7

DISSERTATIO MATHEMATICA,
SPECIMINA QUÆDAM
METHODI TANGENTIUM INVERSÆ
SISTENS.

QUAM

CONS. AMPL. FAC. PHIL. ABOËNS.

PRÆSIDE

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PRO GRADU

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ABOÆ,

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- - - - - - *Methodus directa tangentium ubique facilis, inversa generalis nulla, ejusque loco tantum particulares dari possunt regulæ, quarum qui plures collegerit, is optime de hac Methodo meruisse censebitur.*

JAC. BERNOULLI *Opp. Tom. I. p. 622.*

§. I.

Duplex in Geometria sublimiori occurrit *Methodus* quæ dicitur *Tangentium;* alia *directa*, alia *inversa*. *Illa* rationem monstrat, ex data curvæ indole seu æquatione, ad quodvis ejus punctum inveniendi tangentem, normalem, radium curvaturæ, & quæ sunt reliqua ex tangentibus dependentia. *Hæc* vero ex datis tangentium proprietatibus vel aliis quibusvis curvarum affectionibus, relationem inter harum Coordinatas, adeoque rationem easdem construendi detegere docet. Methodo illi directæ perficiendæ jam dudum operam dederunt Mathematici, & invento calculo differentiali, hæc doctrina illud facillime attigit fastigium, ut in ea nihil fere amplius desiderari videatur. Primum Methodi tangentium inversæ specimen debetur Clarissimo Gallorum Mathematico *de Beaune;* & licet ab illo inde tempore egregia fuerit Mathematicorum industria in hac Geometriæ Curvarum parte excolenda, eam tamen in incunabulis quasi jacere adhuc fatendum est. Id enim huic disciplinæ commune est cum calculo integrali, sine quo in illa parum proficere licet, ut quæ hactenus inventæ sint regulæ, quamvis in casibus quam plurimis, immo innumeris sufficiant, minime tamen generales dici mereantur, quum longe plura occurrant problemata, quorum solutio vulgarium regularum ope minime succedat. Sagacitati igitur Geometrarum relinquitur, in hujusmodi casibus novas, quibus metam contingant, detegere vias. In re tam maxime ardua moliri aliquid, audacissimum certe videbitur; quum vero quædam nobis succurrerint problemata huc pertinentia, quorum partim nullam a Geometris antea factam mentionem viderimus, partim novæ nobis sese obtulerint solutiones, horum pertractationem, speciminis Academici loco, benigna Tua C. L. venia, publicæ jam luci committere audemus.

§. II.

PROBLEMA. (Fig. 1.) *Si Curvæ BM ad quodvis punctum M coordinatæ Orthogonales fuerint* $AP = x$ *&* $PM = y$, *atque ad tangentem ML ex dato puncto A ducta normalis* $AL = v$: *ex data relatione inter v & alterutram coordinatarum x vel y invenire indolem curvæ.*

Demissa LK perpendiculari in axem AP, & ex puncto m ipsi M infinite propinquo ducta tangente Sm, cui ex dato puncto A fiat normalis SA, tangenti ML occurrens in l, si angulus LMP ($= LAK$) dicatur ϕ; erit ang. $SMl = d\phi$ & (posito sinu toto $= 1$) $LK = v \, Sin \, \phi$, atque $AK = v \, Cos \, \phi$, adeoque $PK (= PA + AK) = x + v \, Cos \, \phi$ & $ML = \frac{PK}{Sin \, \phi}$

$= \frac{x + v \, Cos\, \phi}{Sin\, \phi}$, unde, quum ſit $dv = Sl = LM.\, d\phi$, & $y = MP = ML\, Cos\, \phi + LK$, ſequentes obtinentur Æquationes: $dv = \frac{(x + v\, Cos\, \phi)\, d\phi}{Sin\, \phi}$ (A) atque $y = \frac{x\, Cos\, \phi + v}{Sin\, \phi}$ (B), quarum ope ex data relatione ipſarum v & x invenitur natura curvæ BM. Eadem ratione, facta permutatione Coordinatarum, demonſtratur fore: $dv = \frac{(y - v\, Sin\, \phi)\, d\phi}{Cos\, \phi}$ (A′) & $x = \frac{y\, Sin\, \phi - v}{Cos\, \phi}$ (B′), quæ Æquationes inſerviunt curvæ inveniendæ, quando v eſt functio quædam ipſius y.

