DISSERTATIO PHYSICO MATHEMATICA
DE
COMPOSITIONE & RESOLUTIONE MOTUUM VERTIGINIS,

Quam

VENIA AMPL. FAC. PHILOS. ABOËNS.

Publice examinandam proponent

GABRIEL PALANDER,

Math. Appl. Docens & ad R. Acad. Biblioth. Amanuensis

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LAURENTIUS PHILIPP. PALANDER

Fratres, Stip. Regii, Tavastenses.

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§. 1.

Subtilem motuum vertiginis theoriam, cujus per universam Astronomiam Physicam latissime patet usus, ab antiquioribus prorsus fere incultam, circa medium mox exeuntis seculi egregiis acutissimorum, qui hic illic inclaruere, Geometrarum conspirantibus studiis insignius demum promoveri coeptam esse, qui scientiæ Mechanicæ fata vel leviter perlustraverit, facilis agnoscet. Nempe rotationis legum accuratius evolvendarum studium mirum quantum excitavit & auxit a summis viris hac ætate suscepta earum inæqualitatum sollers investigatio, quæ in motu telluris dudum observatæ, præcessio æquinoctiorum & nutatio axis dici solent. Quod consilium eo valuit, ut certa facilique rotationum componendarum & resolvendarum ratione detecta & exposita, virium quarumcunque, ad motus corporum turbandos, effectus accurate æstimari, motuumque vel complicatissimorum phænomena ad calculos facile revocari, jam omnino queant. Singulorum vero, qui hanc Mechanicæ sublimioris partem excolere satagentes motus vertiginis componere & resolvere docuerunt, merita recensere cum longum sit & a nostris rationibus alienum; eorum illustrissimos, PAULlum Frisium, d'Alembert, Eulerum tantum nominasse sufficiat. Quippe dicto Problemati compositionis & resolutionis rotationum cognoscendo impensius vacans, incidi in solutionem, elegantia ac concinnitate summa commendabilem, quam La Grange, Analysta subtilissimus, in eximio opere: *Analytische Mechanik, aus dem Französischen von F. W. A. Murhard. Götting. 1797, p. 31 seqq.*, nuper suppeditavit: ipsamque Summi Viri Analysin, pressius ab illo exhibitam, demonstrationibus passim, ubi referat, muniendam meisque illustrandam rationibus suscepi, haud dubitans fore, ut ea, quæ a tenuitate virium juvenilium proficisci possint, L. B. æqui bonique consulat.

§. II.

Lemma. I. *Si sumtis coordinatis orthogonalibus, sint x', y' & z' constantibus* α, β *&* γ *respective sumtis proportionales; linea his definita, quæ dicatur p, erit recta per initium coordinatarum transiens, & si anguli, sub quibus ipsa p axes x, y & z secabit, dicantur* λ, μ *&* ν *respective, erit* $Cos\lambda = \frac{\alpha}{\sqrt{(\alpha^2 + \beta^2 + \gamma^2)}}$, $Cos\,\mu = \frac{\beta}{\sqrt{(\alpha^2 + \beta^2 + \gamma^2)}}$ *&* $Cos\,\nu = \frac{\gamma}{\sqrt{(\alpha^2 + \beta^2 + \gamma^2)}}$.

Sint p', p'' binæ projectiones orthographicæ ipsius p in plana axium x & y, x & z respective: quarum illam æquatione $\beta x' - \alpha y' = o$, hanc vero æqua-

æquatione $\gamma x' - \alpha z' = o$, exprimendam esse, ex hypothesi constat. Quo facto, cum posito $y' = z' = o$, obtineatur $x' = o$; facile patet, p' & p'' esse rectas in initio coordinatarum concurrentes. Ergo & ipsa p erit recta per initium coordinatarum transiens. Q. E. 1:o D.

