DE MOTU PROJECTILIUM IN AËRE.

SPECIMEN MATHEMATICUM,

Conſ. Ampl. Fac. Phil. Aboënſ.

Auctore

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ABOÆ

Impreſſit Johan. Chriſtophor. Frenckell.

S:Æ R:Æ M:TIS

MAXIMÆ FIDEI VIRO,

GENERALI EXERCITUS EQUESTRIS

DUCI,

PRÆFECTO CLASSIS NAVALIS MINORIS,

NEC NON

EQUITI & COMMENDATORI

REGIORUM ORDINUM,

ILLUSTRISSIMO ATQUE GENEROSISSIMO

LIBERO BARONI,

D:NO AUGUSTINO

EHRENSVÄRD,

MÆCENATI SUMMO

SACRUM.

יהוה אדני הושעה

Quanti in Arte præcipue Pyrobolica sit momenti, veram nosse trajectoriam, quam globus oblique ad horizontem vi quavis data projectus, urgente gravitate uniformi percurrit, nemini hujus artis vel parum gnaro ignotum esse potest. Semota medii resistentia hanc curvam fore Parabolam conicam, suo jam ævo docuit GALILÆUS. Eandem vero in aëre, motui corporis resistente, a figura parabolica notabiliter satis aberrare necesse est. Utcunque enim parva sit aëris densitas, & hinc resistentia ejus singulo temporis momento insensibilis, infinita tamen, ut ita loquar, multitudo hujusmodi resistentiarum momentanearum non potest non, finito quovis tempore, sensibilem producere effectum. Hinc igitur commoti tam Physici, quam Mathematici multam in eo collocarunt operam, ut vim resistendi fluidorum detegerent atque motuum in iisdem factorum Symptomata determinarent. Hic vero quam maxime valet tritum illud: *Ardua, quæ pulcra.* Multa scilicet in hoc capite supersunt quibus etjam posthac Mathematici sua exerceant ingenia. Ne itaque ægre feras, L. B. quod in re tanti momenti publicæ luci committere audeamus leviuscula hæc nostra meletemata, ad usus practicos inprimis adcommodata, de semita globi projecti in aëre seu medio quovis uniformi, (& quidem quiescente), sub hypothesi illa vulgari, quod sit resistentia medii in ratione velocitatis duplicata.

§. I.

Sit ω altitudo, ex qua corpus, urgente vi gravitatis constante g, absque resistentia cadendo, eam acquirat velocitatem, quacum movetur in puncto M trajectoriæ AMB in medio resistente descriptæ & dicatur angulus CMP a linea ad horizontem verticali MP & ad curvam normali CM interceptus Φ, ipsius vero trajectoriæ pars AM, punctum M & datum A interjacens atque tempore t percursa, s ejusque fluxio ds, sitque in loco M medii resistentia $=$ R; erit generatim $\frac{R}{g} = -\frac{d\omega}{ds} - \text{Sin}\ \Phi$.

Demonst. Si enim vis gravitatis g, in directione MP agens, resolvatur in vires binas, quarum una directioni corporis MT contraria cum vi resistentiæ concurrit ad retardandum corporis motum, altera vero eidem directioni normalis ad corporis accelerationem vel retardationem nihil conducit; erit illa $=$ g Sin Φ, (sumto Radio $=$ I,) adeo-

que

que tota vis retardans $= R + g \operatorname{Sin} \phi$. Hæc igitur vis erit directe ut decrementum velocitatis & inverse ut momentum temporis (Princ. Mechan.). Unde, cum sit in puncto M velocitas $= \sqrt{2g\omega} = \frac{ds}{dt}$, erit $- \frac{gd\omega}{ds} = R + g \operatorname{Sin} \phi$ & $\frac{R}{g} = - \frac{d\omega}{ds} - \operatorname{Sin} \phi$. Q. E. D.

§. 2.