Exempl. 1. Si $v = x$, erit $dx = \frac{x\, d\phi\, (1 + Cos\, \phi)}{Sin\, \phi}$ (A), & $y = \frac{x\, (1 + Cos\, \phi)}{Sin\, \phi}$ (B), adeoque $dx = y\, d\phi$. Quum vero ſit $dx = dy\, Tang\, \phi$, erit $\frac{dy}{y} = Cotg\, \phi\, d\phi$, & facta integratione, $y = C\, Sin\, \phi$ (denotante C quantitatem conſtantem arbitrariam). Hoc ipſius y valore ſubſtituto in æqu. B, prodit $x = \frac{C\, Sin\, \phi^2}{1 + Cos\, \phi} = C\, (1 - Cos\, \phi)$. Erit igitur $C^2\, Sin\, \phi^2 = y^2$ & $C^2\, Cos\, \phi^2 = C^2 - 2\, Cx + x^2$; unde (ob $Sin\, \phi^2 + Cos\, \phi^2 = 1$) obtinetur $y^2 = 2\, Cx - x^2$, æquatio ad Circulum.

Exempl. 2. Si $v = x + a$, fit $dx = \frac{x\,d\Phi(1 + Cos\,\Phi)}{Sin\,\Phi}$ $+ a\,Cotang\,\Phi\,d\Phi$ (A), quæ æquatio (dividendo per $2\,Sin\,\tfrac{1}{2}\Phi^2$) transformatur in hanc per se integrabilem:

$$\frac{dx}{2\,Sin\,\tfrac{1}{2}\Phi^2} - \frac{x\,d\Phi\,(1 + Cos\,\Phi)}{2\,Sin\,\tfrac{1}{2}\Phi^2\,Sin\,\Phi} = \frac{a\,Cotg\,\Phi\,d\Phi}{2\,Sin\,\tfrac{1}{2}\Phi^2}$$ seu

$$\frac{dx}{2\,Sin\,\tfrac{1}{2}\Phi^2} - \frac{(\tfrac{1}{2}a + x)\,Cos\,\tfrac{1}{2}\Phi\,d\Phi}{2\,Sin\,\tfrac{1}{2}\Phi^3} = -\frac{a\,d\Phi}{2\,Sin\,\Phi},$$ cujus integrale est $\frac{\tfrac{1}{2}a + x}{2\,Sin\,\tfrac{1}{2}\Phi^2} = C - \tfrac{1}{2}a\,L\,Tang\,\tfrac{1}{2}\Phi$ (designante *L* Logarithmum hyperbolicum), quæ æquatio collata cum $y = \frac{x(1 + Cos\,\Phi) + a}{Sin\,\Phi}$ exhibet indolem curvæ, quæ quidem, nisi fuerit $a = o$ (Exempl. 1.) semper erit transcendens.

Exempl. 3. Si $v = nx + a$, erit $dx = \frac{x\,d\Phi\,(1 + n\,Cos\,\Phi)}{n\,Sin\,\Phi} + \frac{a\,Cos\,\Phi\,d\Phi}{n\,Sin\,\Phi}$ (A), quæ æquatio ope multiplicatoris $\frac{Cos\,\tfrac{1}{2}\Phi^{\frac{1-n}{n}}}{2\,Sin\,\tfrac{1}{2}\Phi^{\frac{1+n}{n}}}$ per se integrabilis redditur. Facta vero integratione prodit: $\frac{a}{1+n}$ $+ x = \frac{2a}{1-n^2}\,Sin\,\tfrac{1}{2}\Phi^2 + 2\,C\,Sin\,\tfrac{1}{2}\Phi^2\,Tang\,\tfrac{1}{2}\Phi^{\frac{1-n}{n}}$

Ex hac æquatione comparata cum $y = \frac{x(n + Cos\,\Phi) + a}{Sin\,\Phi}$ (B), (B), innotescit relatio Coordinatarum x & y adeoque natura ipsius Curvæ BM, quam patet semper fore Algebraicam, nisi fuerit $n = 1$ (Exempl. 2.)

Scholion 1. Si in æquatione supra allata $y = \frac{x\, Cos\, \Phi + v}{Sin\, \Phi}$, pro $Sin\, \Phi$ & $Cos\, \Phi$ substituantur horum valores $\frac{dx}{\sqrt{dx^2 + dy^2}}$ & $\frac{dy}{\sqrt{dx^2 + dy^2}}$, oritur $ydx - xdy = v\sqrt{dx^2 + dy^2}$, adeoque $\frac{dy}{dx} = \frac{xy \pm v\sqrt{x^2 + y^2 - v^2}}{x^2 - v^2}$, unde quoties fuerit v functio aliqua alterutrius ipsarum x & y, facta integratione statim obtinetur ipsa æquatio curvæ. Quum vero in plerisque casibus difficillimam fore constat integrationem æquationis hujus differentialis, commodior sæpissime erit Methodus supra exposita inveniendi relationem inter curvæ coordinatas.

Scholion 2. Ope æquationum illarum A & B, inveniri etiam potest natura curvæ, quoties fuerit v functio quæcunque anguli Φ. Facta enim reductione obtinetur: $x = \frac{dv\, Sin\, \Phi}{d\psi} - v\, Cos\, \Phi$, & $y = \frac{dv\, Cos\, \Phi}{d\Phi} + v\, Sin\, \Phi$, unde ope anguli Φ datur relatio ipsarum x & y, & quidem absque omni integratione; quod unico exemplo illustrasse sufficiat. Si fuerit $v\, Cos\, \Phi$ (seu AK) $= a$, adeoque locus puncti L recta posi-

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tione data KL, erit $\frac{dv}{d\phi} = \frac{a \, Sin \, \phi}{Cos \, \phi^2}$ adeoque $x = a$ $(tg \, \phi^2 - 1)$ & $y = 2 \, a \, tg \, \phi$, unde obtinetur $y^2 = 4a \, (a + x)$. Curva igitur hæc erit Parabola conica, cujus Focus est A & vertex K.

Scholion 3. Generatim etiam eadem Methodo solvi potest problema hoc pro datis quibusvis angulis MPN & MLA. Si scilicet dato Coordinatarum angulo $MPN = \alpha$, & ex dato puncto A ducta AL, quæ cum tangente ML efficiat angulum quemvis constantem $ALM = \lambda$, detur relatio inter $AL = v$ & alterutram Coordinatarum $AP = x$ vel $PM = y$; (posito ang. $LMP = \phi$, adeoque $LAK = \lambda - \alpha + \phi$) inveniuntur æquationes: $dv = \frac{(v \, Sin \, \lambda \, Cos \, \phi + x \, Sin \, \alpha) \, d\phi}{Sin \, \lambda \, Sin \, \phi}$ $= \frac{(y \, Sin \, \alpha - v \, Sin \, \lambda \, Cos \, \overline{\alpha - \phi}) \, d\phi}{Sin \, \lambda \, Sin \, (\alpha - \phi)}$, & $y = \frac{v \, Sin \, \lambda + x \, Sin \, (\alpha - \phi)}{Sin \, \phi}$ vel $x = \frac{y \, Sin \, \phi - v \, Sin \, \lambda}{Sin \, (\alpha - \phi)}$, quarum ope indoles curvæ innotescit. Si vero fuerit v functio aliqua anguli ϕ, absque ulla integratione indagari potest relatio inter x & y ex æquationibus: $x = \frac{Sin \, \lambda}{Sin \, \alpha} \left(\frac{dv \, Sin \, \phi}{d\phi} - Cos \, \phi \right)$, & $y = \frac{Sin \, \lambda}{Sin \, \alpha} \left(\frac{dv \, Sin \, \overline{\alpha - \psi}}{d\phi} + v \, Cos \, \overline{\alpha - \phi} \right)$.

§. III.

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PROBLEMA. (Fig. 1.) *Si ex dato puncto A ad quamvis curvæ BM tangentem ML ducatur normalis AL; ex data relatione ipsarum $AL = v$ & $ML = t$, invenire curvam.*

Sint curvæ BM coordinatæ orthogonales $AP = x$ & $PM = y$, atque ex dato puncto A ducatur tangenti parallela sive normali curvæ MN perpendicularis AQ, ordinatæ PM occurrens in O. Si angulus LMP ($= MNP = AOP = MOQ$) dicatur ϕ, (posito sinu toto $= 1$), erit $OP = x \, Cotg \, \phi$, & $AO = \frac{x}{Sin \, \phi}$, unde $MO = y - x \, Cotg \, \phi$ adeoque $OQ = MO \, Cos \, \phi = y \, Cos \, \phi - \frac{x \, Cos \, \phi^2}{Sin \, \phi}$, & $MQ = MO \, Sin \, \phi$ seu $v = y \, Sin \, \phi - x \, Cos \, \phi$, atque t ($= AO + OQ$) $= y \, Cos \, \phi + x \, Sin \, \phi$, quibus valoribus substitutis in data æquatione pro t & v, & exterminatis functionibus anguli ϕ ope æquationis $\frac{dx}{dy} =$ tang ϕ, obtinetur æquatio differentialis inter x & y, quæ integrata exhibet æquationem curvæ quæsitam.

Facilior autem plerumque fit hujus Problematis solutio ope ordinatarum AM ex dato polo A prodeuntium. Si namque fuerit $AM = u$, atque interceptus ab AM & recta positione data AK angulus $MAK = z$, nec non ang. $AML = \psi$; erit $v = u \, Sin \, \psi$ &

$t = u \, Cos \, \psi$, quibus substitutis in data æquatione, relationem ipsarum t & v exprimente, inveniatur $tg \, \psi = U$ functioni cuidam ipsius u, unde porro, quum in genere sit $udz = du \, tang \, \psi$, obtinetur $z = \int \frac{Udu}{u}$. Ex hac æquatione ulterius, si ita placuerit, erui facile potest alia pro coordinatis orthogonalibus x & y, quoniam est $tang \, z = -\frac{y}{x}$ & $u = \sqrt{x^2 + y^2}$.

Exempl. 1. Si fuerit $mt + nv = a$, erit $m \, Cos \, \psi + n \, Sin \, \psi = \frac{a}{u}$, unde si ponatur $(m^2 + n^2) u^2 - a^2 = s^2$, adeoque $u^2 = \frac{s^2 + a^2}{m^2 + n^2}$, invenitur $tg \, \psi$ seu $U = \frac{mn(a^2 + s^2) \pm (m^2 + n^2) as}{m^2 a^2 - n^2 s^2} = \frac{na \pm ms}{ma \mp ns}$ & $\frac{du}{u} = \frac{sds}{a^2 + s^2}$, adeoq; $\frac{Udu}{u} = dz = \pm \frac{ads}{a^2 + s^2} \pm \frac{mds}{ma \mp ns}$ cujus integrale est $z = C \mp \text{Arc. } Tg \frac{s}{a} - \frac{m}{n} Log \, (ma \mp ns)$. Ex hac æquatione collata cum $u^2 = \frac{s^2 + a^2}{m^2 + n^2}$ facile invenitur constructio curvæ quæsitæ.

Exempl. 2. Si quæratur curva BM talis, ut sit $tv = 2a^2$, erit $u = \frac{2a}{\sqrt{Sin \, 2\psi}}$, & $\frac{du}{u} = - Cotg \, 2\psi \, d\psi$

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adeoque $dz = - Tang\,\psi\, Cotg\, 2\psi d\psi = - d\psi + \frac{d\psi}{2 Cos\,\psi^2}$. Hinc integrando invenitur $z = C - \psi + \frac{1}{2} tg\,\psi$ (adeoque $\varphi + C' = \frac{1}{2} Tg\,\psi$); unde patet constructionem hujus curvæ ex quadratura Circuli pendere.

Scholion. In genere si ex puncto fixo A ducta recta AL cum tangente ML efficiat datum angulum ALM $= \lambda$, & detur relatio ipsarum AL $= v$ & ML $= t$, retentis de cætero prioribus denominationibus, ad investigandam indolem curvæ sequentes conducunt formulæ: $v = \frac{u\, Sin\,\psi}{Sin\,\lambda}$, $t = \frac{u\, Sin\,(\lambda + \psi)}{Sin\,\lambda}$, atque $du\, tg\,\psi = u dz$, quarum veritas vel ex sola inspectione triangulorum *LMA* & *Mrm* facile patet.

§. IV.

Theorema. (Fig. 1.) *Si curvæ BM ad punctum quodvis* M *Radius osculi sit RM, qui* (productus si opus fuerit) *secet in N rectam positione datam AN, & ex dato hujus puncto A ad tangentem ML, nec non ex puncto curvæ M ad rectam AN ducantur perpendiculares AL, MP respective: erit Radius osculi RM ad resectam AN, ut fluxio ipsius AP ad fluxionem ipsius AL.*

Demonstr. Sumta portione curvæ *Mm* infinite parva, si ducantur *Rm*, *AM*, *Am*, atque tangenti *mS* perpendicularis *AS*, nec non ipsis *AN*, *MP*, *MR* & *Am* respective perpendiculares *mp*, *Mq*, *AQ* & *Mr*, sitque $AP = x$, $PM = y$, $AM = u$ & $AL = v$; erit

$RM = \frac{udu}{dv}$ [quoniam ob $\triangle Slm \backsim \triangle MRm$ est Sl $(= dv) : Mm :: Sm \,(= LM) : MR$ atque ob $\triangle Mrm \backsim \triangle MLA$, $Mm : rm \,(= du) :: AM \,(= u) : LM$, ergo ex æquo $dv : du :: u : RM$]. Porro ob $\triangle Mqm \backsim \triangle MPN$, erit $Mq \,(= dx) : qm \,(= dy) :: MP \,(= y) : PN$, quamobrem $PN = \frac{ydy}{dx}$, adeoque $AN = x + \frac{ydy}{dx} = \frac{xdx + ydy}{dx}$. Quum vero sit $u^2 = x^2 + y^2$ & hinc $udu = xdx + ydy$, erit $AN = \frac{udu}{dx}$; quare $RM : AN :: \frac{udu}{dv} : \frac{udu}{dx} :: dx : dv$. Q. E. D.

Aliter & brevius idem sic evincitur: quum ob $\triangle Slm \backsim \triangle MRm$, sit $Sl \,(= dv) : Mm :: Sm \,(= AQ) : RM$ & ob $\triangle Mqm \backsim \triangle AQN$, $Mm : Mq \,(= dx) :: AN : AQ$, erit ex æquo $dv : dx :: AN : RM$. Q. E. D.

§. V.

Problema. (Fig. I.) *Invenire curvam BM, cujus Radius quivis osculi RM ad portionem AN, quam ex data positione recta resecat, sit in data ratione: $RM : AN :: 1 : n$.*

Iisdem ac in §. IV. positis, quum per Theorema allatum sit $RM : AN : dx : dv$, erit $dv = ndx$, adeoque $v = a + nx$ (designante a constantem quamvis arbi-

arbitrariam); unde indoles curvæ facile invenitur (§. II.). Si itaque Ang. MNA dicatur φ, pro $n >$ vel < 1 erit $\frac{a}{1+n} + x = \frac{2a\,Sin\,\frac{1}{2}\varphi^2}{1-nn} + 2C\,Sin\,\frac{1}{2}\varphi^2$ $Tang\,\frac{1}{2}\varphi\,\frac{1-n}{n}$ atque $y = \frac{x\,(n + Cos\,\varphi) + a}{Sin\,\varphi}$ (§. II. Exempl. 3.). Si vero fuerit RM = AN seu $n = 1$, erit (§. II. Ex. 2.) $\frac{1}{2}a + x = 2\,Sin\,\frac{1}{2}\varphi^2\,(C - \frac{1}{2}a\,L\,tg\,\frac{1}{2}\varphi)$ & $y = \frac{x\,(1 + Cos\,\varphi) + a}{Sin\,\varphi}$, nisi sumatur $a = o$, quo in casu curva quæsita erit circulus (§. II. Ex. 1.)

Scholion 1. Quoniam sumta x uniformiter fluente, seu $d^2x = o$, sit Radius curvaturæ RM $= \frac{(dx^2 + dy^2)^{\frac{3}{2}}}{-dx\,d^2y}$ & AN $= x + \frac{ydy}{dx}$, methodo vulgari pro curva quæsita obtinetur æquatio differentio-differentialis: $n\,(dx^2 + dy^2)^{\frac{3}{2}} + (xdx + ydy)\,d^2y = o$, quæ quidem integrationem admittit, calculum vero prolixiorem supponit, quare solutionem a nobis allatam huic præferendam autumamus.

Scholion 2. Si resecta AN dicatur p & normalis MN s, adsumta dp constante, erit generatim Radius osculi RM $= s - \frac{dp - ds^2}{d^2s}$ (descripto enim centro R arcu Nt, & posita $dp = Nn$, erit $tn = ds = dp\,Cos\,\varphi$, adeoque $Cos\,\varphi = \frac{ds}{dp}$ & $d\varphi = -$

 d^2s

$\frac{d^2 s}{\sqrt{dp^2 - ds^2}}$, nec non $Nt = dp \, Sin \, \Phi = \sqrt{dp^2 - ds^2}$.
Est vero $RM - s = NR = \frac{Nt}{d\Phi} = \frac{dp^2 - ds^2}{-d^2 s}$; Ergo $RM = s - \frac{dp^2 - ds^2}{d^2 s}$). Hoc adhibito Radii osculi valore, præsens Problema, in quo ponitur $RM : p :: 1 : n$, ad sequentem redigitur æquationem differentio-differentialem: $(p - ns). d^2 s + n (dp^2 - ds^2) = o$, cujus mox patet primum integrale fore: $(p - ns) ds - (C + s - np) dp = o$, quæ æquatio differentialis primi gradus per Methodos cognitas ulterius integrari potest. Hac peracta integratione invenitur relatio ipsarum s & p, unde quum sit $\frac{ds}{dp} = Cos \, \Phi$, facile innotescit ratio construendi curvam quæsitam ope æquationum $y = s \, Sin \, \Phi$ & $x = p - s \, Cos \, \Phi$.

§. VI.

THEOREMA. (Fig. 2.) *Si Curva BP sit tractoria ad AM (ita ut ad quodvis illius punctum P ducta tangens PM, huic occurrens in M, sit constans seu PM = a) & ad puncta correspondentia P & M utriusque lineæ ducantur normales PR & MC: punctum occursus harum R erit centrum circuli tractoriam in P osculantis.*

Demonstr. Sumta portione curvæ Pp infinite parva, ducatur ex p alia tangens tractoriæ pm & cen-

centro P per M describatur arcus circuli Mn. Quum igitur sint anguli AMR, MPR & Mnm singuli recti, ideoque PRM = PMA = Mmn, erit $\triangle\, Mmn \backsim \triangle$ MRP, quamobrem $Mn : nm :: MP : PR$. Sunt vero (hypoth.) tangentes PM & pm æquales adeoque etiam $pm = Pn$, unde $Pp = mn$. Erit itaque $Mn : Pp :: MP : PR$ & $\frac{Mn}{MP} = \frac{Pp}{PR}$. Sed $\frac{Mn}{MP}$ = ang. MPm, & junctis R, p, pariter est $\frac{Pp}{PR}$ = ang. PRp; quare anguli MPm & PRp æquantur, adeoque etiam ang. Rpm = RPM = recto. Unde quum Normales ad curvam sint ambæ PR & pR, & quidem infinite vicinæ, punctum occursus harum R erit centrum circuli, curvam BP in P osculantis. Q. E. D.

Cor. 1. Si fuerit curvæ AM ad punctum M Radius curvaturæ MC = R, & ponatur AM = z, nec non MR = y; erit $\frac{aRdy}{\sqrt{y^2 - a^2}} = (R - y)\,dz$. Ducta enim Cm, cui in r occurrat pR producta, & centro C descripto arcu Rq; ob angulos Rqr & RPM rectos nec non ang. Rrq = PRM, erit $\triangle\, Rqr \backsim \triangle$ RPM, unde $Rq : rq\ (= dy) ::$ PM $(= a)$: PR $(= \sqrt{y^2 - a^2})$ & $Rq = \frac{ady}{\sqrt{y^2 - a^2}}$. Porro ob similitudinem sectorum RCq & MCm, est Mm $(= dz) : Rq \left(= \frac{ady}{\sqrt{y^2 - a^2}}\right)$

: : CM (= R) : CR (= R − y); quare $\frac{aRdy}{\sqrt{y^2 - a^2}}$ = (R − y) dz, ex qua æquatione, data Curva AM adeoque relatione ipſarum R & z, ſi inveniatur y, hujus ope facile conſtrui poterit tractoria BP.

Cor. 2. Quum ob △ Mmn ∽ △ MRP ſit Mm: Mn : : MR : MP, adeoque $\frac{Mm}{MR} = \frac{Mn}{MP}$ = ang. MPn = (dem) ang. PRp, atque $\frac{Mm}{MC}$ = ang. MCm; ſequitur eſſe ang. PRp : ang. MCm : : MC : MR.

§. VII.

PROBLEMA. *Invenire Evolutam Tractoriæ ſimplicis.*

Si (Fig. 2.) BP fuerit Tractoria ſimplex, (pro qua igitur erit AM linea recta, adeoque MR & mr parallelæ), erunt AM & MR coordinatæ orthogonales Evolutæ DR. Retentis igitur iisdem ac in §. præced. denominationibus, ob Radium R infinitum adeoque R − y = R, erit (§. VI. Cor. 1.) $\frac{ady}{\sqrt{y^2 - a^2}}$ = dz, unde $\frac{C + z}{a} = \int \frac{dy}{\sqrt{y^2 - a^2}} = Log\, \frac{y + \sqrt{y^2 - a^2}}{a}$ ſeu a N $\frac{C + z}{a} = y + \sqrt{y^2 - a^2}$, deſignante N numerum, cujus Logarithmus hyperbolicus eſt = 1.

Cor. 1. Si curvæ hujus ad punctum R sit Radius osculi KR, erunt sectores RKr, PRp similes, quare KR : PM :: Rr : Mn & ob $\triangle$ Rrq $\backsim$ $\triangle$ MRP, PM : MR :: Rq ($=$ Mm) : Rr; ergo ex æquo KR : MR :: Mm : Mn. Quumque ob $\triangle$ Mnm $\backsim$ $\triangle$ MPR sit Mm : Mn :: MR : MP, erit KR : MR :: MR : MP, adeoque Radius osculi KR $= \frac{y^2}{a}$.

Cor. 2. Quadratura hujus curvæ facilis inventu est. Quum enim sit $dz = \frac{ady}{\sqrt{y^2 - a^2}}$, erit $\int y\,dz = \int \frac{aydy}{\sqrt{y^2 - a^2}} = a\sqrt{y^2 - a^2} + C' = 2\triangle \text{MPR} + C'$, denotante C' spatium quodvis constans.

§. VIII.

PROBLEMA. *Invenire Tractoriam Circuli.*

Sit (Fig. 2.) AM Circulus centro C radio CM $= b$ descriptus, cujus quæritur Tractoria BP, pro qua sit longitudo tangentis PM $= a$. Positis ut antea arcu AM $= z$ & MR $= y$, erit $dz = \frac{abdy}{(b - y)\sqrt{y^2 - a^2}}$ (§. VI. Cor. 1.), quæ formula, facto $y - \sqrt{y^2 - a^2} = v$ seu $y = \frac{a^2 + v^2}{2v}$, ad rationalitatem perducitur; hac enim substitutione obtinetur $dz = \frac{2abdv}{a^2 - 2bv + v^2}$,

cujus

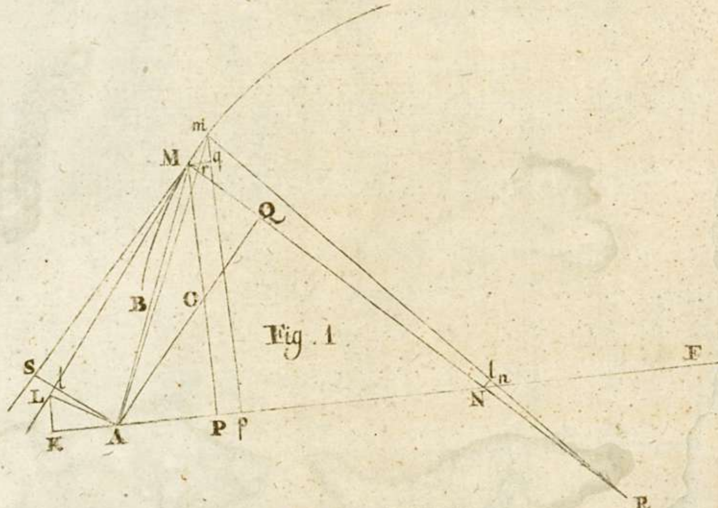

Fig. 1

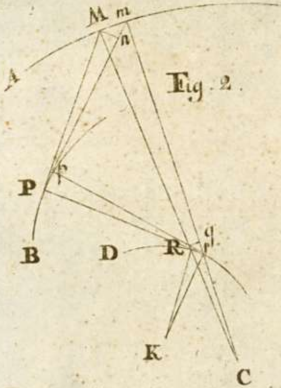

Fig. 2