DISSERTATIO MATHEMATICA

ANALYSEOS SUBLIMIORIS ALGEBRÆ ELEMENTARI CONNECTENDÆ SPECIMEN

EXHIBENS.

CUJUS PARTICULAM III.

CONS. AMPL. FACULT. PHILOS. ABOËNS.

P. P.

GABRIEL PALANDER,

Adjunctus Fac. Philos. Ord.

ET

JOHANNES HERTELL,

Satacundensis.

In Audit. Phys. die XX Dec. MDCCCVI.

h. p. m. s.

ABOÆ,

Typis FRENCKELLIANIS.

5

$y(1)..y(n-1).y(n+1)..y(r) = f_1(n)(x,x_1).y(1)..y(n-1).y(n+1)..y(r)$. In qua formula si quantitatibus $y(1)$, $y(2)$,.. $y(n-1)$ successerint $y_1(1)$, $y_1(2)$,.. $y_1(n-1)$ respective, emerget $P_n = f(n)(x,x_1).y_1(1)..y_1(n-1), y n+1,..y(r)$. Definitis ope hujus formulæ collectisque terminis P_1, P_2..P_r, obtinebitur $F_r(x,x_1) = f_1(1)(x,x_1)y(2).y(r) + f_1(2)(x,x_1)y_1(1).y(3)..y(r) + .. f_1(r)(x,x_1).y_1(1), y_1(2),..y_1(r-1)$.

31. Probavimus supra (n:o 16), esse $f_1(x,x_1)$ functionem, quæ, facto $x=x_1$, nec evanescat nec fiat infinite magna, sed abeat in functionem finitam $f'(x)$ qvantitatis x, post eliminatam alteram x_1, residuæ; sive, ut paucis rem exprimam, esse $f_1(x,x_1) = f'(x_1) + V = f'(x) + (x-x_1)^s P$ denotante V functionem, quæ evanescit posito $x=x_1$, ideoque exprimendam per $(x-x_1)^s P$ (n:o 15). Erit ergo, ob symmetriam functionis primæ $f_1(x,x_1)$ (n:o 19), $f'(x) + (x-x_1)^s P = f'(x_1) + (x_1-x)^s P'$, designante P' eam functionem, in quam mutatur P transponendis inter se x & x_1. Unde eruitur $\frac{f'(x)-f'(x_1)}{x-x_1} = -(x-x_1)^{s-1} P - (x_1-x)^{s-1} P'$. Quæ functio, quia est functio prima ipsius $f'(x)$, ita comparata sit oportet (n:o 16), ut, facto $x = x_1$, finita permaneat. Ex analysi vero hujusce conditionis sponte emanat, esse $s = 1$. Quare erit $f_1(x,x_1) = f'(x) + (x-x_1) P$.

32. Sit e. gr. $f(x) = x^m$, existente indice m numero integro positivo. Qvo pacto obtinebitur (vid.

 n:o 20)

n:o 20) $f'(x) = mx^{m-1}$ et $V = f_{,}(x, x_{,}) - f_{,}(x) = (x^{m-1} + x^{m-2} x_{,} + \ldots x_{,}^{m-1}) - mx^{m-1} = x^{m-2}(x_{,} - x) + x^{m-3}(x_{,}^{2} - x^{2}) + \ldots x(x_{,}^{m-2} - x^{m-2}) + (x_{,}^{m-1} - x^{m-1})$; cujus formulae singula membra, factorem formae $x^{r} - x_{,}^{r}$ continentia, facillimo reducuntur negotio ad formam $(x - x_{,}) P$.

33. Denotet $f_{,}(x_{2}, x_{,})$ eam functionem, in quam transit functio prima $f_{,}(x, x_{,})$ suffecta quantitate x_{2} in locum ipsius x; functio $f_{2}(x, x_{,}, x_{2}) = \frac{f_{,}(x, x_{,}) - f_{,}(x_{2}, x_{,})}{x - x_{2}}$ (vel $= \frac{f(x, x_{,}) - f(x_{,}, x_{2})}{x - x_{2}}$) appellabitur *functio derivata secundi ordinis* vel brevius *functio secunda* functionis originariae $f(x)$. Atque generatim, si fuerit $f_{r}(x, x_{,}, \ldots x_{r})$ *functio derivata ordinis* r functionis originariae $f(x)$, functio $f_{r+1}(x, x_{,}, \ldots x_{r+1}) = \frac{f_{r}(x, x_{,}, \ldots x_{r}) - f_{r}(x_{r+1}, x_{,}, \ldots x_{r})}{x - x_{r+1}}$ veniet nomine *functionis derivatae ordinis* $r+1$.