LEMMA. *In curva quavis AMB sibi semper parallela moveatur MP & sit CM $= r =$ Radio osculi ad punctum M, Ang. CMP $= \phi$ & ipsa curva AM puncta M & quodvis datum A interjacens $= s$; posito Sinu toto $= 1$, erit $r = \pm \frac{ds}{d\phi}$*, ubi signum $+$ valet, si crescente (l. decrescente) uno ipsorum s & ϕ etjam crescat (l. decrescat) alterum, signum vero $-$, si crescente uno decrescat alterum.

Patet hæc propositio ex iis, quæ de curvaturis traduntur in Elementis Methodi fluxionum passim.

§. 3.

LEMMA. *Iisdem positis ac in §. 1. sit C centrum circuli curvam AMB in M osculantis & radius osculi CM $= r$; ducta CD normali ad MP, erit $2\omega =$ CM $= r \operatorname{Cos} \phi$.*

Constat ex iis, quæ demonstrat MAC. LAURIN *Traité des fluxions* §. 419.

Coroll. Erit igitur $\frac{R}{g} = -\frac{\mathrm{Cos}\,\phi.dr}{2ds} - \frac{3}{2}\,\mathrm{Sin}\,\phi = \frac{\mathrm{Cos}\,\phi dr}{2rd\phi} - \frac{3}{2}\,\mathrm{Sin}\,\phi$; nam, ob $2\omega = r\,\mathrm{Cos}\,\phi$, est $d\omega = \frac{1}{2}\,\mathrm{Cos}\,\phi dr - \frac{1}{2}\,r\,\mathrm{Sin}\,\phi d\phi = (\S.\ 2.)\ \frac{1}{2}\,\mathrm{Cos}\,\phi dr + \frac{1}{2}\,\mathrm{Sin}\,\phi ds$, quo valore ipsius $d\omega$ substituto in æquatione: $\frac{R}{g} = -\frac{d\omega}{ds} - \mathrm{Sin}\,\phi$ (§. 1.), obtinetur: $\frac{R}{g} = -\frac{\mathrm{Cos}\,\phi dr}{2ds} - \frac{3}{2}\,\mathrm{Sin}\,\phi = \frac{\mathrm{Cos}\,\phi.dr}{2rd\phi} - \frac{3}{2}\,\mathrm{Sin}\,\phi$.

§. 4.

Si jam per hypothesin assumtam sit $\frac{R}{g} = \frac{\omega}{a}$ obtinetur (§. 3. *Cor.*) pro trajectoria, quam globus in aëre projectus describit, æquatio fluxionalis $\frac{dr}{rd\phi} = \frac{r}{a} + 3\,\mathrm{Tg}\,\phi = -\frac{dr}{ds}$, vel $\frac{dr}{r^2} = \frac{3\,\mathrm{Tg}\,\phi d\phi}{r} + \frac{d\phi}{a}$ ex qua, si ponatur $\frac{1}{r} = pq$, adeoque $p = \frac{1}{qr}$, & $-\frac{dr}{r^2} = pdq + qdp$, eruitur $q = \mathrm{Cos}\,\phi^3$, $-ap = A + \int \frac{d\phi}{\mathrm{Cos}\,\phi^3} = A + \frac{\mathrm{Sin}\,\phi}{2\mathrm{Cos}\,\phi^2} + \frac{1}{2}\,\mathrm{Log.}$

Tg

$\mathrm{Tg}(45^\circ + \tfrac{1}{2}\varphi)$, $-\frac{1}{r} = \frac{\mathrm{Cos}\varphi^3}{a} \cdot \left(A + \int \frac{d\varphi}{\mathrm{Cos}\varphi^3}\right)$

& $-r = \frac{a}{\mathrm{Cos}\varphi^3 \cdot \left(A + \int \frac{d\varphi}{\mathrm{Cos}\varphi^3}\right)}$. Si itaque AP in linea horizontali ſumta dicatur x & huic orthogonaliter inſiſtens PM y, erit $dx = -r\,\mathrm{Cos}\varphi d\varphi = \frac{a d\varphi}{\mathrm{Cos}\varphi^2 \cdot \left(A + \int \frac{d\varphi}{\mathrm{Cos}\varphi^3}\right)}$ & $dy = -r\,\mathrm{Sin}\,\varphi d\varphi = \frac{a\mathrm{Sin}\varphi d\varphi}{\mathrm{Cos}\varphi^3 \left(A + \int \frac{d\varphi}{\mathrm{Cos}\varphi^3}\right)}$.

