2^{n-1} - \frac{1}{2} . 0^n$ ($= 2^{n-1}$), excepto ${}^4F_n = 0$);

$${}^5F_n = \frac{(1+\sqrt{-1})^n + (1-\sqrt{-1})^n}{2};$$

$${}^6F_n = \frac{(1+\sqrt{-1})^n - (1-\sqrt{-1})^n}{2\sqrt{-1}};$$

${}^7F_n = {}^5F_n + {}^6F_n$; ${}^8F_n = {}^5F_n - {}^6F_n$.

Generalia vero summandæ seriei F_n, ex terminis formæ $f(r)\,A_{n.r}$ compositæ, prorsus desiderantur præcepta, in hac tractatiuncula, pro modulo virium, a nobis exponenda.

Quod attinet ad seriem formæ F'_n, ejus nulla, ne quidem specialiora, exstant summandi specimina. Quippe quæ, a nobis generaliter pertractanda, si ad casus supra allatos adplicetur, ogdoada exhibebit valorum specialium: ${}^1F'_n$, ${}^2F'_n$, ${}^3F'_n$, ${}^4F'_n$, ${}^5F'_n$, ${}^6F'_n$, ${}^7F'_n$ & ${}^8F'_n$.

§. 2.

THEOREMA I. $A_{n,r} = A_{\overline{n-1},r} + A_{\overline{n-1},\overline{r-1}}$.

Quia est $A_{n,r} = \dfrac{n.\overline{n-1}..\overline{n-r+1}}{1.2..r} A_{n,0}$, erit

$$A_{\overline{n-r},1} = \frac{\overline{n-1}\;\overline{n-2}..\overline{n-r+1}.\overline{n-r}}{1.2..(r-1).r} A_{\overline{n-1},0},$$ atque

$$A_{\overline{n-1},\overline{r-1}} = \frac{\overline{n-1}.\overline{n-2}..\overline{n-r+1}}{2.2..(r-1)} A_{\overline{n-1},0}$$

$$= \frac{\overline{n-1}.\overline{n-2}..\overline{n-r+1}.r}{1.2..(r-1).r} A_{\overline{n-1},0}.$$ Unde

$$A_{\overline{n-r}}. + A_{\overline{n-1},\overline{r.1}} = \frac{\overline{n-1}.\overline{n-2}..\overline{n+r-1}}{1.2..r}.(n-r) A_{\overline{n-1},0}$$

$$+ \frac{\overline{n-1}.\overline{n-2}..\overline{n-r+1}}{1.2..r}.r A_{\overline{n-1},0}$$

$$= \frac{n.\overline{n-1}.\overline{n-2}..\overline{n-r+1}}{1.2..r} A_{\overline{n-1},0}$$

$$= A_{n,r},\ \text{ob}\ A_{\overline{n-1},0} = A_{n,0} = 1.$$

THEOREMA II. $A_{\overline{i+u},r} = A_{i,r} A_{i,0} + A_{i,\overline{r-1}} A_{i,1} + A_{i,\overline{r-2}} A_{i,2} +$ &c., designante u numerum quemvis integrum positivum.

Quippe

Quippe est (THEOR. I.) $A_{\overline{n+u.r}} = A_{\overline{n+u-1.r}} + A_{\overline{n+u-1.r-1}}$
$= A_{\overline{n+u-1.r}} A_{1.0} + A_{\overline{n+u-1.r-1}} A_{\overline{1.1}}$
$= (A_{\overline{n+u-2.r}} + A_{\overline{n+u-2.r-1}}) A_{1.0}$
$+ (A_{\overline{n+u-2.r-1}} + A_{\overline{n+u-2.r-2}}) A_{1.1}$
$= A_{\overline{n+u-2.r}} A_{1.0} + A_{\overline{n+u-2.r-1}} (A_{1.0} + A_{1.1})$
$+ A_{\overline{n+u-2.r-2}} A_{1.1}$
$= A_{\overline{n+u-2.r}} A_{2.0} + A_{\overline{n+u-2.r-1}} A_{\overline{2.1}} + A_{\overline{n+u-2.r-2}} A_{\overline{2.2}}$
$= (A_{\overline{n+u-3.r}} + A_{\overline{n+u-3.r-1}}) A_{2.0}$
$+ (A_{\overline{n+u-3.r-1}} + A_{\overline{n+u-3.r-2}}) A_{2.1}$
$+ (A_{\overline{n+u-3.r-2}} + A_{\overline{n+u-3.r-3}}) A_{2.2}$
$= A_{\overline{n+u-3.r}} A_{2.0} + A_{\overline{n+n-3.r-1}} (A_{2.0} + A_{2.1})$
$+ A_{\overline{n+u-3.r-2}} (A_{2.1} + A_{2.2}) + A_{\overline{n+u-3.r-3}} A_{2.2}$
$= A_{\overline{n+u-3.r}} A_{3.0} + A_{\overline{n+u-3.r-1}} A_{\overline{3.1}} + A_{\overline{n+u-3.r-2}} A_{\overline{3.2}}$
$+ A_{\overline{n+u-3.r-3}} A_{\overline{3.3}}$. Unde, cum hi valores functionis $A_{\overline{n+u}.r}$ pertineant ad formam hancce:

$A_{\overline{n+u-v.r}} A_{v.0} + A_{\overline{n+u-v.r-1}} A_{\overline{v.1}} + A_{\overline{n+u-v.r-2}} A_{\overline{v.2}}$
$+ .. + A_{\overline{n+u-v.r-v}} A_{\overline{v.v}}$, sumto pro v numero quovis integro positivo; habebitur, facto in hac formula $v = u$,

$$A_{\overline{n+u}.r} = A_{n.r} A_{u.0} + A_{n.\overline{r-1}} A_{u.1} + \,..\, + A_{n.\overline{r-u}} A_{u.u}$$
$$= A_{n.r} A_{n.0} + A_{n.\overline{r-1}} A_{n.1} + A_{n.\overline{r-2}} A_{u.2} \text{ \&c.}$$

Scholion. In nostra supponitur demonstratione, evanescere $A_{u.r}$ sumto $r > u$. Cujus adsumtionis veritas

ritas

ritas sequenti ratione constat. Quia est (Hyp.) in formula: $A_{u.r} = \frac{u.\overline{u-1}\ldots\overline{u-r+1}}{1.2\ldots r}$, u uumerus integer positivus & $u-r$ numerus integer negativus, erit ultimus factor numeratoris $u-r+1$ aut $=0$, aut numerus integer negativus. Si illud, manifestum est evanescere formulam. Sin hoc, inter factores numeratoris, seriem constituentis terminorum, quorum primus ultimusque sunt numeri integri, positivus ille u, negativus hic $u-r+1$, proxime vero insequentem præcedens quisque unitate excedat, dabilis est intermedius quidam, qui sit $=0$. Quamobrem & ipsa formula pariter evanescat necesse est.

§. 3.

Theorema III. Si transeat, exterminatis, vi Theorematis II, quantitatibus, $A_{\overline{n+u}.0}$, $A_{\overline{n+u}.1}$, $A_{\overline{n+u}.2}$, &c. series $F_{\overline{n+u}}$ in $G_{n.u} = B_{n.u.0}A_{n.0} + B_{n.u.1}A_{n.1} + B_{n.u.2}A_{n.2} +$ &c. nec non $F'_{\overline{n+u}}$ in $G'_{n.u} = B'_{n.u.0}A_{n.0} + B'_{n.u.1}A_{n.1} + B'_{n.u.2}A_{n.2} +$ &c.; erunt termini generales serierum $G_{n.u}$, $G'_{n.u}$:

$$B_{n.u.r}A_{n.r} = [f(r)A_{u.0} + f(r+1)A_{u.1} + f(r+2)A_{u.2} + \text{\&c.}]A_{n.r}, \text{ \&}$$

$$B'_{n.u.r}A_{n.r} = [(a+rb)f(r)A_{u.0} + (a+\overline{r+1}.b)f(r+1)A_{u.1} + (a+\overline{r+2}.b)f(r+2)A_{u.2} + \text{\&c.}]A_{n.r}.$$

Quia

Quia est $F_{\overline{n+u}} = [f(r-1)A_{\overline{n+u}.\overline{r-1}} + f(r-2)A_{\overline{n+u}.\overline{r-2}} + \&c.]$ $+ [f(r)A_{\overline{n+u}.r} + f(r+1)A_{\overline{n+u}.\overline{r+1}} + f(r+2)A_{\overline{n+u}.\overline{r+2}} + \&c.]$, nec non $F'_{\overline{n+u}} = [(a+\overline{r-1}.b)\, f(r-1)\, A_{\overline{n+u}.\overline{r-1}}$ $+ (a+\overline{r-2}.b)\, A_{\overline{n+u}.\overline{r-2}} + \&c.] + [(a+rb)f(r)A_{\overline{n+u}.r}$ $+ (a+\overline{r+1}.b)\, f(r+1)\, A_{\overline{n+u}.\overline{r+1}}\ \&c.]$, pro quovis valore indicis r; erit, vi THEOR. II, denotantibus R & R' summas terminorum a quantitate $A_{n.r}$ immunium in seriebus $G_{n.u}$ & $G'_{n.u}$ respective,

$$G_{n.u} = B_{n.u.r}\, A_{n.r} + R$$
$$= [f(r-1)\,(A_{n.\overline{r-1}}A_{u.0} + A_{n.\overline{r-2}}.A_{u.1} + A_{n.\overline{r-3}}A_{u.2} + \&c.]$$
$$+ f(r-2)\,(A_{n.\overline{r-2}}A_{u.0} + A_{n.\overline{r-3}}.A_{u.1} + A_{n.\overline{r-4}}A_{u.2} + \&c.)$$
$$+ \&c.] + [f(r)(A_{n.r}A_{u.0} + A_{n.\overline{r-1}}A_{u.1} + A_{n.\overline{r-2}}A_{u.2} + \&c.)$$
$$+ f(r+1)\,(A_{n.\overline{r+1}}A_{u.0} + A_{n.r}A_{u.1} + A_{n.\overline{r-1}}A_{u.2} + \&c.)$$
$$+ f(r+2)\,(A_{n.\overline{r+2}}A_{u.0} + A_{n.\overline{r+1}}A_{u.1} + A_{r.r}A_{u.2} + \&c.)$$
$$+ \&c.]$$
$$= [f(r)\, A_{u.0} + f(r+1)\, A_{u.1} + f(r+2)\, A_{u.2} + \&c.]\, A_{n.r}$$
$+ R$, atque

$$G'_{n.u} = B'_{n.u.r}\, A_{n.r} + R'$$
$$= (a+\overline{r-1}.b)\, f(r-1)\, (A_{n.\overline{r-1}}A_{u.0} + A_{n.\overline{r-2}}A_{u.1} + \&c.)$$
$$+ (a+\overline{r-2}.b)\, f(r-2)\, A_{n.\overline{r-2}}\, A_{u.0} + A_{n.\overline{r-3}}A_{u.1} + \&c.)$$
$$+ \&c.] + [(a+rb)f(r)(A_{n.r}A_{u.0} + A_{n.\overline{r-1}}\, A_{u.1} + \&c.)$$
$$+ (a+\overline{r+1}.b)\, f(r+1)\, (A_{n.\overline{r+1}}\, A_{u.0} + A_{n.r}A_{u.1} + \&c.)$$
$$+ (a+\overline{r+2}.b)\ f(r+2)\ (A_{n.\overline{r+2}}\, A_{u.0} + A_{n.\overline{r+1}}\, A_{u.1}$$
$$+ A_{n.r}\, A_{u.2} + \&c.) + \&c.]$$