34. Ut uno rem exemplo illustrem, sit $f(x) = x^{4}$. Hujusce quantitatis functio prima $f_{,}(x, x_{,})$ est $= \frac{x^{4} - x_{,}^{4}}{x - x_{,}} = x^{3} + x^{2}x_{,} + xx_{,}^{2} + x_{,}^{3}$ (n:o 20). Hinc vero habebitur $f_{2}(x, x_{,}, x_{2}) = \frac{(x^{3} + x^{2}x_{,} + xx_{,}^{2} + x_{,}^{3}) - (x_{2}^{3} + x_{2}^{2}x_{,} + x_{2}x_{,}^{2} + x_{,}^{3})}{x - x_{2}} = x^{2} + xx_{2} + x_{2}^{2} + (x + x_{2})x_{,} + x_{,}^{2} = x^{2} + x(x_{,} + x_{2}) + x_{,}^{2} + x_{,}x_{2} + x_{2}^{2}$.

$x_, x_2 + x_2^2$. Unde porro elicitur $f_3 (x, x_,, x_2, x_3) =$
$\frac{1}{x - x_3} [(x^2 + x(x_, + x_2) + x_,^2 + x_, x_2 + x_2^2) - (x_3^2 + x_3(x_, + x_2) + x_,^2 + x_, x_2 + x_2^2)] = x + x_, + x_2 + x_3$. Erit denique
$f_4 (x, x_,, x_2, x_3, x_4) = \frac{(x + x_, + x_2 + x_3) - (x_4 + x_, + x_2 + x_3)}{x - x_4} = 1 = x^0$.
Pergendo ulterius offendimus $f_5 (x, x_,, .. x_5) = \frac{x^0 - x_5^0}{x - x_5} = 0$.

35. *Theorema.* Si fuerit $f_r (x, x_,, .. x_r)$ functio symmetrica quantitatum $x, x_,, .. x_r$; dico etiam fore $f_{r+1} (x, x_,, .. x_{r+1})$ symmetrice compositam ex quantitatibus $x, x_,, .. x_{r+1}$.

Jam vero est $f_{r+1} (x, x_,, .. x_{r+1}) = \frac{f_r (x, x_,, .. x_r) - f_r (x_{r+1}, x_,, .. x_r)}{x - x_{r+1}}$ (n:o 33); quippe quam functionem esse symmetrice compositam respectu quantitatum $x_,, x_2, .. x_r$, ex data functionis $f_r (x, x_,, .. x_r)$ symmetria sponte promanat. Quo autem constet, reliquas quantitates x et x_{r+1} æquam cum his participare sortem, fiat, convenienter notioni supra (n:o 33) definitæ, $f_r (x, x_,, .. x_r) = \frac{f_{r-1} (x, x_,, .. x_{r-1}) - f_{r-1} (x_r, x_,, .. x_{r-1})}{x - x_r}$.
Unde emergit $f_r (x_{r+1}, x_,, .. x_r) = \frac{f_{r-1} (x_{r+1}, x_,, .. x_{r-1}) - f_{r-1} (x_r, x_,, .. x_{r-1})}{x_{r+1} - x_r}$, atque dehinc

$$f_{r+1}(x,x_{,},..x_{r+1}) = \frac{f_{r-1}(x,x_{,},..x_{r-1}) - f_{r-1}(x_r,x_{,},..x_{r-1})}{(x-x_r)(x-x_{r+1})}$$

$$-\frac{f_{r-1}(x_{r+1},x_{,},..x_{r-1}) - f_{r-1}(x_r,x_{,},..x_{r-1})}{(x_{r+1}-x_r)(x-x_{r+1})} =$$

$$\frac{1}{(x_{r+1}-x_r)(x-x_{r+1})(x_r-x)}[(x_{r+1}-x_r)f_{r-1}(x,x_{,},..x_{r+1}) +$$

$$(x-x_{r+1})f_{r-1}(x_r,x_{,},..x_{r-1}) + (x_r-x)f_{r-1}(x_{r+1},x_{,},..x_{r-1})]$$