Ex his vero æquationibus non niſi per quadraturas curvarum eruitur relatio inter x & y, unde exacta trajectoriæ AMB deſcriptio nimis perplexa evadit; & ſi vel maxime aliquod excogitari poſſet artificium, quo ex allatis æquationibus functiones ipſius anguli φ exterminentur & relatio inter x & y inveniatur, hanc tamen credimus tam intricatam fore, ut vix quidquam ad praxin adcommodatum exinde deduci queat. Præſtabit igitur loco hujus trajectoriæ adhibere curvam aliquam ſimpliciorem, quæ tam prope ad illam accedat, ut errores ex hac aſſumtione oriundi in praxi minimi fiant. Hyperbolæ, & quidem conicæ, huic ſcopo quam proxime ſatisfacientis conſtructiones ingenioſiſſimas tradit magnus ille NEWTON in *Princ. Math. Phil. Nat. L. II. prop. X.*

Quo-

Quomodo vero ex allata æquatione differentiali calculo analytico eruatur methodus generalis pro casu quovis dato determinandi hujusmodi hyperbolas in seqq. ostendemus.

Scholion. Formulæ allatæ, in quibus functiones anguli φ occurrunt generatim valent pro arcu tam adscendente, quam descendente, modo observetur, quod si pro illo angulus φ positivus ponatur, pro hoc negativus fiet, quia tum ad alteram partem ipsius MP cadit. Hoc in sequentibus etiam observatum volumus, quoties hujus anguli φ mentio fit.

§. 5.

Sit AMB Hyperbola conica, centro F asymptotis FG & FE descripta, cujus vertex K, potentia seu ipsius NK quadratum $= c^2$ *atque angulus asymptotorum GFE* $= \alpha$, *& sumta semper QD parallela asymptoto FE, dicatur pars curvæ AM, punctum datum A & dictam parallelam QD interjacens s, QM u, QF z, osculi in M radius CM r, angulus CMD* φ *& tangens MT* τ; *erit* $z = c\sqrt{(\operatorname{Cos}\varphi : \operatorname{Cos}\overline{\alpha+\varphi})}$, $u = c\sqrt{(\operatorname{Cos}\overline{\alpha-\varphi} : \operatorname{Cos}\varphi)}$, $\tau = c \operatorname{Sin}\alpha : \sqrt{(\operatorname{Cos}\varphi . \operatorname{Cos}\overline{\alpha+\varphi})}$, $r = \tau^3 : 2c^2 \operatorname{Sin}\alpha$ & $\frac{dr}{r\,d\varphi} = -\frac{dr}{ds} = \frac{1}{2}\operatorname{tg}\varphi + \frac{3}{2}\operatorname{tg}(\alpha+\varphi)$.

Demonst. Est enim per naturam hujus hyperbolæ $uz = c^2$, quare erit $-dz : du :: z : u :: z^2 : c^2 :: c^2 : u^2$ & $-dz : du :: \operatorname{Cos}\varphi : \operatorname{Cos}(\alpha+\varphi)$,

$(\alpha+\phi)$, adeoque $z^2 : c^2 :: \mathrm{Cos}\,\phi : \mathrm{Cos}\,(\alpha+\phi) :: c^2 : u^2$, unde singulæ formulæ allatæ facili calculo eruentur.

§. 6.

