THEOREMA PECULIARE

AD LINEAS GEOMETRICAS ORDINIS CUJUSQUE PARIS 2 n, AEQUATIONE HUJUSCE FORMAE: $(x^2+y^2)^n + C_{2n-1.0}\, x^{2n-1} + C_{2n-2.1}\, x^{2n-2} y + \ldots + C_{1.2n-2}\, x y^{2n-2} + C_{0.2n-1}\, y^{2n-1} + \ldots\ldots + C_{1.0}\, x + C_{0.1}\, y + C_{0.0} = 0$ *DEFINITAS,*

SPECTANS.

CONS. AMPL. FAC. PHILOS. AB.

P. P.

GABRIEL PALANDER,
Fac. Philos. Adj. Ord.

ET

GUSTAVUS JOH. INGELIUS,
Satacundensis.

In Audit. Phys. die III Junii MDCCCVII.

h. a. m. s.

ABOÆ,
Typis FRENCKELLIANIS.

Docuit olim EUCLIDES (*Elem. Geom.* III. 35, 36) insignem hanc Circuli proprietatem: *Si binæ rectæ AP & AP' ei occurrant, illa in Punctis P_1 & P_2, hæc in punctis P_1' & P_2', esse constanter $AP_1 \times AP_2 = AP_1' \times AP_2'$.* De quo generalius constituendo Theoremate, ita nimirum, ut lineas altiorum quoque ordinum complectatur, motam post eum a nemine deprehendimus quæstionem. Quare concepimus animo, materiam hancce, cui pertractandæ quidquam tribuatur otii, minime indignam, nostra qualicunque versatam manu edendi publice consilium.

1. Sit igitur, sumtis coordinatis orthogonalibus $Op = x$ & $Pp = y$, æquatio generalis pro linea Geometrica ordinis r, per punctum P transeunte,

$$F(x, y)^r = \Phi(x, y) + F(x, y)^{r-1} =$$
$$x^r + C_{r-1,1} x^{r-1} y + \ldots + C_{1,r-1} x y^{r-1} + C_{0,r} y^r + F(x, y)^{r-1} = 0,$$

exprimente videlicet $F(x,y)^{r-1}$ functionem ejus naturæ, ut sit $F(x,y)^{r-1} = 0$ æquatio generalis pro linea ordinis $r-1$. Fiat porro recta $AP = u$, angulusque, quo ad axem abscissarum inclinatur $= v$; ducta denique perpendiculariter in eundem axem recta Aa, ponatur

$Oa =$

$Oa = \alpha$ & $Aa = \beta$. Quibus poſitis liquet fore $x - \alpha = u\,\text{Coſ.}\,v$ & $y - \beta = u\,\text{Sin.}\,v$. Unde $x = u\,\text{Coſ.}\,v + \alpha$ & $y = u\,\text{Sin.}\,v + \beta$. Qui ſi ſubſtituantur valores in æquatione $F(x, y)^r = o$; obtinebitur $F(u\,\text{Coſ.}\,v + \alpha, u\,\text{Sin.}\,v + \beta)^r = \Phi(u\,\text{Coſ.}\,v + \alpha, u\,\text{Sin.}\,v + \beta) + F(u\,\text{Coſ.}\,v + \alpha, u\,\text{Sin.}\,v + \beta)^{r-1} = (u\,\text{Coſ.}\,v + \alpha)^r + C_{r-1,1}(u\,\text{Coſ.}\,v + \alpha)^{r-1}(u\,\text{Sin.}\,v + \beta) + \ldots + C_{1,r-1}(u\,\text{Coſ.}\,v + \alpha)(u\,\text{Sin.}\,v + \beta)^{r-1} + C_{o,r}(u\,\text{Sin.}\,v + \beta)^r + F(\text{Coſ.}\,v + \alpha, u\,\text{Sin.}\,v + \beta)^{r-1} = u^r(\text{Coſ.}\,v^r + C_{r-1,1}\,\text{Coſ.}\,v^{r-1}\,\text{Sin.}\,v + \ldots + C_{1,r-1}\,\text{Coſ.}\,v\,\text{Sin.}\,v^{r-1} + C_{o,r}\,\text{Sin.}\,v^r) + f(u, v) + \alpha^r + C_{r-1,1}\,\alpha^{r-1}\beta + \ldots + C_{1,r-1}\,\alpha\beta^{r-1} + C_{o,r}\,\beta^r + F(\alpha, \beta)^{r-1} = u^r.\,\Phi(\text{Coſ.}\,v, \text{Sin.}\,v) + f(u, v) + F(\alpha, \beta)^r = o$, comprehenſis brevitatis gratia ſub formula $f(u, v)$ terminis omnibus formæ $C.u^{m+m'}\,\text{Coſ.}\,v^m.\,\text{Sin.}\,v^{m'}$, in quibus fuerit $m + m' < r$.

Hinc vero exſurgit æquatio: $u^r + \dfrac{f(u, v)}{\Phi(\text{Coſ.}\,v, \text{Sin.}\,v)} + \dfrac{F(\alpha, \beta)^r}{\Phi(\text{Coſ.}\,v, \text{Sin.}\,v)} = (u - u_1)(u - u_2) \ldots (u - u_r) = o$, ſumtis $u_1, u_2, \ldots u_r$ pro radicibus æquationis inventæ. Quippe quæ, cum ejusdem ſit ordinis ac ipſa linea propoſita, aperte prodit, lineam quamvis ordinis r a recta qualibet AP in tot ſecari punctis $P_1, P_2, \ldots P_r$, quot numerus r continet unitates, dum ſcilicet radices ſingulæ $u_1, u_2, \ldots u_r$, reales manebunt ſimulque inæquales. Deinde vero patet fore $AP_1 \,.\, AP_2 \ldots AP_r = u_1\,u_2 \ldots u_r = \dfrac{\pm F(\alpha, \beta)^r}{\Phi(\text{Coſ.}\,v, \text{Sin.}\,v)}$, admiſſo ſigno ſuperiori in caſu numeri r paris.

Quod

Quod si jam altera eidem lineæ occurrat recta $AP' = u'$, cum axe abscissarum faciens angulum v'; pari omnino efficietur ratiocinio esse

$$u'^r + \frac{f(u', v')}{\varphi(Cos. v', Sin. v')} + \frac{F(\alpha, \beta)^r}{\varphi(Cos. v', Sin. v')} = (u'-u'_1)(u'-u'_2)\ldots$$

$$(u'-u'_r) = o, \text{ \& dehinc } \frac{\pm F(\alpha, \beta)^r}{\varphi(Cos. v', Sin. v')} = u'_1 \, u'_2 \ldots u'_r$$

$$= AP_1' \,.\, AP_2' \ldots AP_r'.$$

2. Quibus sic constitutis, jam quærere jubet: *quo generali dignoscatur charactere ea linearum familia, in qua est* $AP_1 \,.\, AP_2 \,. \ldots AP_r = AP_1' \,.\, AP_2' \ldots AP_r'$? Mox vero patet, huic conditioni non nisi una hac satisfieri posse ratione, quod nimirum reddatur $\frac{F(\alpha, \beta)^r}{\varphi(Cos. v, Sin. v)} = \frac{F(\alpha, \beta)^r}{\varphi(Cos. v', Sin. v')}$

Quare sit oportet $\varphi(Cos. v, Sin. v) = \varphi(Cos. v', Sin. v')$ quantitas constans pro quibusvis valoribus angulorum v & v'. Facto autem $v' = o$, aperte constat fore $\varphi(Cos. v', Sin. v') = Cos. v'^r + C_{r-1,1} Cos. v'^{r-1} Sin. v' + \ldots + C_{1,r-1} Cos. v' Sin. v'^{r-1} + C^{o,r} Sin. v'^r$ unitati æqualem. Erit ergo constanter $\varphi(Cos. v, Sin. v) = 1 = (Cos. v^2 + Sin. v^2)^{\frac{r}{2}}$.

Quia vero functio $(Cos. v^2 + Sin. v^2)^{\frac{r}{2}}$ serie finita terminorum formæ $C\, Cos. v^{r-s} Sin. v^s$ exhiberi

hiberi nequit, nisi adhibito pro r numero pari; idcirco liquet, nullam omnino lineam ordinis imparis $2n+1$ quæsita illa gaudere adfectione, quod sit $u_1\, u_2 \ldots u_{2n+1}$ quantitas constans.

Fiat igitur $r = 2n$; eritque $\varphi\,(Cos.\,v,\, Sin.\,v) = (Cos.\,v^2 + Sin.\,v^2)^n$. Unde $\varphi\,(x, y) = (x^2 + y^2)^n$. Quo adhibito valore obtinebitur demum æquatio generalis pro familia linearum quæsita:

$(x^2 + y^2)^n + F\,(x, y)^{2n-1} = 0.$

3. Sit e. gr. $n = 2$. Quo substituto valore in æquatione $(x^2 + y^2)^n + F\,(x, y)^{2n-1} = 0$ emergit $(x^2 + y^2)^2 + F\,(x, y)^3 = (x^2 + y^2)^2 + C_{3.0}\, x^3 + C_{2.1}\, x^2 y + C_{1.2}\, x y^2 + C_{0.3}\, y^3 + C_{2.0}\, x^2 + C_{1.1}\, x y + C_{0.2}\, y^2 + C_{1.0}\, x + C_{0.1}\, y + C_{0.0} = 0$, æquatio generalis pro lineis quarti ordinis ita adfectis, ut sit $u_1\, u_2\, u_3\, u_4$ quantitas constans, scilicet $= \dfrac{F\,(\alpha, \beta)^4}{(Cos\,v^2 + Sin.\,v^2)^2}$ $= (\alpha^2 + \beta^2)^2 + C_{3.0}\, \alpha^3 + C_{2.1}\, \alpha^2 \beta + C_{1.2}\, \alpha \beta^2 + C_{0.3}\, \beta^3 + C_{2.0}\, \alpha^2 + C_{1.1}\, \alpha \beta + C_{0.2}\, \beta^2 + C_{1.0}\, \alpha + C_{0.1}\, \beta + C_{0.0}$.

4. Fiat jam $n = n_1 + n_2 + \ldots + n_m$ & ponatur simul $(x^2 + y^2)^n + F\,(x, y)^{2n-1} = [(x^2 + y^2)^{n_1} + F\,(x, y)^{2n_1 - 1}]\; [(x^2 + y^2)^{n_2} + F\,(x, y)^{2n_2 - 1}] \ldots [(x^2 + y^2)^{n_m} + F\,(x, y)^{2n_m - 1}]$. Unde, exæquatis inter se coefficientibus terminorum homologorum in

utroque

utroque æquationis menbro, tot exſurgunt æquationes, quot in functione $F(x, y)^{2n-1}$ deprehenderis coefficientes indeterminatos, h. e. $n(2n+1)$, qui numerus conficit ſummam ſeriei $1+2+..2n$. Simul vero obſervandum, numerum coefficientium in functionibus $F(x, y)^{2n_1-1}, F(x, y)^{2n_2-1}, .. F(x, y)^{2n_m-1}$ ſingulatim ſpectatis esſe $n_1(2n_1+1)$, $n_2(2n_2+1)$.. $n_m(2n_m+1)$ reſpective, univerſim autem ſumtorum fore $= n_1(2n_1+1)+n_2(2n_2+1)+..+n_m(2n_m+1)$ $= 2(n_1^2+n_2^2+..+n^2_m)+(n_1+n_2+..+n_m)$. Aſt cum hic numerus ſit inferior numero iſtarum æquationum, quibus coefficientium prioris membri dependentia a coefficientibus poſterioris deſcribitur; in propatulo eſt, eliminatis dictarum ope æquationum coefficientibus poſterioris membri, perventum iri ad novas æquationes, quarum numerus $= n(2n+1) - 2(n_1^2+n_2^2+..+n_{2m})^2 - (n_1+n_2+..+n_m) = 2n^2+n-2(n_1^2+n_2^2+..+n^2_m)-(n_1+n_2+..+n_m) = 2(n_1+n_2+..+n_m)^2 - 2(n_1^2+n_2^2+..+n^2_m)$, quæque coefficientium functionis $F(x, y)^{2n-1}$ mutuas determinant relationes. Quare functio $(x^2+y^2)^n+F(x, y)^{2n-1}$ haud erit reductibilis ad formam $((x^2+y^2)^{n_1}+F(x, y)^{2n_1-1})(x^2+y^2)^{n_2}+F(x, y)^{2n_2-1})..((x^2+y^2)^{n_m}+F(x, y)^{2n_m-1})$ niſi quoties coefficientes, in functione $(x^2+y^2)^n+F(x, y)^{2n-1}$ obvii, iſtis congruenter æquationibus fuerint definiti. Quod ubi evenerit, ex Theoria curvarum conſtat, æquationem $(x^2+y^2)^n+F(x, y)^{2n-1}=0$ unam non exprimere lineam ordinis $2n$, ſed ſyſtema quoddam