Hæc vero functionis propositæ forma aperte prodit, eam esse symmetricam respectu quantitatum x, x_r et x_{r+1}. Quare, cum præterea nuper demonstratum sit, unam ex his trinis, nimirum x_r, æqua fungi vice eum reliquis $x_{,}, x_2, ... x_{r-1}$; erit $f_{r+1}(x,x_{,},..x_{r+1})$ functio symmetrice constituta ex quantitatibus $x, x_{,}, .. x_{r+1}$.

36. Cum demonstratum sit (n:o 19), esse functionem primam $f_{,}(x, x_{,})$, symmetrice compositam ex quantitatibus x et $x_{,}$; erit quoque, vi Theorematis n:o præced. exhibiti, functio secunda $f_2(x,x_{,},x_2)$ symmetrica respectu quantitatum x, $x_{,}$ et x_2. Hinc simili evincitur ratiocinio, esse $f_3(x,x_{,},x_2,x_3)$ functionem symmetricam quantitatum x, $x_{,}$, x_2 et x_3. Qua pergendo via successive ad $f_4(x,x_{,}..x_4)$, $f_5(x,x_{,}..x_5)$, perveniendum demum erit ad functionem derivatam ordinis cujuscunque r. Erit ergo generatim $f_r(x,x_{,},..x_r)$ functio symmetrica quantitatum $x, x_{,}, .. x_r$.

37. Si

37. Si sit V ejusmodi functio quantitatum $x, x_1, x_2 \ldots x_r$, ut, posito $x = x_1 = x_2 \ldots = x_r$, evanescat; facile patet, eam posse considerari utpote conflatam ex terminis $V_1, V_2, \ldots V_r$ ita comparatis, ut, positis $x_1, x_2, \ldots x_r$ singulatim $= x$, evanescant $V_1, V_2, \ldots V_r$ respective. Quibus sub conditionibus erit (n:o 15) $V_1 = (x - x_1)^{s1} P_1$, $V_2 = (x - x_2)^{s2} P_2$, .. denique $V_r = (x - x_r)^{sr} P_r$, exprimentibus $P_1, P_2, \ldots P_r$ functiones, quae, positis quot et quibusvis quantitatum $x_1, x_2, \ldots x_r = x$, finitae permaneant *).

38. Quod si sit $V = (x - x_1)^{s1} P_1 + (x - x_2)^{s2} P_2 + \ldots (x - x_r)^{sr} P_r$ functio symmetrice composita ex quantitatibus $x_1, x_2, \ldots x_r$; facili probabitur negotio, esse $s_1 = s_2 = \ldots s_r$. Erit enim hoc in casu, transpositis inter se x_1 et x_2, V quoque $= (x - x_2)^{s1} P_1' + (x - x_1)^{s2} P_2' + \ldots (x - x_r)^{sr} P_r'$, denotantibus $P_1', P_2', \ldots P_r'$ eas functiones, in quas transeunt, facta hac transpositione, $P_1, P_2, \ldots P_r$ respective. Unde obtinebitur $(x - x_1)^{sr} P_1 + (x - x_2)^{s2} P_2 + \ldots (x - x_r)^{sr} P_r = (x - x_2)^{s1} P_1' + (x - x_1)^{s2} P_2' + \ldots (x - x_r)^{sr} P_r'$. Fiat jam in hac aequatione $x_2 = x_3 = \ldots x_r = x$, quo ea reducatur ad hanc: $(x - x_1)^{s1}(P_1) = (x - x_1)^{s2}(P_2')$, sive

*) Quam proprietatem omnibus esse communem functionibus, infra (n:o 38 - 40) littera P exprimendis, in antecessum heic indicasse juvabit.

sive $(P_,) = (x - x_,)^{s_2 - s_,}.(P'_2)$ transeuntibus nimirum, vi hujusce determinationis, $P_,$ in $(P_,)$ et P_2, in (P_2'). Quare, cum sit $(P_,)$ suapte natura talis, ut posito $x = x_,$, non evanescat, liquet esse $s_, = s_2$. Simili omnino ratiocinio reliquorum quoque indicum æqualitas demonstrari potest.