$= [(a+rb)$

$$= [(a+rb)f(r)A_{u.0} + (a+\overline{r+1}.b)f(r+1)A_{u.1} + (a+\overline{r+2}.b)f(r+2)A_{u.2} + \&c.]A_{n.r} + R'.$$

Unde obtinebitur

$$B_{n.u.r}A_{n.r} = G_{n.u} - R = [f(r)A_{u.0} + f(r+1)A_{u.1} + f(r+2)A_{u.2} + \&c.]A_{n.r},$$ nec non

$$B'_{n.u.r}A_{n.r} = G'_{n.u} - R' = [(a+rb)f(r)A_{u.0} + (a+\overline{r+1}.b)f(r+1)A_{u.1} + (a+\overline{r+2}.b)f(r+2)A_{u.2} + \&c.]A_{n.r}.$$

Corollar. I. Sit $f(r+ms)=f(r)$, $k(r+ms)=k(r)$, $k(0)=1$, $k(1)=k(2)=..=k(s-1)=0$. Habebitur hoc pacto

$$\begin{aligned}B_{n.u.r} = {} & f(r)[A_{u.0} + A_{u.s} + A_{u.2s}\ \&c.] \\ & + f(r+1)[A_{u.1} + A_{u.\overline{s+1}} + A_{u.\overline{2s+1}}\ \&c.] \\ & + f(r+2)(A_{u.2} + A_{u.\overline{s+2}} + A_{u.\overline{2s+2}}\ \&c.] \\ & + \ldots\ldots\ldots\ldots \\ & + f(r+s-1)[A_{u.\overline{s-1}} + A_{u.\overline{2s-1}} + A_{u.\overline{3s-1}}\ \&c.) \\ = {} & f(r)[k(0)A_{u.0} + k(1)A_{u.1} + k(2)A_{u.2}\ \&c.] \\ & + f(r+1)[k(s-1)A_{u.0} + k(s)A_{u.1} + k(s+1)A_{u.2}\ \&c.] \\ & + f(r+2)[k(s-2)A_{u.0} + k(s-1)A_{u.1} + k(s)A_{u.2}\ \&c.] \\ & + \ldots\ldots\ldots\ldots\ldots\ldots \\ & + f(r+s-1)[k(1)A_{u.0} + k(2)A_{u.1} .. + k(s)A_{u.\overline{s-1}}\ \&c.].\end{aligned}$$

Corollar. 2. Sit $f(r+ms)=(-1)^m.f(r),\ l(r+ms)=(-1)^m.l(r).$ $l(0)=1,\ l(1)=l(2)=..=l(s-1)=0.$ Quo in casu erit

$$B_{n.u.r}=f(r)\,[A_{u.0}-A_{u.s}+A_{u.2s}\ \&c.]$$
$$+f(r+1)\,[A_{u.1}-A_{u.\overline{s+1}}+A_{u.\overline{2s+1}}\ \&c.]$$
$$+f(r+2)\,[A_{u.2}-A_{u.\overline{s+2}}+A_{u.\overline{2s+2}}\ \&c.]$$
$$+\ \ .\ \ .\ \ .\ \ .\ \ .\ \ .\ \ .\ \ .\ \ .$$
$$+f(r+s-1)\,[A_{u.\overline{s-1}}-A_{u.\overline{2s-1}}+A_{u.\overline{3s-1}}\ \&c.]$$
$$=f(r)\,[l(0)\,A_{u.0}+l(1)\,A_{u.1}+l(2)\,A_{u.2}\ \&c.]$$
$$-f(1+1)\,[l(s-1)A_{u.0}+l(s)A_{u.1}+l(s+1)A_{u.2}\ \&c.]$$
$$-f(r+2)\,[l(s-2)\,A_{u.0}+l(s-1)\,A_{u.1}+l(s)\,A_{u.2}\ \&c.]$$
$$-\ \ .\ \ .\ \ .\ \ .\ \ .\ \ .\ \ .\ \ .\ \ .$$
$$-f(r+s-1)\,[l(1)\,A_{u.0}+l(2)\,A_{\ .1}..+l(s)\,A_{u.\overline{s-1}}\ \&c.]$$

Corollar. 3. $B'_{n.u.r}=rb\,B_{n.u.r}+(a\,f(r)\,A_{u.0}$ $+(a+b).f(r+1)\,A_{u.1}+(a+2b)\,f(r+2)\,A_{u.2}+\&c.)$ erit, in casu *Corollarii 1*,

$$=rbB_{n.u.r}$$
$$+f(r)\,[a\,A_{u.0}+(a+sb)\,A_{u.s}+(a+2sb)\,A_{u.2s}\ \&c.]$$
$$+f(r+1)\,[(a+b)\,A_{u.1}+(a+\overline{s+1}.b)\,A_{u.\overline{s+1}}$$
$$+(a+\overline{2s+1}.b)\,A_{u.\overline{2s+1}}\ \&c.]$$
$$+f(r+2)\,[(a+2b)\,A_{u.2}+(a+\overline{s+2}.b)\,A_{u.\overline{s+2}}$$
$$+(a+\overline{2s+2}.b)\,A_{u.\overline{2s+2}}\ \&c.]$$
$$+\ \ .\ \ .\ \ .\ \ .\ \ .\ \ .\ \ .\ \ .\ \ .$$
$$+f(r+s-1)\,[(a+\overline{s-1}.b)\,A_{u.\overline{s-1}}+(a+\overline{2s-1}.b)\,A_{u.\overline{2s-1}}$$
$$+(a+\overline{3s-1}.b)\,A_{u.\overline{3s-1}}\ \&c.]$$

 $=rb$

$= rb\, B_{n.u.r}$
$+ f(r)[a.k(0)A_{u.0} + (a+b).k(1)A_{u.1} + (a+2b)k(2)A_{u.2}$ &c.]
$+ f(r+1)\,[a.k(s-1)\,A_{u.0} + (a+b).k(s)\,A_{u.1}$ &c.]
$+ f(r+2)\,[a.k(s-2)\,A_{u.0} + (a+b)\,k(s-1)A_{u.1}$ &c.
$+ (a+2b)\,k(s)\,A_{u.2}$ &c.] . . . .
$+ f(r+s-1)\,[a.k(1)\,A_{u.0} + (a+b).k(2)A_{u.1} \ldots$
$+ (a+\overline{s-1}.b).k(s)\,A_{u.\overline{s-1}}$. &c.]

nec non in casu *Corollarii 2*,

$= rb\,B_{n.u.r}$
$+ f(r)\,[a\,A_{u.0} - (a+sb)A_{u.s} + (a+2sb)A_{u.2s}$ &c.]
$+ f(r+1)\,[(a+b)\,A_{u.1} — (a+\overline{s+1}.b)\,A_{u.\overline{s+1}}$
$+ (a+\overline{2s+1}.b)\,A_{u.\overline{2s+1}}$ &c.]
$+ f(r+2)\,[(a+2b)\,A_{u.2} — (a+\overline{s+2}.b)\,A_{u.\overline{s+2}}$
$+ (a+\overline{2s+2}.b)\,A_{u.\overline{2s+2}}$ &c.] . . . . . .
$+ f(r+s-1)\,[(a+\overline{s-1}.b)\,A_{u.\overline{s-1}} + (a+\overline{2s-1}.b\,A_{u.\overline{2s-1}}$
$+ (a+\overline{3s-1}.b)\,A_{u.\overline{3s-1}}$ &c.]

$= rb\,B_{n.u.r}$
$+ f(r)\,[a.l(0)\,A_{u.0} + (a+b).l(1)\,A_{u.1}$ &c.]
$- f(r+1)\,[a.l(s-1)\,A_{u.0} + (a+b).l(s)\,A_{u.1}$ &c.]
$- f(r+2)\,[a.l(s-2)\,A_{u.0} + (a+b).l(s-1)\,A_{u.1}$
$+ (a+2b).l(s)\,A_{u.2}$ &c.] . . . .
$- f(r+s-1)\,[a.l(1)\,A_{u.0} + (a+b).l(2)\,A_{u.1} \ldots$
$+ (a+\overline{s-1}.b).l(s)\,A_{u.\overline{s-1}}$ &c.].

§. 4.

§ 4.

Theorema IV. Si fuerint, functiones:

$$f(r+v)\,A_{n.r},\ k(r+v)\,A_{n.r},\ l(r+v)\,A_{n.r};$$
$$(a+rb)\,.\,f(r+v)\,A_{n.r},\ (a+rb)\,.\,k(r+v)\,A_{n.r},$$
$(a+rb)\,.\,l(r+v)\,A_{n.r}$; termini generales serierum, quarum sint summæ respective positæ:

$$F_{n.v},\ K_{n.v},\ L_{n.v};\ F'_{n.v},\ K'_{n.v},\ L'_{n.v};$$

erit, in casu $f(r+s)=f(r)$,

$$F_{\overline{n+u}} = F_{n.0}\,.\,K_{u.0} + F_{n.1}\,.\,K_{u.\overline{s-1}} + F_{n.2}\,.\,K_{u.\overline{s-2}} + \ldots + F_{n.\overline{s-1}}\,.\,K_{u.1},$$ simulque

$$F'_{\overline{n+u}} = F'_{n.0}\,.\,K_{u.0} + F'_{n.1}\,.\,K_{u.\overline{s-1}} + F'_{n.2}\,.\,K_{u.\overline{s-2}} + \ldots + F'_{n.\overline{s-1}}\,.\,K_{u.1}$$
$$+ F_{n.0}\,.\,K'_{u.0} + F_{n.1}\,.\,K'_{u.\overline{s-1}} + F_{n.2}\,.\,K'_{u.\overline{s-2}} + \ldots + F_{n.\overline{s-1}}\,.\,K'_{u.1}$$
$$-\,a\,F_{\overline{n+u}};$$ & in casu $f(r+s)=-f(r)$,

$$F_{\overline{n+u}} = F_{n.0}\,.\,L_{u.0} - [F_{n.1}\,.\,L_{u.\overline{s-1}} + F_{n.2}\,.\,L_{u.\overline{s-2}} + \ldots + F_{n.\overline{s-1}}\,L_{u.1}],$$ simulque

$$F'_{\overline{n+u}} = F'_{n.0}\,.\,L_{u.0} - [F'_{n.1}\,.\,L_{u.\overline{s-1}} + F'_{n.2}\,.\,L_{u.\overline{s-2}} + \ldots + F'_{n.\overline{s-1}}\,L_{u.1}]$$
$$+ F_{n.0}\,.\,L'_{u.0} - [F_{n.1}\,.\,L'_{u.\overline{s-1}} + F_{n.2}\,.\,L'_{u.\overline{s-2}} + \ldots + F_{n.\overline{s-1}}\,L'_{u.1}]$$
$$-\,a\,F_{\overline{n+u}}\,.$$

Quia

Quia sunt functiones: $k(r)\,A_{u.r}$, $l(r)\,A_{u.r}$; $(a+br).k(r)\,A_{u.r}$, $(a+br)\,l(r)\,A_{u.r}$ termini generales serierum respective positarum:

$k(o)\,A_{u.o} + k(1)\,A_{u.1} + k(2)\,A_{u.2}$ &c.,

$l(o)\,A_{u.o} + l(1)\,A_{u.1} + l(2)\,A_{u.2}$ &c.,

$a.k(o)A_{u.o} + (a+b).k(1)A_{u.1} + (a+2b).k(2)A_{u.2}$ &c.,

$a.l(o)A_{u.o} + (a+b).l(1)A_{u.1} + (a+2b).l(2)A_{u.2}$ &c.;

erit:

$f(r)\,(k(o)A_{u.o} + k(1)A_{u.1} + k(2)A_{u.2}$ &c.$) = f(r).K_{u.o}$,

$f(r)\,(l(o)\,A_{u.o} + l(1)A_{u.1} + l(2)\,A_{u.2}$ &c.$) = f(r).L_{u.o}$,

$f(r)\,[a.k(o)\,A_{u.o} + (a+b).k(1)A_{u.1}$ &c.$] = f(r).K'_{u.o}$,

$f(r)\,[a.l(o)A_{u.o} + (a+b).l(1)A_{u.1}$ &c.$] = f(r).L'_{u.o}$,

Reliqui vero omnes termini functionum $B_{n.u.r}$, $B'_{n.u.r}$ sunt aut formæ:

$f(r+v)\,(k(s-v)\,A_{u.o} + k(s-v+1)\,A_{u.1} + k(s-v+2)A_{u.2}$ &c.),

aut formæ:

$f(r+v)\,\,(l(s-v)\,A_{u.o} + l(s-v+1)A_{u.1} + l(s-v+2)A_{u.2}$ &c.),

aut formæ:

$f(r+v)\,\,[a.k(s-v)\,A_{u.o} + (a+b).k(s-v+1)\,A_{u.1} +$ &c.]

aut denique formæ:

$f(r+v)\,\,[a.l(s-v)\,A_{u.o} + (a+b).l(s-v+1)\,A_{u.1} +$ &c.].

Quippe quæ formæ, cum constent ex factore $f(r+v)$ ducto in seriem, cujus terminus generalis est aut

$k(r+s-v)\,A_{u.r}$

$k(r+s-v)A_{u.r}$ aut $l(r+s-v)A_{u.r}$, aut $(a+rb).k(r+s-v)A_{u.r}$, aut denique $(a+rb).l(r+s-v)A_{u.r}$; exhiberi potuerunt per hasce respective positas: $f(r+v)K_{u.\overline{s-v}}$, $f(r+v)L_{u.\overline{s-v}}$, $f(r+v)K'_{u.\overline{r-v}}$, $f(r+v)L'_{u.\overline{s-v}}$. Quod si in his formulis ponatur $v = 1, 2, 3, \ldots (s-1)$, singuli prodibunt, ex quibus conficiuntur $B_{n.u.r}$ & $B_{n.u.r}$, termini. Quibus adhibitis valoribus obtinebitur, in casu $f(r+s) = f(r)$,

$$B_{n.u.r} = f(r)K_{u.0} + f(r+1)K_{u.\overline{s-1}} + f(r+2)K_{u.\overline{s-2}} + \ldots + f(r+s-1)K_{u.1}, \text{ \&}$$

$$B'_{n.u.r} = rb\,B_{n.u.r} + f(r)K'_{u.0} + f(r+1)K'_{u.\overline{s-1}} + f(r+2)K'_{u.\overline{s-2}} + \ldots + f(r+s-1)K'_{u.1};$$

nec non, in casu $f(r+s) = -f(r)$,

$$B_{n.u.r} = f(r)L_{u.0} - (f(r+1)L_{u.\overline{s-1}} + f(r+2)L_{u.\overline{s-2}} + \ldots + f(r+s-1)L_{u.1}), \text{ \&}$$

$$B'_{n.u.r} = rb\,B_{n.u.r} + f(r)L_{u.0} - (f(r+1)L_{u.\overline{s-1}} + f(r+2)L_{u.\overline{s-2}} + \ldots + f(r+s-1)L_{u.1}).$$

Unde erit, in casu illo,

$$B_{n.u.r}A_{n.r} = f(r)A_{n.r}K_{u.0} + f(r+1)A_{n.r}K_{u.\overline{s-1}} + f(r+2)A_{n.r}K_{u.\overline{s-2}} + \ldots + f(r+s-1)A_{n.r}K_{u.1}$$

& in hoc:

$$B_{n.u.r}A_{n.r} = f(r)A_{n.r}L_{u.0} - [f(r+1)A_{n.r}L_{u.\overline{s-1}} + f(r+2)A_{n.r}L_{u.\overline{s-2}} + \ldots + f(r+s-1)A_{n.r}L_{u.1}].$$

 Jam

Jam vero, cum sint quantitates $K_{u.o}$, $K_{u.\overline{s-1}}$ &c. $L_{u.o}$, $L_{u.\overline{s-1}}$ &c., utpote ab indice variabili r immunes, pro constantibus habendae, ideoque quilibet terminus utriusvis seriei, quantitati $B_{n.u.r}\,A_{n.r}$ exaequatae, sub forma $f(r+v)A_{n.r}.C$; erit in uno casu:

$$G_{n.u}=F_{n.o}.K_{u.o}+F_{n.1}.K_{u.\overline{s-1}}+F_{n.2}.K_{u.\overline{s-2}} + \ldots + F_{n.\overline{s-1}}\,K_{u.1};$$

& in altero:

$$G_{n.u}=F_{n.o}.L_{u.o}-[F_{n.1}.L_{u.\overline{s-1}}+F_{n.2}.L_{u.\overline{s-2}} + \ldots + F_{u.\overline{s-1}}.K_{u.1}].$$

Praeterea erit, vi *Theor. III. Coroll. 3*, in casu illo:

$$B'_{n.u.r}\,A_{n.r}=(a+rb)\,B_{n.u.r}\,A_{n.r}-a.B_{n.u.r}\,A_{n.r} + f(r)A_{n.r}\,K'_{u.o}+f(r+1)\,A_{n.r}\,K'_{u.\overline{s-1}} + f(r+2)A_{n.r}K'_{u.\overline{s-2}}\ldots+f(r+s-1)A_{n.r}K'_{u.1};$$

in hoc vero casu:

$$B'_{n.u.r}\,A_{n.r}=(a+rb)\,B_{n.u.r}\,A_{n.r}-a.B_{n.u.r}\,A_{n.r} + f(r)\,A_{n.r}\,L'_{u.o}-[f(r+1)\,A_{n.r}\,L'_{u.\overline{s-1}} + f(r+2)A_{n.r}L_{u.\overline{s-2}}\ldots+f(r+s-1)A_{n.r}\,L_{u.1}].$$

Quia vero est $B_{n.u.r}\,A_{n.r}$ terminus generalis functionis $G_{n.u}$, ex terminis formae $F_{n.v}.C$ compositae; erit $a.B_{n.u.r}\,A_{n.r}$ terminus generalis functionis $a.G_{n.u}$, nec non $(a+rb)\,B_{n.u.r}\,A_{n.r}$ terminus generalis ejus seriei, in quam transit $G_{n.u}$, pro singulis valoribus formulae $F_{n.v}$ substituendo valo-

res

res formulæ $F'_{n.v}$ respective sumtos. Unde colligitur, in casu $f(r+s)=f(r)$ esse

$$G'_{n.u}=F'_{n.0}K_{u.0}+F'_{n.1}.K_{u.\overline{s-1}}+F'_{n.2}.K_{u.\overline{s-2}}+\dots$$
$$+F'_{n.\overline{s-1}}K_{u.1}-a.G_{n.u}+F_{n.0}K'_{u.0}+F_{n.1}K'_{u.\overline{s-1}}$$
$$+F_{n.2}.K'_{u.\overline{s-2}}+\dots+F_{n.\overline{s-1}}K_{u.1},$$

in casu vero $f(r+s)=-f(r)$,

$$G'_{n.u}=F'_{n.0}.L_{u.0}-[F'_{n.1}.L_{u.\overline{s-1}}+F'_{n.2}.L_{u.\overline{s-2}}+\dots$$
$$+F'_{n.\overline{s-1}}L_{u.1}]-a.G_{n.u}+F_{n.0}L'_{u.0}-[F_{n.1}L'_{u.\overline{s-1}}$$
$$+F_{n.2}L'_{u.\overline{s-2}}+\dots+F_{n.\overline{s-1}}L'_{u.1}].$$

Quod si in æquationibus jam inventis ponatur $F_{\overline{n+u}}$ pro $G_{n.u}$ & $F'_{\overline{n+u}}$ pro $G'_{n.u}$, prodibunt formulæ in Theoremate nostro constitutæ.