quoddam ex m_1 lineis inferiorum ordinum, ope æquationum: $(x^2+y^2)^{n_1}+F(x,y)^{2n_1-1}=0$, $(x^2+y^2)^{n_2}+F(x,y)^{2n_2-1}=0, \ldots (x^2+y^2)^{n_m}+F(x,y)^{2n_m-1}=0$, sigillatim describendis *).

5. Quod

---

*) In gratiam tironum observasse juvabit, supponi heic a nobis cum functionem $(y^2+y^2)^n+F(x,y)^{2n-1}$ tum singulos ejus factores ita comparatos, ut evanescere possint. Quod si secus acciderit; æquationum $(x^2+y^2)^n+F(x,y)^{2n-1}=0$, $(x^2+y^2)^{n_1}+F(x,y)^{2n_1-1}=0$, $(x^2+y^2)^{n_2}+F(x,y)^{2n_2-1}=0$, &c. aliqua facta absurda, linea quoque hac exprimenda evadit imaginaria. Sic si fuerit $(x^2+y^2)^n+F(x,y)^{2n-1}=(x^n+F_{n.0}(x,y)^{n-1})^2+\frac{n}{1}(x^{n-1}y+F_{n-1.0}(x,y)^{n-1})^2+\frac{n.(n-1)}{1.2}(x^{n-2}y^2+F_{n-2.2}(x,y)^{n-1})^2+\ldots+\frac{n.(n-1)\ldots 2}{1.2\ldots(n-1)}(xy^{n-1}+F_{1.n-1}(x,y)^{n-1})^2+(y^n+F_{0.n-1}(x,y)^{n-1})^2+A^2$; mox patet absonum fore, statuere $(x^2+y^2)^n+F(x,y)^{2n-1}=0$. Cum enim singuli functionis hujusce termini, utpote quadrati, negativos recipere nequeant valores; non poterit aggregatum ex omnibus evanescere, nisi factis singulis $=0$. Ponendum igitur esset $A^2=0$, quod repugnat. Si e. gr. in æquatione $x^2+y^2+\alpha x+\beta y+\gamma=0$ fuerit γ qvantitas positiva $> \frac{1}{4}(\alpha^2+\beta^2)$; erit, sumto $A=\sqrt{\gamma-\frac{1}{4}(\alpha^2+\beta^2)}$, $x^2+y^2+\alpha x+\beta y+\gamma=(x+\frac{1}{2}\alpha)^2+(y+\frac{1}{2}\beta)^2+\gamma-\frac{1}{4}(\alpha^2+\beta^2)=(x+\frac{1}{2}\alpha)^2+(y+\frac{1}{2}\beta)^2+A^2=0$ æquatio pro Circulo imaginario, eo videlicet, cujus radius $=A\sqrt{-1}$.

5. Quod si in functione $(x^2+y^2)^n + F(x,y)^{2n-1} = [(x^2+y^2)^{n_1} + F(x,y)^{2n_1-1}]\,[(x^2+y^2)^{n_2} + F(x,y)^{2n_2-1}] \ldots [(x^2+y^2)^{n_m} + F(x,y)^{2n_m-1}]$, sumto $m = n$, fiat $n_1 = n_2 = \ldots = n_m = 1$; singulos omnino propositæ functionis factores induere formam $x^2 + y^2 + C_{1.0}\,x + C_{0.1}\,y + C_{0.0}$, vel me non monente, patebit. Quare, cum æquatio: $x^2 + y^2 + C_{1.0}\,x + C_{0.1}\,y + C_{0.0} = 0$ pertineat ad Circulum, exhibebit hoc in casu æquatio $(x^2+y^2)^n + F(x,y)^{2n-1} = 0$ systema ex meris Circulis, quorum numerus n, conflatum. Æquationum vero, quibus coefficientes functionis $F(x,y)_{2n-1}$ inter se cohærent, erit numerus $= 2n^2 - 2n = 2n\,(n-1)$.

6. Sic, ut in exemplo supra (n:o 3) allato maneamus, si ponatur $(x^2+y^2)^2 + C_{3.0}\,x^3 + C_{2.1}\,x^2 y + C_{1.2}\,x y^2 + C_{0.3}\,y^3 + C_{2.0}\,x^2 + C_{1.1}\,xy + C_{0.2}\,y^2 + C_{1.0}\,x + C_{0.1}\,y + C_{0.0} = (x^2 + y^2 + C'_{1.0}\,x + C'_{0.1}\,y + C'_{0.0})\,(x^2 + y^2 + C''_{1.0}\,x + C''_{0.1}\,y + C''_{0.0})$ æquatio $(x^2+y^2)^2 + C_{3.0}\,x^3 + C_{2.1}\,x^2 y + C_{1.2}\,x y^2 + C_{0.3}\,y^3 + C_{2.0}\,x^2 + C_{1.1}\,xy + C_{0.2}\,y_2 + C_{1.0}\,x + C_{0.1}\,y + C_{0.0} = 0$ binorum systemati Circulorum describendo inserviet, quorum unus exprimetur æquatione $x^2 + y^2 + C'_{1.0}\,x + C'_{0.1}\,y + C'_{0.0} = 0$, alter æquatione $x^2 + y^2 + C''_{1.0}\,x + C''_{0.1}\,y + C''_{0.0} = 0$. Jam vero, æquatis sibi invicem coefficientibus terminorum correspondentium in utroque æquationis membro, obtinebitur $C_{3.0} =$

$C'_{0.1}$

$C'_{1.0} + C''_{1.0}$, $C_{2.1} = C'_{0.1} + C''_{0.1}$, $C_{1.2} = C'_{1.0} + C''_{1.0}$, $C_{0.3} = C'_{0.1} + C''_{0.1}$, $C_{2.0} = C'_{0.0} + C''_{0.0} + C'_{1.0} C''_{1.0}$, $C_{1.1} = C'_{0.1} C''_{1.0} + C'_{1.0} C''_{0.1}$, $C_{0.2} = C'_{0.0} + C''_{0.0} + C'_{0.1} C''_{0.1}$, $C_{1.0} = C'_{0.0} C''_{1.0} + C'_{1.0} C''_{0.0}$, $C_{0.1} = C'_{0.0} C''_{0.1} + C'_{0.1} C''_{0.0}$, denique $C_{0.0} = C'_{0.0} C''_{0.0}$. Quod si ope harum decem æquationum eliminentur sex quantitates $C'_{1.0}$, $C'_{0.1}$, $C'_{0.0}$, $C''_{1.0}$, $C''_{0.1}$, $C''_{0.0}$; perveniendum est ad quatuor æquationes, quæ solas contineant quantitates $C_{3.0}$, $C_{2.1}$, &c., coefficientes nimirum terminorum functionem $F(x, y)^3$ componentium. Cujus rei ratio ex formula supra (n:o præced.) exhibita facillime reddi potest. Erit enim, facto $n = 2$, $2n(n-1) = 2 \cdot 2 \cdot (2-1) = 4$.