39. *Theorema.* Si fuerit $f_r(x\, x_, \ldots x_r) = f_r(x) + (x - x_,)\, P_,^{(r)} + (x - x_2) P_2^{(r)} + \ldots (x - x_r).\, P_r^{(r)}$; dico fore $f_{r+1}(x, x_, \ldots x_{r+1}) = f_{r+1}(x) + (x - x_,) P_,^{(r+1)} + (x - x_2) P_2^{(r+1)} + \ldots (x - x^{r+1}) P_{r+1}^{(r+1)}$.

Quia est (Hyp.) $f_r(x, x_, \ldots x_r) = f_r(x) + (x - x_,)\, P_,^{(r)} + (x - x_2)\, P_2^{(r)} + \ldots (x - x_r)\, P_r^{(r)}$; erit quoque $f_r\,(x, x_, \ldots x_{r-1}, x_{r+1}) = f^r\,(x) + (x \cdot x_,)\, P_,^{(r)\prime} + (x - x_2)\, P_2^{(r)\prime} + \ldots (x - x_{r-1})\, P_{r-1}^{(r)\prime} + (x - x_{r+1})\, P_r^{(r)\prime}$. Unde efficitur esse $f_{r+1}\,(x, x_,, \ldots x_{r+1}) = \dfrac{(x - x_,)(P_,^{(r)} - P_,^{(r)\prime})}{x_r - x_{r+1}} + \dfrac{(x - x_2)(P_2^{(r)} - P_2^{(r)\prime})}{x_r - x_{r+1}} + \ldots \dfrac{(x - x_{r-1})(P_{r-1}^{(r)} - P_{r-1}^{(r)\prime})}{x_r - x_{r+1}} + \dfrac{(x - x_r)\,(P_r^{(r)} - P_r^{(r)\prime})}{x_r - x_{r+1}} - P_r^{(r)\prime}$, denotantibus $P_,^{(r)\prime}$, $P_2^{(r)\prime}, \ldots P_{r-1}^{(r)\prime}$, $P_r^{(r)\prime}$, eos valores, quos obtinebunt $P_,^{(r)}, P_2^{(r)}, \ldots P_{r-1}^{(r)}, P_r^{(r)}$, substituendo x_{r+1}, pro x_r. Quod si in hac formula æquentur $x_,, x_2, \ldots x_r$ quantitati x; omnes ejus termini evanescent, excepto ultimo $P_r^{(r)\prime}$, suapte natura tali, ut substituta quanti-