Ut itaque pateat, quo jure pro vera curva projectionis adhiberi possit hyperbola conica, cujus altera asymptotus FE horizonti ad angulos rectos insistit, videndum erit, qualis requiratur ratio resistentiæ ad gravitatem, si projectile in hac curva moveatur. In æquatione igitur $\frac{R}{g} = \frac{\mathrm{Cos}\,\phi\, dr}{2 r d\phi} - \frac{3}{2}\,\mathrm{Sin}\,\phi$ (§. 3. *Cor.*) pro $dr : rd\phi$ substituatur $\frac{3}{2}\,\mathrm{Tg}\,\phi + \frac{3}{2}\,\mathrm{Tg}\,(\alpha+\phi)$, quo facto habetur $\frac{R}{g} = \frac{3}{4}\,\mathrm{Cos}\,\phi.\,\mathrm{Tg}\,(\alpha+\phi) - \frac{3}{4}\,\mathrm{Sin}\,\phi = 3\,\mathrm{Sin}\,\alpha : 4\,\mathrm{Cos}\,(\alpha+\phi)$. Est autem (§. 5.) $\mathrm{Cos}\,(\alpha+\phi) = c^2\,\mathrm{Sin}\,\alpha^2 : \tau^2\,\mathrm{Cos}\,\phi = \tau\,\mathrm{Sin}\,\alpha : 2r\,\mathrm{Cos}\,\phi =$ (§. 3.) $\tau\,\mathrm{Sin}\,\alpha : 4\omega$, unde $R : g = 3\omega : \tau$; adeoque aberratio veræ trajectoriæ ab hyperbola pendet ex variatione tangentis τ. Quum vero ob densitatem aëris vix sensibilem sit τ satis magna, adeoque pro exigua curvæ parte, quæ in rebus practicis necessaria est, constans quam proxime haberi queat; patet hyperbolam hanc in praxi Artilleriæ loco veræ curvæ projectionis non incommode adhiberi posse. Si igitur ex loco A sub angulo elevationis LAB $= \beta$, velocitate $= \sqrt{2gy}$ (vel $= \sqrt{2y}$, assumta $g = 1$) projiciatur globus datus, pro quo sit ut antea index resistentiæ a; sum-

 ta

ta AL $= 3a$, erit (ob $\omega = \frac{\tau^2}{4\omega}$, §. §. 3. 5.) verticalis AH $= \frac{9a^2}{4\gamma} = \frac{1}{2}$ FL, unde determinantur centrum F & asymptoti FE, FG hyperbolæ, in qua idem globus movebitur. Exinde vero, quod AL sumatur $= 3a$, R : g quidem per medium fit aliquanto $> \omega : a$; errores autem hinc oriundos minoris momenti fore judicamus, præsertim cum in motu velociori rationem R : g paulo $> \omega : \alpha$ indicent phænomena.

§. 7.

Explicata sic brevissime natura curvæ, quæ in rebus practicis loco veræ trajectoriæ, a globo in aëre projecto descriptæ, absque notabili errore adhiberi poterit, superest, ut regulas quasdam hinc eruamus ad solvenda varia problemata in praxi Artilleriæ vulgo occurrentia maxime necessarias. Has vero brevitatis studio adferemus absque demonstrationibus, facillima Analysi a Lectore Mathematico detegendis. Si igitur sit (ut in §. 6.) index resistentiæ $= \alpha$, angulus projectionis (seu elevationis tormenti supra horizontem) LAB $= \beta$, projectionis velocitas $= \sqrt{2\gamma}$, atque amplitudo jactus horizontalis AB $= e$, amplitudo quævis alia AR $= s$ & ang. SAE $= \psi$, sequentes obtinebunt regulæ:

I. Datis α, γ & β invenietur amplitudo horizontalis $e =$

$$e = \frac{6a\gamma \operatorname{Sin} 2\beta}{3a + 4\gamma \operatorname{Sin}\beta}.$$

2. Generatim vero, datis iisdem & angulo ψ, erit amplitudo quævis

$$\varepsilon = \frac{12 a\gamma \operatorname{Cos}\beta \operatorname{Sin}\overline{\beta - \psi}}{\operatorname{Cos}\psi\,(3a \operatorname{Cos}\psi + 4\gamma \operatorname{Sin}\overline{\beta - \psi})}.$$

3. Datis a, β, ψ & jactus amplitudine, reperitur

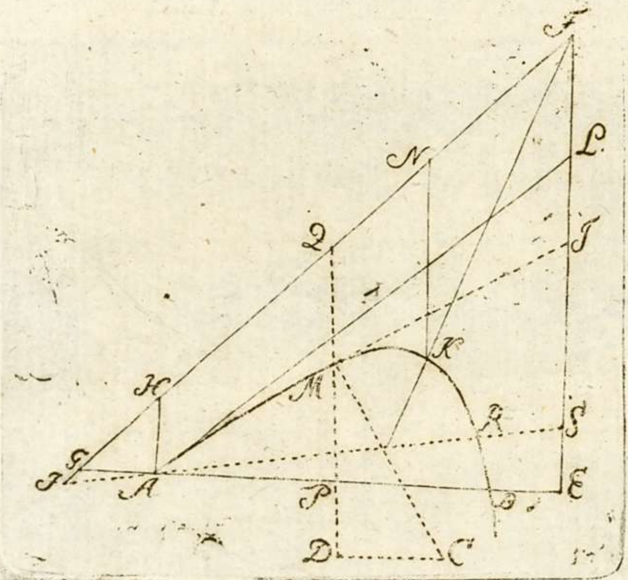
$$\gamma = \frac{3a\varepsilon \operatorname{Cos}\psi^2}{4\operatorname{Sin}\overline{\beta - \psi}\,(3a\operatorname{Cos}\beta - \varepsilon\operatorname{Cos}\psi)} = \frac{3ae}{4\operatorname{Sin}\beta\,(3a\operatorname{Cos}\beta - e)}.$$

Invento vero γ & cognito præterea spatio f ($= 16 \frac{31}{64}$ ped. Svec.), quod grave ex quiete libere cadendo tempore unius minuti secundi conficit, invenietur spatium velocitate constante $= \sqrt{2\gamma}$, tempore $= 1''$ emetiendum $= \sqrt{4fg}$, unde sic ipsa innotescit velocitas.

4. Si fuerit T tempus, quo corpus æquabiliter motum absolvit spatium $= 6a$ velocitate data $= \sqrt{2\gamma}$ & t tempus, quo globus ex loco A velocitate eadem projectus, emetitur arcum AMR erit

$$\frac{t}{T} = \frac{\sqrt{3a\operatorname{Cos}\beta} - \sqrt{(3a\operatorname{Cos}\beta - \varepsilon\operatorname{Cos}\psi)}}{\sqrt{(3a\operatorname{Cos}\beta - \varepsilon\operatorname{Cos}\psi)}}.$$

5. Hinc etiam deduci potest methodus determinandi quantitatem a per experimenta. Projiciantur scilicet duo globi ejusdem voluminis & gravitatis eadem velocitate sub diversis angulis β & B & observen-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ſerventur amplitudines horizontales, quæ ſint e & E reſpective, atque hinc eruatur quantitas a per formulam:

$$3a = \frac{2Ee(\text{Sin}\,B - \text{Sin}\,\beta)}{e\,\text{Sin}\,2B - E\,\text{Sin}\,2\beta}.$$

Et ut hæc quantitas a magis exacta obtineatur, ex pluribus diverſis experimentis colligatur hac methodo valor quidam ipſius medius. Notandum vero, hanc quantitatem a pro diverſa globorum magnitudine atque pondere variare.

6. Poſſet etjam ex formulis ſuperioribus deduci regula, per quam ex datis a, γ & amplitudine invenietur angulus projectionis; quum vero hæc regula, ex æquatione biquadratica eruenda, calculum reddat nimis moleſtum, in rebus practicis ſatius erit ad ſolutionem hujusmodi problematum adhibere artificium Mechanicum ſimpliciſſimum, quod tradit NEWTON in *Princ. Math. Ph. Nat. L.* 2. *Prop.* 10. *Reg.* 8.

Et has quidem regulas allatas ſufficere putamus ſolvendis problematibus fere omnibus in praxi Artilleriæ maxime neceſſariis. Volupe omnino nobis fuiſſet, exemplis quibusdam earundem cum phænomenis experiri convenientiam; Aſt quum deſit nobis copia experimentorum huic fini inſervientium, hæc jam mittere jubemur.

TANTUM.