Coroll. 1. Designantibus $F_{\overline{n+u}.v}$, $F''_{\overline{n+u}.v}$ eas functiones, in quas transeunt $F_{\overline{n+u}}$, $F'_{\overline{n+u}}$ respective, substituto $f(r+v)$ pro $f(r)$ in terminis serierum, ex quibus componuntur, generalibus, liquet fore generatim,

in casu: $f(r+s)=f(r)$;

$$F_{\overline{n+u}.v}=F_{n.v}K_{u.0}+F_{n.\overline{v+1}}K_{u.\overline{s-1}}+F_{n.\overline{v+2}}K_{u.\overline{s-2}}$$
$$+\dots+F_{n.\overline{v+s-1}}K_{u.1};$$

$$F''_{\overline{n+u}.v}-a.F_{\overline{n+u}.v}=F'_{n.v}K_{u.0}+F'_{n.\overline{v+1}}K_{u.\overline{s-1}}$$
$$+F'_{n.\overline{v+2}}K_{u.\overline{s-2}}+\dots+F'_{n.\overline{v+s-1}}K_{u.1}+F_{n.v}K'_{u.0}$$
$$+F_{n.\overline{v+1}}K'_{u.\overline{s-1}}+F_{n.\overline{v+2}}K'_{u.\overline{s-2}}+\dots+F_{n.\overline{v+s-1}}K'_{u.1};$$

in

in casu: $f(r+s) = -f(r)$;

$$F_{\overline{n+u}.v} = F_{n.v} L_{u.o} - [F_{n.\overline{v+1}} L_{u.\overline{s-1}} + F_{n.\overline{v+2}} L_{u.\overline{s-2}} + \ldots + F_{n.\overline{v+s-1}} L_{u.1}];$$

$$F'_{\overline{n+u}.v} - a.F_{\overline{n+u}.v} = F'_{n.v} L_{u.o} - [F'_{n.\overline{v+1}} L_{u.\overline{s-1}} + F'_{n.\overline{v+2}} L_{u.\overline{s-2}} + \ldots + F'_{n.\overline{v+s-1}} L_{u.1} + F_{n.v} L'_{u.o} - [F_{n.\overline{v+1}} L'_{u.\overline{s-1}} + F_{n.\overline{v+2}} L'_{u.\overline{s-2}} + \ldots + F_{n.\overline{v+s-1}} L'_{u.1}].$$

Coroll. 2. Posito $k(r+v)$ pro $f(r+v)$ in *Coroll.* præced. 1 transit $F_{\overline{n+u}.v}$ in $K_{\overline{n+u}.v}$, nec non $F'_{\overline{n+u}.v}$ in $K'_{\overline{n+u}.v}$. Unde colligitur

$$K_{\overline{n+u}.v} = K_{n.v} K_{u.o} + K_{n.\overline{v+1}} K_{u.\overline{s-1}} + K_{n.\overline{v+2}} K_{u.\overline{s-2}} + \ldots + K_{n.\overline{v+s-1}} K_{u.1};$$ nec non

$$K'_{\overline{n+u}.v} - a.K_{\overline{n+u}.v} = K'_{n.v} K_{u.o} + K'_{n.\overline{v+1}} K_{u.\overline{s-1}} + K'_{n.\overline{v+2}} K_{u.\overline{s-2}} + \ldots + K'_{n.\overline{v+s-1}} K_{u.1} + K_{n.v} K'_{u.o} + K_{n.\overline{v+1}} K'_{u.\overline{s-1}} K_{n.\overline{v+2}} + K'_{u.\overline{s-2}} + \ldots + K_{n.\overline{v+s-1}} K'_{u.1}.$$

Coroll. 3. Neque secus, substituendo $l(r+v)$ pro $f(r+v)$ in *Coroll.* 1, obtinebitur

$$L_{\overline{n+u}.v} = L_{n.v} L_{u.o} - [L_{n.\overline{v+1}} L_{u.\overline{s-1}} + L_{n.\overline{v+2}} L_{u.\overline{s-2}} + \ldots + L_{n.\overline{v+s-1}} L_{u.1}];$$ nec non

$$L'_{\overline{n+u}.v} - a.L_{\overline{n+u}.v} = L'_{n.v} L_{u.o} - [L'_{n.\overline{v+1}} L_{u.\overline{s-1}} + L'_{n.\overline{v+2}} L_{u.\overline{s-2}} + \ldots + L'_{n.\overline{v+s-1}} L_{u.1}] + L_{n.v} L'_{u.o} - [L_{n.\overline{v+1}} L'_{u.\overline{s-1}} + L_{n.\overline{v+2}} L'_{u.\overline{s-2}} + \ldots + L_{n.\overline{v+s-1}} L'_{u.1}].$$

§. 5.

THEOREMA V. $K_{\overline{n+1}.v} = K_{n.v} + K_{n.\overline{v+1}};$

$$L_{\overline{n+1}.v} = L_{n.v} + L_{n.\overline{v+1}};$$

$$K'_{\overline{n+1}.v} = K'_{n.v} + K'_{n.\overline{v+1}} + b.K_{n.\overline{v+1}};$$

$$L'_{\overline{n+1}.v} = L'_{n.v} + L'_{n.\overline{v+1}} + b.L_{n.\overline{v+1}}.$$

Sit 1:o $s = 1$.

Est in hoc casu $K_{1.o} = k(o)A_{1.o} + k(1)A_{1.1} = k(o) + k(1) = 2k(o) = 2, L_{1.o} = l(o)A_{1.o} + l(1)A_{1.1} = l(o) + l(1) = l(o) - l(o) = o, K'_{1.o} = a.k(o)A_{1.o} + (a+b).k(1)A_{1.1} = 2a + b, L'_{1.o} = a.l(o)A_{1.o} + (a+b).l(1)A_{1.1} = -b, K_{n.v} + K_{n.\overline{v+1}} = 2K_{n.v}, K'_{n.v} + K'_{n.\overline{v+1}} = 2K'_{n.v}, L_{n.v} + L_{n.\overline{v+1}} = L'_{n.v} + L'_{n.\overline{v+1}} = o.$ Quamobrem erit, posito $u = s = 1$ in §. 4. Coroll. 2, 3;

$K_{\overline{n+1}.v} = K_{n.v}.K_{1.o} = 2K_{n.v} = K_{n.v} + K_{n.\overline{v+1}};$

$L_{\overline{n+1}.v} = L_{n.v}.L_{1.o} = o = L_{n.v} + L_{n.\overline{v+1}};$

$K'_{\overline{n+1}.v} = K'_{n.v}.K_{1.o} + K_{n.v}.K'_{1.o} - a.K_{\overline{n+1}.v}$

$= 2K'_{n.v} + (2a+b)K_{n.v} - 2aK_{n.v} = 2K'_{n.v} + bK_{n.v}$

$= K'_{n.v} + K'_{n.\overline{v+1}} + bK_{n.\overline{v+1}}$, nec non

$L'_{\overline{n+1}.v} = L'_{n.v}.L_{1.o} + L_{n.v}.L'_{1.o} - a.L_{\overline{n+1}.v}$

$= -b.L_{n.v} = L'_{n.v} + L'_{n.\overline{v+1}} + bL_{n.\overline{v+1}}$.

Sit 2:o $s > 1$.

Quia est $K_{1.v} = k(v)A_{1.o} + k(v+1)A_{1.1} = k(v) + k(v+1), L_{1.v} = l(v)A_{1.o} + l(v+1)A_{1.1} = l(v) + l(v+1),$

 $K'_{1.v}$

$K'_{1.v} = a.k(v) + (a+b).k(v+1)$, $L'_{1.v} = a.l(v) A_{1.0} + (a+b).l(v+1) A_{1.1} = a.l(v) + (a+b).l(v+1)$; substituendo pro v numeros: 0, $s-1$, $s-2$, &c., habebitur $K_{1.0} = k(0) + k(1) = k(0) = 1$, $K_{1.\overline{s-1}} = k(s-1) + k(s) = k(s) = k(0) = 1$, $K_{1.\overline{s-2}} = k(s-2) + k(s-1) = K_{1.\overline{s-3}} = .. = K_{1.1} = k(1) + k(2) = 0$; $L_{1.0} = l(0) + l(1) = l(0) = 1$, $L_{1.\overline{s-1}} = l(s-1) + l(s) = l(s) = -l(0) = -1$, $L_{1.\overline{s-2}} = l(s-2) + l(s-1) = L_{1.\overline{s-3}} = .. = L_{1.1} = l(1) + l(2) = 0$; $K'_{1.0} = a.k(0) + (a+b).k(1) = a.k(0) = a$, $K'_{1.\overline{s-1}} = a.k(s-1) + (a+b).k(s) = (a+b).k(s) = a+b$, $K'_{1.\overline{s-2}} = a.k(s-2) + (a+b).k(s-1) = K'_{1.\overline{s-3}} = .. = K'_{1.1} = a.k(1) + (a+b).k(2) = 0$; $L'_{1.0} = a.l(0) + (a+b).l(1) = a.l(0) = a$, $L'_{1.\overline{s-1}} = a.l(s-1) + (a+b).l(s) = -(a+b).l(0) = -(a+b)$, $L'_{1.\overline{s-2}} = a.l(s-2) + (a+b).l(s-1) = L'_{1.\overline{s-3}} = .. = L'_{1.1} = a.l(1) + (a+b).l(2) = 0$. Quibus adhibitis valoribus, ope *Coroll.* 2, 3, §. 4, evincitur esse

$$K_{\overline{n+1}.v} = K_{n.v}.K_{1.0} + K_{n.\overline{v+1}}.K_{1.\overline{s-1}} + K_{n.\overline{v+2}}.K_{1.\overline{s-2}} \text{ \&c.}$$
$$= K_{n.v} + K_{n.\overline{v+1}};$$

$$L_{\overline{n+1}.v} = L_{n.v}.L_{1.0} - L_{n.\overline{v+1}}.L_{1.\overline{s-1}} - L_{n.\overline{v+2}}.L_{1.\overline{s-2}} \text{ \&c.}$$
$$= L_{n.v} + L_{n.\overline{v+1}};$$

$$K'_{\overline{n+1}.v} = K'_{n.v}.K_{1.0} + K'_{n.\overline{v+1}}.K_{1.\overline{s-1}} + K'_{n.\overline{v+2}}.K_{1.\overline{s-2}} \text{ \&c.}$$
$$+ K_{n.v}.K'_{1.0} + K_{n.\overline{v+1}}.K'_{1.\overline{s-1}} \text{ \&c.} - a.K_{\overline{n+1}.v}$$
$$= K'_{n.v}$$

$= K'_{n.v} + K'_{n.\overline{v+1}} + a.K_{n.v} + (a+b).K_{n.\overline{v+1}} - a.K_{\overline{n+1}.v}$
$= K'_{n.v} + K'_{n.\overline{v+1}} + bK_{n.\overline{v+1}}$;

$L'_{\overline{n+1}.v} = L'_{n.v}.L_{1,0} - L'_{n.\overline{v+1}}.L_{1,\overline{s-1}} - L'_{n.\overline{v+2}}.L_{1,\overline{s-2}}$ &c.
$+ L_{n.v}.L'_{1.0} - L_{n.\overline{v+1}}.L'_{1.\overline{s-1}}$ &c. $- a.L_{\overline{n+1}.v}$
$= L'_{n.v} + L'_{n.\overline{v+1}} + a.L_{n.v} + (a+b)L_{n.\overline{v+1}} - aL_{\overline{n+1}.v}$
$= L'_{n.v} + L'_{n.\overline{v+1}} + b.L_{n.\overline{v+1}}$.