tate

tate x pro $x_{,}$, x_{2}, . . x_{r-1} et x_{r+1} transeat in $(P_r^{(r)})$ designante videlicet $(P_r^{(r)})$ functionem illam quantitatis x, in quam transit $P_r^{(r)}$, posita x pro unaquaque quantitatum ab $x_{,}$ ad x_r inclusive. Est igitur $f^{r+1}(x, x_{,}, .. x^{r+1})$ functio ejus naturæ, ut, æquatis quantitati x omnibus reliquis ab $x_{,}$ usque ad x^{r+1} inclusive, transmigret in functionem finitam quantitatis $\bar{x}$ puta $(P_r^{(r)})$, quæ fiat $= f^{r+1}(x)$. Quare erit (n:o 37 & 38) $f^{r+1}(x, x_{,}, .. x^{r+1}) = f^{r+1}(x) + V_{,} + V_{2} + .. V^{r+1} = f^{r+1}(x) + (x - x_{,})^s P_{,}^{(r+1)} + (x - x_{2})^s P_{2}^{(r+1)} + .. (x - x^{r+1})^s P^{r+1(r+1)}$. Quod si in functione hac symmetrica inter se permutentur x et $x_{,}$; erit quoque eadem $= f^{r+1}(x_{,}) + (x_{,} - x)^s P_{,}^{(r+1)'} + (x_{,} - x_{2})^s P_{2}^{(r+1)'} + .. (x_{,} - x^{r+1})^s P^{r+1(r+1)'}$, ubi $P_{,}^{(r+1)'}$, $P_{2}^{(r+1)'}$, . $P^{r+1(r+1)'}$ denotant novos valores functionum $P_{,}^{(r+1)}$, $P_{2}^{(r+1)}$,.. $P^{r+1(r+1)}$ ex hac transpositione oriundos. Comparando jam binas has formas functionis $f^{r+1}(x, x_{,}, .. x^{r+1})$ obtinebitur $f^{r+1}(x) - f^{r+1}(x_{,}) = -[(x - x_{,})^s P_{,}^{(r+1)} + (x - x_{2})^s P_{2}^{(r+1)} + .. (x - x^{r+1})^s P^{r+1(r+1)}] + [(x_{,} - x)^s P_{,}^{(r+1)'} + (x_{,} - x_{2})^s P_{2}^{(r+1)'} + .. (x_{,} - x^{r+1})^s P^{r+1(r+1)}]$. Quæ æquatio, ex comparatione functionum identicarum enata, semper persistet, ut demum cunque quantitates $x_{,}$, x_{2},.. x^{r+1} definiantur. Fiat igitur unaquaque quantitatum x_{2}, . . $x_r = x_{,}$. Quo pacto habebitur $f^{r+1}(x) - f^{r+1}(x_{,}) = -(x - x_{,})^s [(P_{,}^{(r+1)}) + (P_{2}^{(r+1)}) + .. (P_{,+1}^{(r+1)})] + (x_{,} - x)^s (P^{(r+1)'})$, sive

$$\frac{f^{r+1}(x) - f^{r+1}(x_{,})}{x - x_{,}} = -(x_{,} - x)^{s-1} [(P_{,}^{(r+1)}) + (P_{2}^{(r+1)}) + .. (P^{r+1(r+1)})] - (x_{,} - x)$$

$(x_{,} - x)^{s-1}. (P_{,}(r+1)')$. Est vero prius hujusce aequationis membrum, utpote functio prima functionis $f^{r+1}(x)$, ita comparatum (n:o 16), ut, pro $x = x_{,}$, vicem functionis finitae ipsius x subeat. Cui congruenter conditioni erit $s - 1 = 0$, sive $s = 1$. Quo probato propositi veritas Theorematis in aperto est sita.

40. Quia supra (n:o 31) probatum est, esse $f_{,}(x, x_{,}) = f'(x) + (x - x_{,}) P_{,}$; efficitur hinc vi Theorematis praeced. esse $f_{2}(x, x_{,}, x_{2}) = f''(x) + (x - x_{,}) P_{,}(2) + (x - x_{2}) P_{2}(2)$. Unde porro pari ratiocinio evincitur esse $f_{3}(x, x_{,}, x_{2}, x_{3}) = f'''(x) + (x - x_{,}) P_{,}(3) + (x - x_{2}) P_{2}(3) + (x - x_{3}) P_{3}(3)$. Atque cum dehinc eundem sequendo ratiocinandi modum ad functiones derivatas altiorum ordinum progredi liceat; erit generatim $f_{r}(x, x_{,}, .. x_{r}) = f^{r}(x) + (x - x_{,}) P_{,}(r) + (x - x_{2}) P_{2}(r) + .. (x - x_{r}) P_{r}(r)$.

41. Sint propositae quantitates quotvis $a_{,}, a_{2}, .. a_{r}$: fiant ex potestatibus harum positivis integris tot producta formae $a_{,}^{s_{,}} . a_{2}^{s_{2}} .. a_{r}^{s_{r}}$, quot ex his ea lege combinandis, ut sit $s_{,} + s_{2} + .. s_{r} =$ dato numero n, formari possunt; summam ex colligendis hisce productis oriundam, compendiaria exhibendam signandi ratione, liceat ponere $= (a_{,}, a_{2}, .. a_{r})^{n}$. Sic e. gr. facto $r = 3$ et $n = 2$, designet $(a_{,}, a_{2}, a_{3})^{2}$ hanc functionem: