DISSERTATIO MATHEMATICA
ANALYSEOS SUBLIMIORIS ALGEBRÆ ELEMENTARI CONNECTENDÆ SPECIMEN
EXHIBENS.

CUJUS PARTICULAM V.

CONS. AMPL. FACULT. PHILOS. ABOËNS.

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h. a. m. s.

ABOÆ,
Typis FRENCKELLIANIS.

7.

54. Quia est (n:o 19 & 33) $\frac{f(x) - f(x_1)}{x - x_1} = f_1(x, x_1)$, $\frac{f_1(x, x_1) - f_1(x_1, x_2)}{x - x_2} = f_2(x, x_1, x_2)$ atque generatim $\frac{f_r(x, x_1, .. x_r) - f_r(x_1, x_2, .. x_{r+1})}{x - x_{r+1}} = f_{r+1}(x, x_1, .. x_{r+1})$; erit dehinc $f(x) = f(x_1) + (x-x_1) f_1(x, x_1)$, $f_1(x, x_1) = f_1(x_1, x_2) + (x - x_2)(x, x_1, x_2)$, denique $f_r(x, x_1, .. x_r) = f_r(x_1, x_2, .. x_{r+1}) + (x - x_{r+1}) f_{r+1}(x, x_1, .. x_{r+1})$. Quibus usurpatis valoribus exhibebitur $f(x) = f(x_1) + (x-x_1) f_1(x_1, x_2) + (x-x_1)(x-x_2) f_2(x_1, x_2, x_3) + ... \\ ..... + (x-x_1)(x-x_2)...(x-x_r) f_r(x_1, x_2, .. x_{r+1}) + (x-x_1)(x-x_2)...(x-x_{r+1}) f_{r+1}(x, x_1, .. x_{r+1})$. Ex hac vero formula, pro diversa determinatione quantitatum arbitrariarum x, x_1 &c., elegantissima plurima, eademque maxime generalia, facile eliciuntur Theoremata.

55. Fiat e. gr. $f(x) = (e + x)^m$, sumto pro m numero quovis integro positivo. Quo pacto erit (n:o 48) $(e+x)^m = (e+x_1)^m + (x-x_1)(e+x_1, e+x_2)^{m-1} + (x - x_1)(x - x_2)(e + x_1, e + x_2, e + x_3)^{m-2} + .. + (x - x_1)(x - x_2) ... (x - x_r)(e + x_1, e + x_2, .. e + x_{r+1})^{m-r} + (x - x_1)(x - x_2) .. (x - x_{r+1})(e + x, e + x_1, .. e + x_{r+1})^{m-r-1}$.

Qua

Qua continuata serie, donec fuerit $r = m$, habebitur [ob evanescentem formulam $(e + x, e + x_1, . . e + x_{r+1})^{m-r-1} = (e + x, e + x_1, . . e + x_{r+1})^{-1}$ et transeuntem $(e + x_1, e + x_2, . . e + x_{r+1})^{n-r}$ in $(e + x_1, e + x_2, . . e + x_{r+1})^0 = 1$] $(e + x)^m = (e + x_1)^m + (x - x_1)(e + x_1, e + x_2)^{m-1} + (x - x_1)(x - x_2)(e + x_1, e + x_2, e + x_3)^{m-2} + . . (x - x_1)(x - x_2) . . (x - x_m)$.

Sic facto $m = 4$, obtinebitur $(e + x)^4 = (e + x_1) + (x - x_1)(e + x_1, e + x_2)^3 + (x - x_1)(x - x_2)(e + x_1, e + x_2, e + x_3)^2 + (x - x_1)(x - x_2)(x - x_3)(e + x_1, e + x_2, e + x_3, e + x_4)^1 + (x - x_1)(x - x_2)(x - x_3)(x - x_4) = (e + x_1)^4 + (x - x_1)[(e + x_1)^3 + (e + x_1)^2 (e + x_2) + (e + x_1)(e + x_2)^2 (e + x_2)^3] + (x - x_1)(x - x_2)[(e + x_1)^2 + (e + x_1)(e + x_2) + (e + x_1)(e + x_3) + (e + x_2)^2 + (e + x_2)(e + x_3) + (e + x_3)^2] + (x - x_1)(x - x_2)(x - x_3)(e + x_1 + e + x_2 + e + x_3 + e + x_4) + (x - x_1)(x - {}_2)(x - x_3)(x - x_4)$.

56. Ponatur vero jam in formula (n:o 54) proposita $x_2 = \varphi(x_1)$, $x_3 = \varphi(x_2) = \varphi[\varphi(x_1)] = \varphi^2(x_1)$ atque generatim $x_{r+1} = \varphi(x_r) = \varphi^r(x_1)$. Quo facto habebitur $f(x) = f(x_1) + (x - x_1) f_1(x_1, x_2) + (x - x_1)(x - \varphi x_1) f_2(x_1, \varphi x_1, \varphi^2 x_1) + . . + (x - x_1)(x - \varphi x_1) . . (x - \varphi^{r-1} x_1) f_r(x_1, \varphi x_1, . . \varphi^{r} x_1) + (x - x_1)(x - \varphi x_1) . . (x - \varphi^r x_1) f_{r+1}(x, x_1, \varphi x_1, . . \varphi^r x_1) = A$,

$== A$, manentibus ſcilicet indeterminatis in hac æquatione, tam quantitatibus x et x_1, quam ipſa functione $\varphi(x_1)$.

57. Quod ſi in formula A fiat $\varphi(x_1) = \alpha x_1 + \beta$; facillimo eruitur negotio $\varphi^2(x_1) == \alpha^2 x_1 + \alpha\beta + \alpha == \alpha^2 x_1 + \beta(\alpha^2 - 1) : (\alpha - 1)$, atque generatim $\varphi^r(x_1) == \alpha^r x_1 + \beta(\alpha^r - 1) : (\alpha - 1)$. Quare erit $f(x) == f(x_1) + (x - x_1) f_1(x_1, \alpha x_1 + \beta) + (x - x_1)(x - \alpha x_1 - \beta) f_2[x_1, \alpha x_1 + \beta, \alpha^2 x_1 + \beta(\alpha + 1)] + \ldots + (x - x_1)(x - \alpha x_1 + \beta) \ldots [x - \alpha^{r-1} x_1 - \beta(\alpha^{r-1} - 1) : (\alpha - 1)] f_r[x_1, \alpha x_1 + \beta, \ldots, \alpha^r x_1 + \beta(\alpha^r - 1) : (\alpha - 1)] + (x - x_1)(x - \alpha x_1 - \beta) \ldots [x - \alpha^r x_1 - \beta(\alpha^r - 1) : (\alpha - 1)] f_{r+1}(x, x_1, \alpha x_1 + \beta, \ldots \alpha^r x_1 + \beta(\alpha^r - 1) : (\alpha - 1)] == A_1$. Quippe quæ formula, in caſu ſpecialisſimo, ubi fuerit $\alpha == 1$ et $\beta = 0$ ſive $\varphi(x_1) = x_1$, transit in hanc: $f(x) == f(x_1) + (x - x_1) f_1(x_1, x_1) + (x - x_1)^2 f_2(x_1, x_1, x_1) + \ldots + (x - x_1)^r f_r(x_1 \ldots) + (x - x_1)^{r+1} f_{r+1}(x, x_1 \ldots) == f(x_1) + (x - x_1) f'(x_1) + (x - x_1)^2 f''(x_1) + \ldots + (x - x_1)^r f^r(x_1) + (x - x_1)^{r+1} f_{r+1}(x, x_1 \ldots)$ (n:o 40) $== A'_1$.

58. Obſervandum vero heic eſt, formulam A'_1 ſingularem ideo mereri attentionem, quod ſeriem conſtituat ſecundum poteſtates quantitatis $x - x_1$ procedentem. Nempe hæc ejus adfectio eo valet, ut hac mediante formula functio quævis $f(x_1)$ in ſeriem formæ ſupra (n:o 10) expoſitæ converti posſit, quoties nimirum fuerint indices ſinguli r_1, r_2 &c.

 numeri

numeri integri positivi. Facto enim x_{1} = quantitati constanti a, formula nostra abit in hanc $f(x) = f(a) + (x-a) f'(a) + (x-a)^2 f''(a) + \ldots + (x-a)^r f^r(a) + (x-a)^{r+1} f^{r+1}(x, a \ldots) = B$, in qua quantitates $f(a)$, $f'(a)$, $f''(a)$ &c. coëfficientium constantium vices sustinent.

Posito demum $a = o$ emergit $f(x) = f(o) + x f'_{1}(o) + x^2 f''(o) + \ldots + x^r f^r(o) + x^{r+1} f^{r+1}(x, o \ldots) = B'$. Vim vero formularum B et B' ex uno facile cognoscas exemplo. Facto videlicet $f(x) = (e+x)^m$, obtinebitur, ex formula B, $(e+x)^m = (e+a)^m + (x-a)(e+a)^{m-1}\, {}^{m-1}C_2 + (x-a)^2 (e+a)^{m-2}\, {}^{m-2}C_3 + \ldots + (x-a)^m = (e+a)^m + \frac{m}{1}(e+a)^{m-1}(x-a) + \frac{m(m-1)}{1 \cdot 2}(x-a)^2 (e+a)^{m-2} + \ldots + (x-a)^m$ (n:o 55 et 46) pariterque, ex formula B', $(e+x)^m = e^m + \frac{m}{1} e^{m-1} x + \frac{m(m-1)}{1 \cdot 2} e^{m-2} x^2 + \ldots + x^m$, existente indice m numero positivo integro.

59. Quod si ponatur (in formula A'_{1}) $x = x_{1} + e$ et pro x_{1} ubique scribatur x; liquet fore $f(x+e) = f(x) + e f'(x) + e^2 f''(x) + \ldots + e^r f^r(x) + e^{r+1} f^{r+1}(x+e, x \ldots) = C$.

Hanc functionis $f(x+e)$ in seriem explicandæ formam directa elicuimus ratione, quam ipsam LA-

GRANGE elegantisſima, magis licet indirecta, munivit demonſtratione *); in eo tamen nobis viſus paululum reliquisſe deſiderandum, quod probatam ſupra (n:o 6) Analyticam adfectionem functionis $f(x)$, pro $x = o$ evaneſcentis, principii loco præſtruxerit **).

60. Sumta in formula *C* quantitate e negativa, exſurgit $f(x - e) = f(x) - e f'(x) + e^2 f''(x) - \ldots \pm e^r f^r(x) \mp e^{r+1} f^{r+1}(x - e, x \ldots) = C'$, adhibito ſigno ſuperiore pro numero r pari. Quæ formula in eo tantum differt a formula *C*, quod terminos alternos, factorem quippe formæ e^{2n+1} continentes, exhibeat negativos.

61. Si functionum $f(x + e) - f(x)$ et $f(x-e) - f(x)$ fuerit aut *utraque poſitiva* aut *utraque negativa*, ſumta quantitate e indefinite parva; functio $f(x)$ in illo caſu *minima* dicitur, in hoc *maxima*.

62. Sit

*) In *L. c. No* 10, 11. 1 *Th. pagg.* 10 - 14.

**) Exinde nimirum, quod ſit $f(x + i) - f(x) = F(i)$ ejusmodi functio quantitatis i, quæ evaneſcat pro $i = o$, concludit eſſe $F(i)$ formæ $i^r P$, ſumto indice r poſitivo, *Lib. c. N:o* 11. *Th. pag.* 112.

62. Sit m quantitas quaecunque, quae, in locum ipsius x suffecta, efficiat functionem $f(x)$ *minimam*, vel *maximam*. Quo substituto valore in formulis C et C' obtinebitur $f(m + e) - f(m) = e f'(m) + e^2 f''(m) + \ldots + e^r f^r(m) + e^{r+1} f_{r+1}(m + e, m \ldots)$ atque $f(m - e) - f(m) = - e f'(m) + e^2 f''(m) - \ldots \pm e^r f^r(m) \mp e^{r+1} f_{r+1}(m - e, m \ldots)$. Quia autem in utraque terminus primus, indefinite decrescente quantitate e, reliquorum superat summam (n:o 11); sponte hinc fluit, quo adficiatur signo primus ille terminus, idem quoque ipsam functionem, facta quantitate e perexigua, esse recepturam. Quo igitur quantitates $f(m + e) - f(m)$ et $f(m - e) - f(m)$ conditioni (n:o praeced.) definitae subjectae sint, serierum easdem exhibentium primi termini ejusdem sint nominis, necesse est. Incipiet ergo utraque series vel ab $e^2 f''(m)$, vel ab $e^4 f^{iv}(m)$ vel denique a termino quodam formae $e^{2n} f^{2n}(m)$. Quippe quod fieri nequit, nisi admiseris saltem $f'(m) = o$. Unde erit m radix aequationis $f'(x) = o$. Quod si praeterea evenerit, ut hoc adhibito valore evanescat $f''(x)$, non potest non $f'''(m)$ quoque evanescere. Atque generatim, si evanuerint functiones $f''(m)$, $f'''(m)$ &c, usque ad $f^{2n}(m)$ inclusive, erit simul $f^{2n+1}(m) = o$. Habebitur ergo $f(m + e) - f(m)$ vel $= e^2 f'(m) + e^3 f'''(m) + \ldots + e^r f^r(m) + e^{r+1} f_{r+1}(m + e, m \ldots)$ vel $= e^4 f^{iv}(m) + e^5 f^v(m) + \ldots + e^r f^r(m) + e^{r+1} f_{r+1}(m + e, m, \ldots)$, vel denique $= e^{2n} f^{2n}(m) + e^{2n+1} f^{2n+1}(m) + \ldots + e^r f^r(m)$

$e^r f^r (m) + e^{r+1} f_{r+1} (m + e, m..)$. Per se vero patet, si fuerit primi termini factor $f^{2n} (m)$ quantitas positiva, esse $f(n)$ *minimum* functionis $f(x)$: sin minus, *maximum*.