Quum præterea ex Geometria notissimum sit, puncti cujusvis coordinatis x' y' & z' determinati ab initio coordinatarum distantiam æquari $\sqrt{(x'^2 + y'^2 + z'^2)}$; facillimo habebitur negotio $Cos\,\lambda = \frac{x'}{\sqrt{(x'^2 + y'^2 + z'^2)}} = \frac{\alpha}{\sqrt{(\alpha^2 + \beta^2 + \gamma^2)}}$, $Cos\,\mu = \frac{y'}{\sqrt{(x'^2 + y'^2 + z'^2)}} = \frac{\beta}{\sqrt{(\alpha^2 + \beta^2 + \gamma^2)}}$ & $Cos\,\nu = \frac{z'}{\sqrt{(x'^2 + y'^2 + z'^2)}} = \frac{\gamma}{\sqrt{(\alpha^2 + \beta^2 + \gamma^2)}}$ Q. E. 2:o D.

Coroll. Si anguli, quibus ad axem x projectæ p' & p'' inclinantur, sint ε & ε' respective; erit $tg\,\varepsilon \left(= \frac{y'}{x'}\right) = \frac{\beta}{\alpha}$ & $tg\,\varepsilon' \left(= \frac{z'}{x}\right) = \frac{\gamma}{\alpha}$.

§. III.

LEMMA II. *Si superficies plana æquatione* $\alpha x'' + \beta y'' + \gamma z'' - \delta \sqrt{(\alpha^2 + \beta^2 + \gamma^2)} = o$ *expressa dicatur* S; *recta* p *plano* S *ad angulos rectos insistet, & erit* δ *plani hujus ab initio coordinatarum distantia.*

Si ſint s & s' interſectiones ſuperficiei planæ S ex occurſu planorum per axes x & y, x & z, transeuntium oriundæ; ex principiis Geometriæ prono fluit alveo, eas æquationibus $\alpha x''' + \beta y'' - \delta\sqrt{(\alpha^2+\beta^2+\gamma^2)} = o$, $\alpha x'''' + \gamma z'''' - \delta\sqrt{(\alpha^2+\beta^2+\gamma^2)} = o$, reſpective poſitis exhiberi. Quum vero poſito $y''' = z'''' = o$, prodeat $x''' = x'''' = \delta\sqrt{(\alpha^2+\beta^2\gamma^2+)}:\alpha$, hoc ipſo indicio apertum eſt, ipſas s, s' in axe x, ad diſtantiam $\delta\sqrt{(\alpha^2+\beta^2+\gamma^2)}:\alpha$ ab initio coordinatarum, concurrere. Dicantur porro anguli, quos cum axe x rectæ s, s' faciunt, η, η' reſpective: quo pacto obtinebitur $tg\ \eta = \frac{y'''}{x''' - \delta\sqrt{(\alpha^2+\beta^2+\gamma^2)}:\alpha} = -\frac{\alpha}{\beta}$ atque $tg\ \eta' = \frac{z''''}{x'''' - \delta\sqrt{(\alpha^2+\beta^2+\gamma^2)}:\alpha} = -\frac{\alpha}{\gamma}$. Quos tangentium valores cum iis comparando, qui pro angulis ε, ε', ſub quibus projectiones ipſius p eundem axem x ſecant, ſupra (*Coroll. Lemm. I.*) eruti ſunt, deprehendimus eſſe $tg\ \eta'$ ($= -$ *Cotg* ε) $= tg\ (\varepsilon \pm 90^\circ)$ nec non $tg\ \eta'$ ($= -$ *Cotg* ε') $= tg\ (\varepsilon' \pm 90^\circ)$. Unde $\eta - \varepsilon = \eta' - \varepsilon' = \pm 90^\circ$. Cum autem s & p' ſint in plano axium x & y' pariterque s' & p'' plano per axes x & z transeunte contineantur, manifeſtum eſt, angulum $\eta - \varepsilon$ illarum, angulum vero $\eta' - \varepsilon'$ harum mutuam inclinationem exhibere. Quare erit p' in s & p'' in s' ad angulos rectos. Cum præterea angulos η & η' a δ prorſus non pendere, inſpectis formulis tangentium, liquido pateat;

at; tribuendo ipsi δ diversos quotvis valores, infinita numero existent plana inter se parallela: quippe quorum cum planis axium x & y, x & z oriundæ intersectiones binis s, s' respective collocatis erunt parallelæ. Quod si fiat $\delta = o$, rectæ s & s' cum projectionibus ipsius p in ipso coordinatarum initio concurrent. Quo in casu ex antea demonstratis facile probabitur, rectam p utrique intersectionum s & s' perpendiculariter insistere. Erit ergo recta p plano per has intersectiones transeunti, ideoque etiam singulis huic parallelis, æquatione generali $\alpha x'' + \beta y'' + \gamma z'' - \delta \surd(\alpha^2 + \beta^2 + \gamma^2) = o$ expressis planis, ad angulos rectos. Q. E. 1:o D.

Quoniam est $1 : Cos\lambda : : \frac{(\alpha^2 + \beta^2 + \gamma^2)^{\frac{1}{2}}\delta}{\alpha} : \delta$ (*Lemm.* 1.), & ex supra demonstratis portio axeos x inter initium coordinatarum & planum S intercepta $= \delta \surd(\alpha^2 + \beta^2 + \gamma^2) : \alpha$; erit pars rectæ p inter idem punctum & planum interposita, quæ plani S ab initio coordinatarum distantiam exprimit, $= \delta$. Q. E. 2:o D.

Coroll. Si puncti cujuslibet in plano S siti a recta p distantia dicatur r; erit $r = \surd(x''^2 + y''^2 + z''^2 - \delta^2)$.

§ IV.

PROBLEMA. *Si punctum quodvis, coordinatis orthogonalibus x, y & z definitum, ternis viribus rotatri-*

 cibus

cibus urgeatur, quarum una circa axem x celeritate angulari α, altera circa axem y celeritate β, tertia circa axem z celeritate γ seorsim circumvolvi possit; ex iisdem viribus simul impressis oriundas coordinatarum variationes momentaneas determinare.

Quo primum cognoscantur variationes coordinatarum ex revolutione puncti circa axem x oriundæ, e re est, radium vectorem $= \sqrt{(y^2 + z^2)}$, qui brevitatis gratia dicatur ϱ, angulumque φ, ipso ϱ & axe y comprehensum assumamus; quibus positis erit $y = \varrho \, Cos \, \varphi$ & $z = \varrho \, Sin \, \varphi$. Quoniam vero facile patet, ex rotatione, in quam inquirimus, nullam sive ipsius x sive radii ϱ variationem existere; rotatio elementaris $d\varphi$ circa axem x dabit has variationes: $dy = - \varrho \, Sin \, \varphi d\varphi = - z d\varphi$ & $dz = \varrho \, Cos \varphi d\varphi = y d\varphi$, ob $d. \, Cos \varphi = - Sin \varphi d \varphi$ & $d. \, Sin \varphi = Cos \varphi d \varphi$.

Similiter in plano, cui axis y verticaliter insistit, sumto radio vectore $\varrho' = \sqrt{(x^2 + z^2)}$ anguloque ψ tali, ut sit $z = \varrho' \, Cos \, \psi$ & $x = \varrho'$ Sin ψ; erunt rotationi elementari $d\psi$ circa axem y debitæ variationes: $dz = - x d\psi$ & $dx = z d\psi$.

Facto denique $x = \varrho'' \, Cos \omega$ & $y = \varrho'' \, Sin \omega$; haud absimili ratiocinio efficitur, motum vertiginis elementarem $d\omega$ circa axem z concipiendum afficere coordinatas x & y variationibus: $dx = - y d\omega$ & $dy = x d\omega$.

Quod

Quod si ternæ hæ rotationes simul absolvendæ concipiantur; ex principiis calculi differentialis constat, variationes ipsarum x, y & z ex motu vertiginis composito oriundas summis variationum, quæ ex singulis rotationibus exstiterunt, æquari, ideoque esse $dx = zd\psi - yd\omega$, $dy = xd\omega - zd\psi$ & $dz = yd\phi - zd\psi$. Præterea ob synchronismum rotationum, sit dt tempusculum vertigini elementari impensum: quo pacto erit $d\phi = \alpha dt$, $d\psi = \beta dt$ & $d\omega = \gamma dt$. Unde, his substitutis valoribus, liquet esse $\frac{dx}{dt} = \beta z - \gamma y$, $\frac{dy}{dt} = \gamma x - \alpha z$ & $\frac{dz}{dt} = \alpha y - \beta x$.

§. V.

Theorema. *Iisdem, ac in Problemate præcedenti, positis, motus vertiginis compositus, ex conjuncta ternarum virium rotatricium actione oriundus, continuabitur circa rectam p (Lemm. 1 definitam) celeritate angulari* $= \sqrt{(\alpha^2 + \beta^2 + \gamma^2)}$.

Primo quidem rectam p quiescere ideoque rotationis compositæ axem esse, exinde facile constat, quod coordinatarum x', y' & z', quibus quodvis hujus punctum determinatur, variationes momentaneæ omnino evanescant: quippe quæ erunt $\frac{dx'}{dt} = \beta z' - \gamma y' = 0$, $\frac{dy'}{dt} = \gamma x' - \alpha z' = 0$ & $\frac{dz'}{dt} = \alpha y' - \beta x' = 0$.

Deinde vero quia ex Geometria habetur ſpatiolum a puncto quovis plani S, cui axis rotationis p ad angulos rectos inſiſtit (*Lemm. II.*), tempuſculo dt confectum $= \surd(dx''^2 + dy''^2 + dz''^2)$, & eſt diſtantia ejus ab axe rotationis $= r$ (*Coroll. Lemm. II.*); ex principiis motus patet eſſe, celeritatem hujus puncti angularem, quæ dicatur $c = \frac{\surd(dx''^2 + dy''^2 + dz''^2)}{r\,dt}$.

Eſt autem $\frac{dx''^2 + dy''^2 + dz''^2}{dt^2} = (\beta z'' - \gamma y'')^2 + (\gamma x'' - \alpha z'')^2 + (\alpha y'' - \beta x'')^2 = (\alpha^2 + \beta^2 + \gamma^2) \times (x''^2 + y''^2 + z''^2 - \delta^2) - (\alpha x'' + \beta y'' + \gamma z'' + \delta \surd(\alpha^2 + \beta^2 + \gamma^2)) \times (\alpha x'' + \beta y'' + \gamma z'' - \delta \surd(\alpha^2 + \beta^2 + \gamma^2)) = (\alpha^2 + \beta^2 + \gamma^2) r^2$. Unde $c = \surd(\alpha^2 + \beta^2 + \gamma^2)$.

Coroll. Quia eſt $c = \surd(\alpha^2 + \beta^2 + \gamma^2)$; erit (*Lemm. I.*) $Coſ\lambda = \frac{\alpha}{c}$, $Coſ\mu = \frac{\beta}{c}$ & $Coſ\nu = \frac{\gamma}{c}$; unde $\alpha = c\,Coſ\lambda$, $\beta = c\,Coſ\mu$ & $\gamma = c\,Coſ\nu$.

§. VI.

PROBLEMA. *Datis binis pluribusve rotationibus circa totidem diverſos axes p', p'' &c. in puncto ſibimet invicem occurrentes, quarum celeritates angulares ſint c', c'' &c. reſpective; motum vertiginis, qui ex his componitur, determinare.*

Sum-

Sumto in puncto concursus axium p', p'' &c. initio coordinatarum orthogonalium, faciant rectæ p', p'' &c. cum axe x angulos λ', λ'' &c., cum axe y μ', μ'' &c., cum axe z ν', ν'' &c. respective: quibus positis, resolvantur primo, secundum Coroll. Theor. præcedentis, singulæ celeritates angulares in ternas circa axes coordinatarum, ponendo

$$\alpha' = c' \operatorname{Cos} \lambda', \beta' = c' \operatorname{Cos} \mu' \ \& \ \gamma' = c' \operatorname{Cos} \nu',$$
$$\alpha'' = c'' \operatorname{Cos} \lambda'', \beta'' = c'' \operatorname{Cos} \mu'' \ \& \ \gamma'' = c'' \operatorname{Cos} \nu'', \text{ \&c.}$$

Dein vero colligendis iis, quæ ad eosdem pertinent axes, celeritatibus, statuatur

$$\alpha \ (= \alpha' + \alpha'' + \ldots) = c' \operatorname{Cos} \lambda' + c'' \operatorname{Cos} \lambda'' + \ldots,$$
$$\beta \ (= \beta' + \beta'' + \ldots) = c' \operatorname{Cos} \mu' + c'' \operatorname{Cos} \mu'' + \ldots, \ \&$$
$$\gamma \ (= \gamma' + \gamma'' + \ldots) = c' \operatorname{Cos} \nu' + c'' \operatorname{Cos} \nu'' + \ldots.$$

Quo facto quæratur ternis α, β & γ æquivalens unica rotatio, quæ per Theorema præcedens fiet circa axem p celeritate angulari $c = \sqrt{(\alpha^2 + \beta^2 + \gamma^2)} = \sqrt{\left(c'^2 + c''^2 + \ldots + c'c'' \times (\operatorname{Cos}(\lambda' + \lambda'') + \operatorname{Cos}(\lambda' - \lambda'') + \operatorname{Cos}(\mu' + \mu'') + \operatorname{Cos}(\mu' - \mu'') + \operatorname{Cos}(\nu' + \nu'') + \operatorname{Cos}(\nu' - \nu'')) + \ldots\right)}$, substitutis ipsarum α, β & γ valoribus & reductione per formulam Trigonometricam: $\operatorname{Cos}(a+b) + \operatorname{Cos}(a-b) = 2 \operatorname{Cos} a \operatorname{Cos} b$, facta. Inventa vero c erit $\operatorname{Cos} \lambda = \frac{\alpha}{c} = \frac{c' \operatorname{Cos} \lambda' + c'' \operatorname{Cos} \lambda'' + \ldots}{c}$, $\operatorname{Cos} \mu = \frac{\beta}{c} = \frac{c' \operatorname{Cos} \mu' + c'' \operatorname{Cos} \mu'' + \ldots}{c}$ & $\operatorname{Cos} \nu = \frac{\gamma}{c} = \frac{c' \operatorname{Cos} \nu' + c'' \operatorname{Cos} \nu'' + \ldots}{c}$,

ex-

exprimentibus λ, μ & ν inclinationes rectæ p ad axes x, y & z.

Coroll. Si fuerint bini axes rotationis p' & p'' in plano axium coordinatarum x & y constitutæ; erit, ob $\lambda' + \mu' = \lambda'' + \mu'' = \nu' = \nu'' = 90°$, $c = \sqrt{(c'^2 + c''^2 + 2c'c'' Cos (\lambda' - \lambda''))}$ nec non $Cos \mu = Sin \lambda = \frac{c' Sin \lambda' + c'' Sin \lambda''}{c}$ & $Cos \nu = 0$. Ergo jacebit axis rotationis compositæ p in plano axium x & y, ad rectas p', p'' utrinque collocatas angulis $\lambda' - \lambda$, $\lambda - \lambda''$, inclinata. Quoniam præterea ex cognitis facile probari potest, esse $Sin (\lambda' - \lambda) = \frac{c''}{c} . Sin (\lambda' - \lambda'')$ & $Sin (\lambda - \lambda'') = \frac{c'}{c} . Sin (\lambda' - \lambda'')$; liquet esse $Sin (\lambda' - \lambda) : Sin (\lambda - \lambda'') : Sin (\lambda' - \lambda'') :: c'' : c' : c$. Unde, si sumantur $p' : p'' :: c' : c''$, constat diagonalem parallelogrammi lateribus p' & p'' comprehensi, & coincidere cum ipsa p & esse celeritati rotationis compositæ c proportionalem, adeo ut tam axis positionem quam ipsam motus quantitatem exprimat.

Schol. Mira sane est ac omni attentione digna, quam in Coroll. præcedenti deteximus, inter motus vertiginis & liberos convenientia. Nempe ut bini motus liberi, lateribus parallelogrammi exhibiti, unicum componunt motum, cujus tam directio quam quantitas diagonali exprimitur, ita binis quoque rotationibus, quarum celeritates angulares axibus rotationum exprimuntur, semper æquivalet unica, circa diagonalem parallelogrammi, quod axibus continetur, celeritate angulari, quæ ipsi diagonali erit proportionalis, absolvenda. Unde omnibus iis, quæ de componendis & resolvendis motibus liberis in elementis Mechanicis traduntur, ad motus vertiginis translatis, rotationum resolvendarum facillima constat ratio: cui igitur Problemati ope formularum pro $Cos \lambda$, $Cos \mu$ & $Cos \nu$ inventarum solvendo diutius inhærere, supervacaneum est.