§. 6.

THEOREMA VI.

$K_{\overline{n+s}.v} = 2K_{n.v} + K_{n.\overline{v+1}}A_{s.1} + K_{n.\overline{v+2}}A_{s.2} + \ldots$
$+ K_{n.\overline{v+s-1}}.A_{s.\overline{s-1}}$;
$L_{\overline{n+s}.v} = L_{n.\overline{v+1}}A_{s.1} + L_{n.\overline{v+2}}A_{s.2} + \ldots$
$+ L_{n.\overline{v+s-1}}.A_{s.\overline{s-1}}$.

Facto $u = s$ in formulis $K_{\overline{n+u}.v}$, $L_{\overline{n+u}.v}$ (§. 4. *Coroll.* 2, 3) habetur
$K_{\overline{n+s}.v} = K_{n.v}.K_{s.0} + K_{n.\overline{v+1}}.K_{s,\overline{s-1}} + K_{n.\overline{v+2}}.K_{s,\overline{s-2}}$
$+ \ldots + K_{n.\overline{v+s-1}}.K_{s.1}$; nec non
$L_{\overline{n+s}.v} = L_{n.v}.L_{s.0} - [L_{n.\overline{v+1}}.L_{s.\overline{s-1}} + L_{n.\overline{v+2}}L_{s.\overline{s-2}}$
$+ \ldots + L_{n.\overline{v+s-1}}L_{s.1}]$. Unde, cum sit
$K_{s.v} = k(v)A_{s.0} + k(v+1)A_{s.1} + \ldots + k(v+s-2).A_{s.\overline{s-2}}$
$+ k(v+s-1).A_{s.\overline{s-1}} + k(v+s).A_{s.s}$; nec non
$L_{s.v} = l(v)A_{s.0} + l(v+1)A_{s.1} + \ldots + l(v+s-2)A_{s.\overline{s-2}}$
$+ l(v+s-1)A_{s.\overline{s-1}} + l(v+s)A_{s.s}$; ideoque

$K_{s.0}$

$K_{s.0} = k(0)\,A_{s.0} + k(1)\,A_{s.1} + .. + k(s)\,A_{s.s} = k(0)\,A_{s.0} + k(s)\,A_{s.s} = 2$, $K_{s.\overline{s-1}} = k(s-1)\,A_{s.0} + k(s)\,A_{s.1} + \&c. = k(s)\,A_{s.1} = A_{s.1}$, $K_{s.\overline{s-2}} = k(s-2)\,A_{s.0} + k(s-1)\,A_{s.1} + k(s)\,A_{s.2}\,\&c. = A_{s.2}$, $K_{s.\overline{s-3}} = k(s)\,A_{s.3} = A_{s.3}$, .. $K_{s.1} = k(s)\,A_{s.\overline{s-1}} = A_{s.\overline{s-1}}$; $L_{s.0} = l(0)\,A_{s.0} + l(1)\,A_{s.1} + .. + l(s)\,A_{s.s} = 0$, $L_{s.\overline{s-1}} = l(s-1)\,A_{s.0} + l(s)\,A_{s.1}\,\&c. = -A_{s.1}$, $L_{s.\overline{s-2}} = l(s)\,A_{s.2} = -A_{s.2}$, .. $L_{s.1} = l(s)\,A_{s.\overline{s-1}} = -A_{s.\overline{s-1}}$; his ſubſtitutis valoribus veritas Theorematis haud difficulter demonſtratur.

Coroll. 1. Sit $s = 1$. Quo pacto erit $K_{\overline{n+s}.v} = K_{\overline{n+1}.v} = 2\,K_{n.v}$, & $L_{\overline{n+s}.v} = L_{\overline{n+1}.v} = 0$ (cfr. præced. §. 5.).

Coroll. 2. Sit $s = 2$. Unde $K_{\overline{n+s}.v} = K_{\overline{n+2}.v} = 2\,K_{n.v} + K_{n.\overline{v+1}}.\,A_{2.1} = 2\,(K_{n.v} + K_{n.\overline{v+1}}) = 2\,K_{\overline{n+1}.v}$ (*Theor. V.*), & $L_{\overline{n+s}.v} = L_{\overline{n+2}.v} = L_{n.\overline{v+1}}.\,A_{2.1} = 2\,L_{n.\overline{v+1}}$.

Coroll. 3. Poſito $s = 3$ habetur $K_{\overline{n+s}.v} = K_{\overline{n+3}.v} = 2\,K_{n.v} + K_{n.\overline{v+1}}\,A_{3.1} + K_{n.\overline{v+2}}.\,A_{3.2} = 2\,K_{n.v} + 3\,K_{n.\overline{v+1}} + 3\,K_{n.\overline{v+2}}$, nec non $L_{\overline{n+s}.v} = L_{\overline{n+3}.v} = L_{n.\overline{v+1}}\,A_{3.1} + L_{n.\overline{v+2}}\,A_{3.2} = 3\,L_{n.\overline{v+1}} + 3\,L_{n.\overline{v+2}} = 3\,L_{\overline{n+1}.\overline{v+1}}$.

§. 7.

THEOREMA VII.

$$K'_{\overline{n+s}.v} = 2K'_{n.v} + bs.K_{n.v} + (K'_{n.\overline{v+1}} + b.K_{n.\overline{v+1}})A_{s.1}$$
$$+ (K'_{n.\overline{v+2}} + 2b.K_{n.\overline{v+2}})A_{s.2} + \ldots$$
$$+ (K'_{n.\overline{v+s-1}} + (s-1)b.K_{n.\overline{v+s-1}})A_{s.\overline{s-1}};$$

$$L'_{\overline{n+s}.v} = -bs.L_{n.v} + (L'_{n.\overline{v+1}} + b.L_{n.\overline{v+1}})A_{s.1}$$
$$+ (L'_{n.\overline{v+2}} + 2b.L_{n.\overline{v+2}})A_{s.2} + \ldots$$
$$+ (L'_{n.\overline{v+s-1}} + (s-1)b.L_{n.\overline{v+s-1}})A_{s.\overline{s-1}}.$$

Inserto s pro u in formulis $K'_{\overline{n+u}.v}$, $L'_{\overline{n+u}.v}$ (§. 4. *Coroll.* 2. 3.) hæ exſurgunt æquationes:

$$K'_{\overline{n+s}.v} = K'_{n.v}.K_{s.0} + K'_{n.\overline{v+1}}.K_{s.\overline{s-1}} + \ldots$$
$$+ K'_{n.\overline{v+s-1}}.K_{s.\overline{s-1}} + K_{n.v}.K'_{s.0} + K_{n.\overline{v+1}}.K'_{s.\overline{s-1}}$$
$$+ \ldots + K_{n.\overline{v+s-1}}.K'_{s.1} - a.K_{\overline{n+s}.v};$$

$$L'_{\overline{n+s}.v} = L'_{n.v}.L_{s.0} - [L'_{n.\overline{v+1}}.L_{s.\overline{s-1}} + \ldots$$
$$+ L'_{n.\overline{v+s-1}}.L_{s.1}] + L_{n.v}.L'_{s.0} - [L_{n.\overline{v+1}}.L'_{s.\overline{s-1}}$$
$$+ \ldots + L_{n.\overline{v+s-1}}.L'_{s.1}] - a.L_{\overline{n+s}.v}.$$

Quod ſi in hisce formulis ſubſtituantur valores quantitatum: $K_{s.0}$, $K_{s.\overline{s-1}}$, $K_{s.\overline{s-2}}$, .. $K_{s.1}$; $L_{s.0}$, $L_{s.\overline{s-1}}$, $L_{s.\overline{s-2}}$, .. $L_{s.1}$; in præcedenti §. 6. conſtituti, & determinentur $K'_{s.0}$, $K'_{s.\overline{s-1}}$, $K'_{s.\overline{s-2}}$, .. $K'_{s.1}$; $L'_{s.0}$, $L'_{s.\overline{s-1}}$, $L'_{s.\overline{s-2}}$, .. $L'_{s.1}$, ope æquationum:

$K'_{s.v} = a.k(v)\,A_{s.0} + (a+b).k(v+1)\,A_{s.1} + \dots$
$+ (a+bs).k(v+s)\,A_{s.s}$;
$L'_{s.v} = a.l(v)\,A_{s.0} + (a+b).l(v+1)\,A_{s.1} + \dots$
$+ (a+bs).l(v+s)\,A_{s.s}$;
nostrum per se constat Theorema.

Coroll. 1. $s=1$. $K'_{\overline{n+s}.v} = K'_{\overline{n+1}.v} = 2K'_{n.v} + bK_{n.v}$, & $L'_{\overline{n+s}.v} = L'_{\overline{n+1}.v} = -b.L_{n.v}$.

Coroll. 2. $s=2$; $K'_{\overline{n+s}.v} = K'_{\overline{n+2}.v} = 2K'_{n.v} + 2bK_{n.v} + (K'_{n.\overline{v+1}} + b.K_{n.\overline{v+1}})A_{2.1} = 2(K'_{n.v} + K'_{n.\overline{v+1}} + b.K_{n.\overline{v+1}}) + 2b.K_{n.v} = 2K'_{\overline{n+1}.v} + 2b.K_{n.v}$ (Theor. V), nec non $L'_{\overline{n+s}.v} = L'_{\overline{n+2}.v} = -2b.L_{n.v} + (L'_{n.\overline{v+1}} + b.L_{n.\overline{v+1}})A_{2.1} = 2L'_{n.\overline{v+1}} + 2b.(L_{n.\overline{v+1}} - L_{n.v})$.

§. 8.

Theorema VIII. Si fuerit $f(r+s)$ vel $=f(r)$ vel $=-f(r)$; erit, in illo casu:

$F_{n.v} = f(v).K_{n.0} + f(v+1).K_{n.\overline{s-1}} + \dots + f(v+s-1).K_{n.1}$;
$F'_{n.v} = f(v).K'_{n.0} + f(v+1)K'_{n.\overline{s-1}} + \dots + f(v+s-1).K'_{n.1}$;

in hoc vero casu:

$F_{n.v} = f(v).L_{n.0} - [f(v+1)L_{n.\overline{s-1}} + \dots + f(v+s-1).L_{n.1}]$;
$F'_{n.v} = f(v).L'_{n.0} - [f(v+1).L'_{n.\overline{s-1}} + \dots + f(v+s-1).L'_{n.1}]$.

Facto

Facto $n = o$, & deinde posito n pro u in formulis: $F_{n + u.v}$, $F_{n + u.v}$; in §. 4. *Coroll.* 1. determinatis, habebitur, in casu priori: $F_{n.v} = F_{o.v}.K_{n.o} + F_{o.\overline{v+1}}.K_{n.\overline{s-1}} + F_{o.\overline{v+2}}.K_{n.\overline{s-2}} + .. + F_{o.\overline{v+s-1}}.K_{n.1}$; $F'_{n.v} + a.F_{n.v} = F'_{o.v}.K_{n.o} + F'_{o.\overline{v+1}}.K_{n.\overline{s-1}} + F'_{o.\overline{v+2}}.K_{n.\overline{s-2}} + ... + F'_{o.\overline{v+s-1}}.K_{n.1} + F_{o.v}.K'_{n.o} + F_{o.\overline{v+1}}.K'_{n.\overline{s-1}} + F_{o.\overline{v+2}}.K'_{n.\overline{s-2}} + .. + F_{o.\overline{v+s-1}}.K'_{n.1}$; nec non in casu altero: $F_{n.v} = F_{o.v}.L_{n.o} - [F_{o.\overline{v+1}}.L_{n.\overline{s-1}} + F_{o.\overline{v+2}}.L_{n.\overline{s-2}} + .. + F_{o.\overline{v+s-1}}.L_{n.1}]$; $F'_{n.v} + a.F_{n.v} = F'_{o.v}.L_{n.o} - [F'_{o.\overline{v+1}}\, L_{n.\overline{s-1}} + F'_{o.\overline{v+2}}.L_{n.\overline{s-2}} + .. + F'_{o.\overline{v+s-1}}.L_{n.1}] + F_{o.v}L'_{n.o} - [F_{o.\overline{v+1}}.L'_{n.\overline{s-1}} + F_{o.\overline{v+2}}.L'_{n.\overline{s-2}} + .. + F_{o.\overline{v+s-1}}.L'_{n.1}]$. Quia vero functionum: $F_{n.v}$, $F'_{n.v}$; ex terminis formae: $f(r+v).A_{n.r}$, vel formae: $(a+rb).f(r+v).A_{n.r}$ compositarum, in casu $n = o$, utraque cum primo exaequatur suae seriei termino: $f(v).A_{o.o} = f(v)$, vel $a.f(v)\,A_{o.o} = a.f(v)$, indici $r = o$ debito; habebitur $F_{o.v} = f(v)$, $F_{o.\overline{v+1}} = f(v+1)$, $F_{o.\overline{v+2}} = f(v+2)$, &c.; $F'_{o.v} = a.f(v)$, $F'_{o.\overline{v+1}} = a.f(v+1)$, $F'_{o.\overline{v+2}} = a.f(v+2)$, &c. Quibus substitutis valoribus exsurgunt aequationes Theorema nostrum componentes.

§. 9.

Exempla, quibus Theoremata praecedentia ad summandas series, tum formae $F_{n.v}$, tum formae $F'_{n.v}$, adplicandi methodus commonstratur.

Exempl. 1.

Exempl. 1. Sit $f(r+1)=f(r)$. Est in hoc exemplo $K_{\overline{n+1}.v}=2K_{n.v}$ (*Theor. VI. Cor. 1*), ideoque $K_{\overline{n+2}.v}=2K_{\overline{n+1}.v}=2^2.K_{n.v}$, atque generatim $K_{\overline{n+u}.v}=2^u.K_{n.v}$. Unde, vi *Theor. VIII*, $F_{\overline{n+u}.v}=f(v).K_{\overline{n+u}.o}=f(v).2^u.K_{n.o}$, ideoque $F_{u.v}=f(v).2^u$, sive posito n pro u, $F_{n.v}=f(v).2^n$. Est præterea $K'_{\overline{n+1}.v}=2K'_{n.v}+bK_{n.v}$ (*Theor. VII. Cor. 1*), ideoque $K'_{\overline{n+2}.v}=2K'_{\overline{n+1}.v}+bK_{\overline{n+1}.v}=2^2K'_{n.v}+2bK_{n.v}+bK_{\overline{n+1}.v}=2^2K'_{n.v}+2bK_{\overline{n+1}.v}$, $K'_{\overline{n+3}.v}=2K'_{\overline{n+2}.v}+bK_{\overline{n+2}.v}=2^3K'_{n.v}+3bK_{\overline{n+2}.v}$, atque generatim $K'_{\overline{n+u}.v}=2^uK'_{n.v}+bu.K_{\overline{n+u}.v}$ $=2^u.(K'_{n.v}+\frac{1}{2}bu.K_{n.v})$. Unde, facto $n=o$, & posito n pro u, $K'_{n.v}=2^n.(K'_{o.v}+\frac{1}{2}bn.K_{o.v})=2^n.(a.k(v)$ $+\frac{1}{2}nb.k(v))=2^n.(a+\frac{1}{2}nb).k(v)$. Quare erit (*Theor. VIII*) $F'_{n.v}=f(v).K'_{n.o}=2^n.(a+\frac{1}{2}nb)f(v).k(o)$ $=2^n.(a+\frac{1}{2}nb).f(v)$. Quod si fiat $f(v)=1$, habebitur ${}^1F_n=2^n$ & ${}^1F'_n=2^n.(a+\frac{1}{2}nb)$.

Exempl. 2. Sit $f(r+1)=-f(r)$. Unde $L_{\overline{n+1}.v}=o$ (*Theor. VI. Cor. 1.*) & $L'_{\overline{n+1}.v}=-bL_{n.v}$ (*Theor. VII. Cor. 1*), ideoque $L'_{\overline{n+2}.v}=-bL_{\overline{n+1}.v}=o$. Quamobrem erit, vi *Theor. VIII*, $F_{\overline{n+1}.v}=f(v).L_{\overline{n+1}.o}$ $=o$, $F'_{\overline{n+1}.v}=f(v).L'_{\overline{n+1}.v}=-b.f(v).L_{n.o}$

&

& $F'_{\overline{n+2}.v} = f(v) \,.\, L'_{\overline{n+2}.o} = -b \,.\, f(v) \,.\, L_{\overline{n+1}.o}$ $= o$. Hinc, positis pro n numeris o, 1, 2, &c., habebitur $F_{1.v} = o$, $F'_{1.v} = -b \,.\, f(v) L_{o.o} = -b \,.\, f(v)$, $F_{2.v} = o$, $F'_{2.v} = o$, &c. atque generatim $F_{n.v} = F'_{n.v} = o$, exceptis $F_{o.v} = f(v)$, $F'_{o.v} = a \,.\, f(v)$, & $F'_{1.v} = -b \,.\, f(v)$. Facto jam $f(v) = 1$ exsurgit ${}^2F_n = {}^2F'_n = o$, si exceperis ${}^2F_o = 1$, ${}^2F'_o = a$, & ${}^2F'_1 = -b$.

Exempl. 3. Ponatur $f(r+2) = f(r)$. Quo pacto erit $K_{\overline{n+2}.v} = 2 K_{\overline{n+1}.v}$ (*Theor.* *VI.* *Cor.* 2), $K_{\overline{n+3}.v} = 2 K_{\overline{n+2}.v} = 2^2 \,.\, K_{\overline{n+1}.v}$, $K_{\overline{n+4}.v} = 2^3 \,.\, K_{\overline{n+1}.v}$, atque generatim $K_{\overline{n+u}.v} = 2^{u-1} \,.\, K_{\overline{n+1}.v}$, ideoque, facto $n = o$ & posito n pro u, $K_{n.v} = 2^{n-1} \,.\, K_{1.v}$. Unde, vi *Theor.* *VIII*, obtinebitur $F_{n.v} = f(v) \,.\, K_{n.o} + f(v+1) K_{n.1} = 2^{n-1} \,.\, (f(v) \,.\, K_{1.o} + f(v+1) \,.\, K_{1.1}) = 2^{n-1} \,.\, (f(v) + f(v+1))$, ob $K_{1.o} = K_{1.1} = 1$, si exceperis $F_{o.v} = f(v)$. Est vero præterea $K'_{\overline{n+2}.v} = 2 K'_{\overline{n+1}.v} + 2b\, K_{n.v}$ (*Theor.* *VII.* *Cor.* 2), $K'_{\overline{n+3}.v} = 2 K'_{\overline{n+2}.v} + 2b\, K_{\overline{n+1}.v} = 2^2 \,.\, [K'_{\overline{n+1}.v} + b\,(K_{n.v} + \tfrac{1}{2} K_{\overline{n+1}.v})]$, $K'_{\overline{n+4}.v} = 2 K'_{\overline{n+3}.v} + 2b\, K_{\overline{n+2}.v} = 2^3 \,.\, [K'_{\overline{n+1}.v} + b \,.\, (K_{n.v} + K_{\overline{n+1}.v})]$, $K'_{\overline{n+5}.v} = K'_{\overline{n+4}.v} + 2 K_{\overline{n+3}.v} = 2^4 \,.\, [K'_{\overline{n+1}.v} + b\,(K_{n.v} + \tfrac{3}{2} K_{\overline{n+1}.v})]$ atque generatim $K'_{\overline{n+u+2}.v} = 2^{u+1} \,.\, [K'_{\overline{n+1}.v} + b\,(K_{n.v} + \tfrac{1}{2} \,.\, u \,.\, K_{\overline{n+1}.v})]$. Unde, vi *Theor.* *VIII.*, posito nimirum

mirum $n = 0$, obtinebitur $F'_{u+2.v} = 2^{u+1}.[K''_{1.0} + b.(K_{0.0} + \frac{1}{2}u.K_{1.0})].f(v) + 2^{u+1}.[K'_{1.1} + b(K_{0.1} + \frac{1}{2}u.K_{1.1})].f(v+1)$. Quare, cum sit $K'_{1.0} = a.k(0) + (a+b).k(1) = a.k(0) = a$, $K'_{1.1} = a.k(1) + (a+b).k(2) = (a+b).k(2) = a+b$, $K_{0.0} = 1$, $K_{0.1} = 0$, & $K_{1.0} = K_{1.1} = 1$; habebitur, posito n pro u, $F'_{n+2.v} = 2^{n+1}.[a + \frac{1}{2}.\overline{n+2}.b].f(v) + 2^{n+1}.[a + \frac{1}{2}\overline{n+2}.b].f(v+1) = 2^{n+1}.[a + \frac{1}{2}\overline{n+2}.b].(f(v) + f(v+1))$, sive $F'_{n.v} = 2^{n-1}.[a + \frac{1}{2}nb].(f(v) + f(v+1))$, exceptis $F'_{0.v} = a.f(v)$, & $F'_{1.v} = a.f(v) + (a+b).f(v+1)$.

Sit 1:o $f(v) = 1^v = f(v+1)$. Unde $F_{n.v} = {}^1F_n = 2.2^{n-1} = 2^n$, & $F'_{n.v} = {}^1F'_n = 2.2^{n-1}.[a + \frac{1}{2}n.b] = 2^n.(a + \frac{1}{2}nb)$, ut in *Exempl.* 1.

Sit 2:o $f(v) = (-1)^v = 1$. Unde $f(v+1) = (-1)^{v+1} = -1$, ideoque $F_{n.v} = {}^2F_n = 0$, excepto ${}^2F_0 = 1$, nec non $F'_{n.v} = 0$, exceptis ${}^2F'_0 = a$, & ${}^2F'_1 = -b$, ut in *Ex.* 2.

Fiat 3:o $f(v) = \frac{1}{2}(1 + (-1)^v) = 1$. Unde $f(v+1) = 0$, ideoque $F_{n.v} = {}^3F_n = 2^{n-1}$, excepto ${}^3F_0 = 1$, atque $F'_{n.v} = {}^3F'_n = 2^{n-1}.(a + \frac{1}{2}nb)$, si exceperis ${}^3F'_0 = a$ & ${}^3F'_1 = a$.

Ponatur 4:o $f(v) = \frac{1}{2}(1 - (-1)^v) = 0$. Quo pacto erit $f(v+1) = 1$, $F_{n.v} = {}^4F_n = 2^{n-1}$, nec non $F'_{n.v} =$

$F'_{n.v} = {}^4F'_n = 2^{n.1} . (a + \frac{1}{2}.nb)$, exceptis ${}^4F_0 = 0$, ${}^4F'_0 = 0$, & ${}^4F_1 = a + b$.

Exempl. 4. Sit $f(r+2) = -f(r)$. Quo pacto, vi *Theor VI. Cor.* 2, habebitur $L_{\overline{n+2}.v} = 2\,L_{n.\overline{v+1}}$, $L_{\overline{n+2}.\overline{v+1}} = 2\,L_{n.\overline{v+2}} = -2\,L_{n.v}$, & $L_{\overline{n+4}.v} = 2\,L_{\overline{n+2}.\overline{v+1}} = -4\,L_{n.v}$. Hinc vero colligitur esse $L_{\overline{n+8}.v} = -4\,L_{\overline{n+4}.v} = (-4)^2 . L_{n.v}$, $L_{\overline{n+12}.v} = -4\,L_{\overline{n+8}.v} = (-4)^3 . L_{n.v}$, atque generatim $L_{\overline{n+4m}.v} = (-4)^m . L_{n.v}$. Unde, cum sit $L_{0.v} = l(v)$, ideoque (*Theor. V*) $L_{1.v} = L_{0.v} + L_{0.\overline{v+1}} = l(v) + l(v+1)$, $L_{2.v} = 2.l(v+1)$, & $L_{3.v} = 2.l(v+1) + 2.l(v+2) = 2.l(v+1) - 2.l(v)$; vi *Theor. VIII*, erit $F_{4m.v} = f(v).L_{4m.0} - f(v+1).L_{4m.1} = (-4)^m.[f(v).l(0) - f(v+1).l(1)] = (-4)^m . f(v)$, $F_{\overline{4m+1}.v} = f(v).L_{\overline{4m+1}.0} - f(v+1).L_{\overline{4m+1}.1} = (-4^m.[f(v).(l(0)+l(1)) - f(v+1).(l(1) + l(2))] = (-4)^m . [f(v) . l(0) - f(v+1) . l(2)] = (-4)^m . [f(v) + f(v+1)]$, $F_{\overline{4m+2}} = f(v).L_{\overline{4m+2}.0} - f(v+1).L_{\overline{4m+2}.1} = (-4)^m.f(v)\,L_{2.0} - (-4)^m.f(v+1)\,L_{2.1} = 2.(-4)^m.f(v).l(1) - 2.(-4)^m.f(v+1).l(2) = 2.(-4)^m.f(v+1)$, nec non $F_{\overline{4m+3}.v} = f(v).L_{\overline{4m+3}.0} - f(v+1).L_{\overline{4m+3}.1} = (-4)^m.f(v).L_{3.0} - (-4)^m.f(v+1).L_{3.1} = -2.(-4)^m.[f(v) - f(v+1)]$.

Est

Est vero (*Theor. VII. Cor.* 2) $L'_{\overline{n+2}.v} = 2L'_{n}.\overline{v+1} + 2b(L_{n}.\overline{v+1} - L_{1}.v)$, ideoque $L'_{\overline{n+2}}.\overline{v+1} = -2L'_{n}.v - 2b(L_{n}.v + L_{n}.\overline{v+1}) = -2(L'_{n}.v + bL_{\overline{1+1}.v})$, & $L'_{\overline{n+4}.v} = 2L'_{\overline{n+2}}.\overline{v+1} + 2b(L_{\overline{n+2}}.\overline{v+1} - L_{\overline{n+2}.v}) = -4(L'_{n}.v + bL_{\overline{n+1}.v}) - 2b(2L_{n}.v + 2L_{n}.\overline{v+1}) = -4(L'_{n}.v + 2b.L_{\overline{n+1}.v})$. Unde $L'_{\overline{n+8}.v} = -4(L'_{\overline{n+4}.v} + 2bL_{\overline{n+5}.v}) = (-4)^2.(L'_{n}.v + 4b.L_{\overline{n+1}.v})$ (ob $L_{\overline{n+5}.v} = -4L_{\overline{n+1}.v}$); $L'_{\overline{n+12}.v} = -4(L'_{\overline{n+8}.v} + 2b.L_{\overline{n+9}.v}) = (-4)^3.(L'_{n}.v + 6b.L_{\overline{n+1}.v})$ (ob $L_{\overline{n+9}.v} = (-4)^2.L_{\overline{n+1}.v}$); atque generatim $L'_{\overline{n+4m}.v} = (-4)^m.(L'_{n}.v + 2mb.L_{\overline{n+1}.v})$. Unde, substitutis pro $L'_{n}.v$ & $L_{\overline{n+1}.v}$, quos posito $n = 0, 1, 2, 3$ recipiunt, valoribus, ope *Theor. V* definiendis, obtinebitur $L'_{4m.v} = (-4)^m.(L'_{0}.v + 2mb.L_{1}.v) = (-4)^m.[a.l(v) + 2mb(l(v) + l(v+1))]$; $L'_{\overline{4m+1}.v} = (-4)^m.(L'_{1}.v + 2mb.L_{2}.v) = (-4)^m.[a.l(v) + (a+b).l(v+1) + 4mb.l(v+1)] = (-4)^m.[a.l(v) + (a + \overline{4m+1}.b).l(v+1)]$; $L'_{\overline{4m+2}.v} = (-4)^m.(L'_{2}.v + 2mb.L_{3}.v) = 2(-4)^m.[(a + \overline{2m+1}.b).l(v+1) - \overline{2m+1}.b.l(v)]$; nec non $L'_{\overline{4m+3}.v} = (-4)^m.(L'_{3}.v + 2mb.L_{4}.v) = 2.(-4)^m.[a.l(v+1) - (a + \overline{4m+3}.b).l(v)]$. Quamobrem, vi *Theor. VIII*, erit $F'_{4m.v} = f(v).L_{4m.0} - f(v+1).L'_{4m.1} = (-4)^m.[(a+2mb).f(v) + 2mb.f(v+1)]$; $F'_{\overline{4m+1}.v} = f(v).L'_{\overline{4m+1}.0} - f(v+1).L'_{\overline{4m+1}.1} = (-4)^m.[a.f(v) + (a + \overline{4m+1}.b).f(v+1)]$; $F'_{\overline{4m+2}.v} = f(v)$.

$= f(v) . L'_{\overline{4m+2}.0} - f(v+1) . L'_{\overline{4m+2}.1} = -2 . (-4)^m .$ $[\overline{2m+1} . b f(v) - (a + \overline{2m+1} . b) . f(v+1)]$; nec non $F'_{\overline{4m+3}.v} = f(v) . L'_{\overline{4m+3}.0} - f(v+1) . L'_{\overline{4m+3}.1} =$ $-2 . (-4)^m . [(a + \overline{4m+3} . b) . f(v) - a . f(v+1)]$.

Sit 1:o $f(v) = Cos. vq = 1$. Unde $f(v+1) = Cos. \overline{v+1} . q = 0$, $F_{4m.v} = {}^5F_{4m} = (-4)^m$, $F_{\overline{4m+1}.v} = {}^5F_{\overline{4m+1}} = (-4)^m$, $F_{\overline{4m+2}.v} = {}^5F_{\overline{4m+2}} = 0$, $F_{\overline{4m+3}.v} = {}^5F_{\overline{4m+3}} = -2 . (-4)^m$: nec non $F'_{4m.v} = {}^5F'_{4m} = (-4)^m . (a + 2mb)$, $F'_{\overline{4m+1}} = {}^5F'_{\overline{4m+1}} = (-4)^m . a$, $F'_{\overline{4m+2}.v} = {}^5F'_{\overline{4m+2}} = -(-4)^m . (\overline{4m+2} . b)$, & $F'_{\overline{4m+3}.v} = {}^5F'_{\overline{4m+3}} = -2 . (-4)^m . (a + \overline{4m+3} . b)$.

Fiat 2:o $f(v+1) = Sin . \overline{v+1} . q = 1$. Quo pacto erit $f(v) = Sin . vq = 0$; ideoque $F_{4m.v} = {}^6F_{4m} = 0$, $F_{\overline{4m+1}.v} = {}^6F_{\overline{4m+1}} = (-4)^m$, $F_{\overline{4m+2}.v} = {}^6F_{\overline{4m+2}} = 2 . (-4)^m$, $F_{\overline{4m+3}.v} = {}^6F_{\overline{4m+3}} = 2 . (-4)^m$; nec non $F'_{4m.v} = {}^6F'_{4m} = (-4)^m . 2mb$, $F'_{\overline{4m+1}.v} = {}^6F'_{\overline{4m+1}} = (-4)^m . (a + \overline{4m+1} . b)$, $F'_{\overline{4m+2}.v} = {}^6F'_{\overline{4m+2}} = (-4)^m . (2a + \overline{4m+2} . b)$, & $F'_{\overline{4m+3}.v} = {}^6F'_{\overline{4m+3}} = (-4)^m . 2a$.

Ponatur 3:o $f(v) = Cos. vq + Sin . vq = 1$, & $f(v+1) = Cos \overline{v+1} . q + Sin \overline{v+1} . q = 1$. Unde $F_{4m.v} = {}^7F_{4m} = (-4)^m$, $F_{\overline{4m+1}.v} = {}^7F_{\overline{4m+1}} = 2 . (-4)^m$, $F_{\overline{4m+2}.v} = {}^7F_{\overline{4m+2}} = 2 . (-4)^m$, $F_{\overline{4m+3}.v} = {}^7F_{\overline{4m+3}} = 0$;

 nec

nec non $F'_{4m.v} = {}^7F'_{4m} = (-4)^m.(a + 4mb)$, $F'_{\overline{4m+1}.v} = {}^7F'_{\overline{4m+1}} = (-4)^m.(2a + \overline{4m+1}.b)$, $F'_{\overline{4m+2}.v} = {}^7F'_{\overline{4m+2}} = (-4)^m.2a$, & $F'_{\overline{4m+3}.v} = {}^7F'_{\overline{4m+3}} = -2.(-4)^m.(\overline{4m+3}.b)$.

Quod si 4:0 fiat $f(v) = Cos.vq - Sin.vq = 1$, & $f(v+1) = Cos.\overline{v+1}.q - Sin.\overline{v+1}.q = -1$; prodibit $F_{4m.v} = {}^8F_{4m} = (-4)^m$, $F_{\overline{4m+1}.v} = {}^8F_{\overline{4m+1}} = 0$, $F_{\overline{4m+2}.v} = {}^8F_{\overline{4m+2}} = -2.(-4)^m$, $F_{\overline{4m+3}.v} = {}^8F_{\overline{4m+3}} = (-4)^{m+1}$; $F'_{4m.v} = {}^8F'_{4m} = (-4)^m.a$, $F'_{\overline{4m+1}.v} = {}^8F'_{\overline{4m+1}} = -(-4)^m.(\overline{4m+1}.b)$, $F'_{\overline{4m+2}.v} = {}^8F'_{\overline{4m+2}} = -2.(-4)^m.(a + \overline{4m+2}.b)$, & $F'_{\overline{4m+3}.v} = {}^8F'_{\overline{4m+3}} = -2(-4)^m.(2a + \overline{4m+3}.b)$.

Exempl. 5. Ponatur $f(r+3) = f(r)$. Fiat brevitatis gratia $K_{n.v} + K_{n.\overline{v+1}} + K_{n.\overline{v+2}} = X_n$, $K'_{n.v} + K'_{n.\overline{v+1}} + K'_{n.\overline{v+2}} = X'_n$, $\dfrac{2^{3m} - (-1)^{3m}}{2^3 + 1} = C_m$, $\tfrac{3}{2}((m-1).2^{3.\overline{m-1}} - (m-2).2^{3.\overline{m-2}} + \ldots - (-1)^{m-1}.2^3) = D_m$, $C_m + 3(C_{m-1} - C_{m-2} + \ldots - (-1)^{m-1}.C_1) = E_m$. Quo pacto obtinebitur, vi *Theor.* VI. *Cor.* 3, $K_{\overline{n+3}.v} = 3X_n - K_{n.v}$, & prohinc $K_{\overline{n+3m}.v} = 3C_m.X_n + (-1)^m.K_{n.v}$, ideoque (*Theor.* VIII) $F_{\overline{n+3m}.v} = 3C_m.X_n.(f(v) + f(v+1) + f(v+2))$

$+ (-1).[f(v)$

$+(-1)^m.[f(v).K_{n.0}+f(v+1).K_{n.2}+f(v+2).K_{n.1}]$.
Unde, facto $n = 0, 1, 2$;

$$F_{3m.v} = 3C_m.X_0.[f(v)+f(v+1)+f(v+2)] + (-1)^m.f(v);$$

$$F_{\overline{3m+1}.v} = 3C_m.X_1.[f(v)+f(v+1)+f(v+2)] + (-1)^m.f(v)+f(v+1);$$

$$F_{\overline{3m+2}.v} = 3C_m.X_2.[f(v)+f(v+1)+f(v+2)] + (-1)^m.f(v)+2f(v+1)+f(v+2).$$

Præterea, ope *Theor. VII*, habebitur $K'_{\overline{n+3}.v} = 3X'_n + 3b(X_n+K_{n.\overline{v+2}}) - K'_{n.v}$; ideoque $K'_{\overline{n+3m}.v} = 3C_m.X'_n + (-1)^m.K'_{n.v} + 3b[X_n.(D_m+E_m) - (-1)^m.m.K_{n.\overline{v+2}}]$. Quamobrem, vi *Theor. VIII*, erit $F'_{\overline{n+3m}.v} = 3[C_m.X'_n + b.X_n(D_m+E_m)].[f(v)+f(v+1)+f(v+2)] + (-1)^m.[(K'_{n.0}-3mb.K_{n.2}).f(v) + (K'_{n.2}-3mb.K_{n.1}).f(v+1) + (K'_{n.1}-3mb.K_{n.0}).f(v+2)]$. Unde emergit

$$F'_{3m.v} = 3[C_m.X'_0+b.X_0.(D_m+E_m)].[f(v)+f(v+1)+f(v+2)] + (-1)^m.[a.f(v)-3mb.f(v+2)];$$

$$F'_{\overline{3m+1}.v} = 3[C_m.X'_1+b.X_1.(D_m+E_m)].[f(v)+f(v+1)+f(v+2)] + (-1)^m.[(a-3mb).f(v) + (a+b).f(v+1)-3mb.f(v+2)];$$

$$F'_{\overline{3m+2}.v} = 3[C_m.X'_2+b.X_2.(D_m+E_m)].[f(v)+f(v+1)+f(v+2)] + (-1)^m.[(a-6mb).f(v) + (2a-(3m-2).b).f(v+1)+(a-(3m-2).b).f(v+2)].$$

Fa-

Facile vero patet, in formulis jam inventis esse $X_0 = 1$, $X'_0 = a$; $X_1 = 2$, $X'_1 = 2a + b$, $X_2 = 2^2$, $X'_2 = 2^2 . (a + b)$; atque generatim, sumto pro n numero quovis integro positivo, $X_n = 2^n$ & $X'_n = X_n . (a + \frac{1}{2} nb) = 2^n . (a + \frac{1}{2} nb)$.

Sit *e. gr.* $f(r + v) = 2 - \frac{4}{3} [Sin.^2 (\frac{4}{3} . rq) + Sin.^2 (\frac{4}{3} . \overline{r+1} . q)]$, nec non $m = n = 2$. Quibus positis erit $f(v) = 2 - \frac{4}{3} . Sin.^2 (\frac{4}{3} . q) = 1$, $f(v+1) = 2 - \frac{4}{3} . [Sin.^2 (\frac{4}{3} . q) + Sin.^2 (\frac{8}{3} . q)] = 0$, $f(v+2) = 2 - \frac{4}{3} . Sin.^2 (\frac{8}{3} . q) = 1$; $X_n = X_2 = 4$, $X'_n = X'_2 = 4(a+b)$; $C_m = C_2 = \frac{2^6 - 1}{2^3 + 1} = 7$, $D_m = D_2 = \frac{3}{2} . 2^3 = 12$, $E_m = E_2 = C_2 + 3\, C_1 = 10$. Unde $F_{\overline{3m+2}.v} = F_{8.v} = 3 . 7 . 4 . 2 + (-1)^2 . 2 = 170$, & $F'_{\overline{3m+2}.v} = F'_{8.v} = 3 . [7 . 4 (a+b) + 4 . 22 . b] . 2 + (-1)^2 . [(a - 12 b) + (a - 4b)] = 170 . a + 680 . b$.

§. 10.

Methodi a nobis adumbratæ in Analysi usum unico illustrasse juvabit, ope ejusdem facile solvendo, quod finem opellæ facturi adposuimus, Problemate.

Problema. Determinare summam seriei:

$$(2p-5) . \frac{2p}{3 . 4} + (2p-9) . \frac{2p . \overline{2p-1} . \overline{2p-2}}{3 . 4 . 5 . 6} + \ldots + (3-2p).$$

$+ (3-2p) . \frac{\overline{2p} . \overline{2p-1} .. 5 . 4}{3 . 4 .. (2p)} = S$, denotante p numerum quemvis integrum positivum.

Utrique æquationis membro, ducto in $\frac{(2p+3).(2p+2).(2p+1)}{1 . 2}$, addatur: $(2p+3)+(2p-1)$.

$$\frac{(2p+3).(2p+2)}{1 . 2} - (1+2p).\frac{\overline{2p+3}.\overline{2p+2}..3.2}{1.2..(2p+2)}.$$

Quo pacto obtinebitur $\frac{(2p+3).(2p+2).(2p+1)}{1 . 2}.S$

$$+ (2p+3) + (2p-1).\frac{(2p+3).(2p+2)}{1.2} - (1+2p).$$

$$\frac{(2p+3)(2p+2)..3.2}{1.2..(2p+2)} = (2p+3)+(2p-1).\frac{\overline{2p+3}.\overline{2p+2}}{1.2}$$

$$+ (2p-5).\frac{\overline{2p+3}.\overline{2p+2}.\overline{2p+1}.2p}{1.2.3.4} + .. + (3-2p).$$

$$\frac{\overline{2p+3}.\overline{2p+2}..5.4}{1.2..(2p)} - (1+2p).\frac{\overline{2p+3}.\overline{2p+2}..2}{1.2..(2p+2)}$$

$= a . A_{n,0} + (a+2b) A_{n,2} + (a+4b) A_{n,4}$ &c. posito $a = 2p+3$, $n = 2p+3$, & $b=-2$. Quippe cujus seriei summa (§. 9. *Exempl.* 3. *Cas.* 3) est $= {}^3F'_n = 2^{n-1}.(a+\frac{1}{2}nb) = 0$, ob $\frac{1}{2}nb = -(2p+3) = -a$.

Quamobrem erit $\frac{\overline{2p+3}\,.\,\overline{2p+2}\,.\,\overline{2p+1}}{1\,.\,2}\,.\,S + (2p+3)$
$+ (2p-1)\,.\,\frac{(2p+3)\,(2p+2)}{1\,.\,2} - (1+2p)\,.\,\frac{(2p+3)\,.\,(2p+2)\,..\,3\,.\,2}{1\,.\,2\,...\,(2p+2)}$
$= 0$, ideoque $S + \frac{p-1}{p+1} = 0$, ſive $S = \frac{1-p}{1+p}$.

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