63. Sit e. gr. functio $f(x) = A + a(b + cx)^s + a'(b' + c'x)^s$ (sumto pro s numero quovis integro positivo), cujus quæratur minimum vel maximum.

Jam vero facile eruitur $f_1 (x, x_1) =$
$\frac{a[(b + cx)^s - (b + cx_1)^s]}{x - x_1} + \frac{[a'(b' + c'x)^s - (b' + c'x_1)^s]}{x - x_1}$
$= \frac{ac[(b + cx)^s - (b + cx_1)^s]}{(b + cx) - (b + cx_1)} + \frac{a'c'[(b' + c'x)^s - (b' + c'x_1)^s]}{(b' + c'x) - (b' + c'x_1)}$
$= ac(b + cx, b + cx_1)^{s-1} + a'c'(b' + c'x, b' + c'x_1)^{s-1}$,
$f_2 (x, x_1, x^2) = ac^2 (b + cx, b + cx_1, b + cx_2)^{s-2}$
$+ a'c'^2 (b' + c'x, b' + c'x_1, b' + c'x_2)^{s-2}$, atque generatim $f_r (x, x_1, .. x_r) = ac^r (b + cx, b + cx_1, .. b + cx_r)^{s-r} + a'c'^r (b' + c'x', b' + c'x_1, .. b' + c'x_r)^{s-r}$
(n:o 47). Unde $f^I (x) = s\, a\, c\, (b + cx)^{s-1} + a'c'$
$(b' + c'x)^{s-2}$), $f^{II} (x) = \frac{s\, (s - 1)}{1 . 2} (a c^2 (b + c x)^{s-2} + a'c'^2$
$(b' + c'x)^{s-2}$), atque generatim $f^r (x) = \frac{s\, (s - 1)..(s - r + 1)}{1 . 2 . . r}$
$(a c^r (b + c x)^{s-r} + a' c'^r (b' + c' x)^{s-r})$. Hinc, determinandæ quantitati m inservitura, hæc habetur æquatio $s\, (a\, c\, (b + c\, m)^{s-1} + a' c' (b' + c' m)^{s-1}) = 0$, sive $a\, c\, (b + c\, m)^{s-1} = - a' c' (b' + c' m)^{s-1}$. Sit jam
1:o $s = 2n$. Quo pacto evincitur esse $(ac)^{\frac{1}{2n-1}} (b + cm)$
$= - (a'c')$

$= - (a' c')^{\frac{1}{2n-1}} (b' + c' m)$, ideoque $m =$

$$\frac{- b (a c)^{\frac{1}{2n-1}} - b' (a' c')^{\frac{1}{2n-1}}}{c (a c)^{\frac{1}{2n-1}} + c' (a' c')^{\frac{1}{2n-1}}}.$$ Si autem 2:o ſit $s = 2n+1$; erit, extracta radice ordinis n ex utroque membro æquationis $a c (b + c m)^{2n} = - a' c' (b' c' m)^{2n}$, $(a c)^{\frac{1}{n}} (b + c m)^2 = - (a' c')^{\frac{1}{n}} (b' + c' m)^2$. Unde porro elicitur $(a c)^{\frac{1}{2n}} (b + c m) = \pm\sqrt{-1}\, (a' c')^{\frac{1}{2n}} (b' + c'm)$ atque $m =$

$$\frac{- b (a c)^{\frac{1}{2n}} \pm b' (a' c')^{\frac{1}{2n}} \sqrt{-1}}{c (a c)^{\frac{1}{2n}} \mp c' (a' c')^{\frac{1}{2n}} \sqrt{-1}}.$$ Quæ formula in eo caſu, ubi producta ac et $a'c'$ eodem adficiuntur ſigno, imaginarium exhibet valorem quantitatis m. Quo conſtat indicio, ipſius tunc functionis nullum dari minimum vel maximum.

Quod ſi fiat $a' = o$; functio propoſita $f(x)$ evadit $= A + a (b + c x)^s$. In hoc vero caſu formulæ nuper datæ exhibent $m = - \frac{b}{c}$. Qui ſi adhibeatur valor in formulis $f''(m) = \frac{s (s - 1)}{1 . 2} a c^2 (b + c m)^{s-2}$, $f'''(m) = \frac{s (s - 1) (s - 2)}{1 . 2 . 3} a c^3 (b + c m)^{s-3}$ &c. usque

ad