DE

INTEGRATIONE FLUXIONUM

FORMÆ: $(\text{Sin}\, Z)^m . (\text{Cos}\, Z)^n dZ.$

SPECIMEN ACADEMICUM,

QUOD,

Conf. Ampl. Fac. Philof. Aboënf.

PRÆSIDE

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ABOÆ

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Dygdädla *Madame*

Madame ANNA LINDQVIST, Född FRENDEEN.

Min Huldesta Moder.

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Min Huldesta Mors

ödmjukl:ydigste Son,
JOH. HENR. LINDQVIST.

De Integratione Fluxionum

Formæ: $(\text{Sin}\,Z)^m\,(\text{Cos}\,Z)^n\,dZ$.

In Actis Reg. Acad. Scient. Holm. pro A. 1758. *p.* 194 seqq. exhibuit Cel. Dn. MALLET quatuor formulas pro integrandis fluxionibus formæ: $\overline{\text{Sin}\,Z.}^m\,\overline{\text{Cos}\,Z.}^n\,dZ$ vel $s^m\,x^n\,dZ$, denotante *s* Sinum atque *x* Cosinum anguli seu arcus Circularis, cujus radius $= 1$. Postea generali atque unica Analysi Dn. PRÆSES tum eadem Theoremata tum bina alia elicuit, quæ quidem in casibus quibusdam non contemnendum calculi compendium præstant. Hæc cum mihi benigne communicaret, operæ pretium duxi, aliquantulum hac in re desudare, brevissimamque ejus expositionem, Speciminis Academici loco, publicæ committere luci.

§. 1. Nihil difficultatis habet integratio hujusmodi formarum differentialium, quoties fuerit *m* vel $n = +1$; est enim, (neglectis ubique constantibus), $\int s x^n dZ = -\frac{x^{n+1}}{n+1}$ & $\int s^m x dZ = \frac{s^{m+1}}{m+1}$. Quinimo cum sit generatim $s^m x^n dZ = -x^n.\overline{1-xx}^{(m-1):2}.dx = s^m.\overline{1-ss}^{(n-1):2}ds$: sequitur, quoties *m* (vel *n*)

est

est numerus positivus impar, posse, resolvendo $\overline{1-xx}^{(m-1):2}$ (vel $\overline{1-ss}^{(n-1):2}$) in seriem per theorema binomiale Newtonianum, $s^m x^n dZ$ exhiberi finito terminorum numero $= m+1:2$ (vel $n+1:2$), quorum singuli, utpote formæ $y^e dy$, facillime integrantur.

§. 2. Ut autem aliis etiam casibus convenientes investigentur integrandi formulæ, (nisi forte omnibus in universum æque sufficiens aliqua dari possit), & inprimis inquiratur, annon integrale quærendum membratim seu per partes ita reperiri queat, ut quæ, inuento forte integrali aliquo partiali, dein adhuc integranda supersit fluxio, cum proposita ejusdem generis seu formæ ac proinde pariter resolubilis, simplicior tamen, sit; atque sic, uniformi tenore seu constante quadam lege opus urgendo, tandem deueniatur ad integrandam fluxionem in suo genere simplicissimam: ponatur partiale illud integrale esse algebraicum formæ $as^k x^l$. Sumendo itaque α) ipsius $s^k x^l$ fluxionem & β) substituendo in ea $1-ss$ pro xx, vel γ) $1-xx$ pro ss, obtinetur $d(s^k x^l) = (\alpha)\ dZ.\, s^{k-1} x^{l-1}(kx^2 - ls^2) = (\beta)$ $dZ.(ks^{k-1} x^{l-1} - \overline{k+l}.\, s^{k+1} x^{l-1}) = (\gamma.)\ dZ.$ $(\overline{k+l}.\, s^{k-1} x^{l+1} - ls^{k-1} x^{l-1})$. Statuantur jam successive in valore α, $k \pm 1 = m$, $l \mp 1 = n$; in β, $k \pm 1 = m$, $l - 1 = n$; in γ, $k - 1 = m$, $l \pm 1 = n$: tum

inte-

integrando & reducendo prodibunt formulæ seu Theoremata sex sequentia:

$$\text{A)}\ \int s^m x^n dZ = -\frac{s^{m-1}x^{n+1}}{n+1} + \frac{m-1}{n+1}\int s^{m-2}x^{n+2} dZ.$$

$$\mathfrak{A})\ \int s^m x^n dZ = \frac{s^{m+1}x^{n-1}}{m+1} + \frac{n-1}{m+1}\int s^{m+2}x^{n-2} dZ.$$

$$\text{B)}.\ \int s^m x^n dZ = -\frac{s^{m-1}x^{n+1}}{m+n} + \frac{m-1}{m+n}\int s^{m-2}x^n dZ.$$

$$\mathfrak{B})\ \int s^m x^n dZ = \frac{s^{m+1}x^{n+1}}{m+1} + \frac{m+n+2}{m+1}\int s^{m+2}x^n dZ.$$

$$\text{C)}\ \int s^m x^n dZ = \frac{s^{m+1}x^{n-1}}{m+n} + \frac{n-1}{m+n}\int s^m x^{n-2} dZ.$$

$$\mathfrak{C}).\ \int s^m x^n dZ = -\frac{s^{m+1}x^{n+1}}{n+1} + \frac{m+n+2}{n+1}\int s^m x^{n+2} dZ.$$

Harum formularum, quas sic quidem præbuit allata Analysis generalis, posteriores quatuor sunt ipsa illa Theoremata D:NI MALLET, quorum ab initio memini.

§. 3. Potest quælibet harum formularum converti in seriem; cujus autem singuli termini commodissime invenientur, & præterea, quibus casibus apta sit vel minus, optime judicari poterit, exhibito prius termino seriei generali. Et quidem attenta ta formulæ inspectio facile dabit terminum generalem summatorium (h. e. signo integrationis $\int$ affectum), qui, ad ductum ipsius formulæ resolvatur in bina sua membra, algebraicum atque summatorium. Hæc junctim sumta demum exponent formam seriei generalem. Hoc, inquam, modo obtinetur terminus seriei generalis

Algebraicus, ordine $\overline{r+1}$:*us*. Summatorius.

ex A)... $-\dfrac{\overline{m-1}.\overline{m-3}\dots\overline{m-2r+1}}{\overline{n+1}.\overline{n+3}\dots\overline{n+2r+1}}.s^{m-2r-1}x^{n+2r+1}$

$+Q\,\overline{m-2r-1}.\int s^{m-2r-2}x^{n+2r+2}dZ$

𝔄)... $+\dfrac{\overline{n-1}.\overline{n-3}\dots\overline{n-2r+1}}{\overline{m+1}.\overline{m+3}\dots\overline{m+2r+1}}.s^{m+2r+1}x^{n-2r-1}$

$+Q\,\overline{n-2r-1}.\int s^{m+2r+2}x^{n-2r-2}dZ$

B)... $-\dfrac{\overline{m-1}.\overline{m-3}\dots\overline{m-2r+1}}{\overline{m+n}.\overline{m+n-2}\dots\overline{m+n-2r}}.s^{m-2r-1}x^{n+1}$

$+Q\,\overline{m-2r-1}.\int s^{m-2r-2}x^{n}dZ.$

𝔅)... $+\dfrac{\overline{m+n+2}.\overline{m+n+4}\dots\overline{m+n+2r}}{\overline{m+1}.\overline{m+3}\dots\overline{m+2r+1}}.s^{m+2r+1}x^{n+1}$

$+Q\,\overline{m+n+2r+2}.\int s^{m+2r+2}x^{n}dZ.$

C)... $+\dfrac{\overline{n-1}.\overline{n-3}\dots\overline{n-2r+1}}{\overline{m+n}.\overline{m+n-2}\dots\overline{m+n-2r}}.s^{m+1}x^{n-2r-1}$

$+Q.\,\overline{n-2r-1}\int s^{m}x^{n-2r-2}dZ.$

ℭ)...

$$\text{C)} \ldots - \frac{\overline{m+n+2}.\overline{m+n+4}\ldots\overline{m+n+2r}}{\overline{n+1}.\overline{n+3}\ldots\overline{n+2r+1}}. s^{m+1} x^{n+2r+1}$$

$$+ Q.\overline{m+n+2r+2}.\int s^m x^{n+2r+2} dZ.$$

& substituendo pro r successive 1, 2, 3,... 9, &c. prodit terminus seriei Algebraicus 2:dus, 3:ius, 4:tus, ... 10.mus &c. nec non residuus summatorius, quem ingrediens Q coëfficientem notat termini illius Algebraici, ubique tamen positive accipiendum.

§. 4. Quamvis generales sint istæ formulæ (§. 2.) eatenus, ut recte se habeant, sive positivi sive negativi fuerint numeri m, n, sive alter positivus, alter negativus: non tamen existimandum, singulas singulis casibus æque convenire nedum sufficere. Sic v. g. obtinetur quidem $\int s^4 x^3 dZ = \frac{s^5 x^2}{5} + \frac{2s^7}{35}$ per Theor. A, & $= \frac{s^5 x^2}{7} + \frac{2s^5}{35}$ per Theor. C; reliqua vero Theoremata, saltem solitarie adhibita, hoc in casu omnino sunt inepta. Plurimis autem in casibus, binis simul utendum est formulis, ut integrale quæsitum ad formam simplicissimam reducatur, quapropter, præcipue si paucissimis, quam fieri possit, terminis id præstare velis, delectus Theorematum adhibendus erit; quod unico probasse sufficiat exemplo. Valor fluentis $\int \frac{s^8 dZ}{x^9}$ triplici inprimis inis ratione obtinetur; scil. quia $m = 8$ & $n = -9$, 1:o ex C prodit $\int \frac{s^8 dZ}{x^9} = \frac{s^9}{8x^8} - \frac{s^9}{48x^6} + \frac{s^9}{64x^4} - \frac{5s^9}{128x^2} + \frac{35}{128} \cdot \int \frac{s^8 dZ}{x}$, adeoque substituto ex Theor. B. pro $\int \frac{s^8 dZ}{x}$ ejus valore $= -\frac{s^7}{7} - \frac{s^5}{5} - \frac{s^3}{3} - s + \int \frac{dZ}{x}$, obtinetur valor ipsius $\int \frac{s^8 dZ}{x^9}$ & quidem 9 terminis expressus. 2:o Inverse fit per B, $\int \frac{s^8 dZ}{x^9} = \frac{s^7}{x^8} - \frac{7}{3} \cdot \frac{s^5}{x^8} + \frac{7}{3} \cdot \frac{s^3}{x^8} - \frac{s}{x^8} + \int \frac{dZ}{x^9}$ & per C fit $\int \frac{dZ}{x^9} = \frac{s}{8x^8} + \frac{7}{48} \cdot \frac{s}{x^6} + \frac{35}{192} \cdot \frac{s}{x^4} + \frac{35}{128} \cdot \frac{s}{x^2} + \frac{35}{128} \cdot \int \frac{dZ}{x}$, unde itidem 9 terminis exprimitur totus valor ipsius $\int \frac{s^8 dZ}{x^9}$. Sed 3:o A uno negotio & solummodo 5 terminis dat quæsitum integrale, scil. $\int \frac{s^8 dZ}{x^9} = \frac{s^7}{8x^8} - \frac{7}{48} \cdot \frac{s^5}{x^6} + \frac{35}{192} \cdot \frac{s^3}{x^4} - \frac{35}{128} \cdot \frac{s}{x^2} + \frac{35}{128} \cdot \int \frac{dZ}{x}$. Postrema igitur methodus hoc in casu cæteris merito præferenda est.

§. 5. Antequam vero regulas istas adferam, quæ in casu quovis speciali de commodissima horum Theorematum (§. 2.) & inde eruendarum se-

serierum (§. 3.) adplicatione observandæ sunt, sequentes præmittere placebit generales animadversiones circa indolem harum serierum:

α). Coëfficiens termini cujusvis ingreditur, ceu factor, terminum proxime sequentem, adeoque secuturos omnes.

β) Evanescente igitur termino quodam, abrumpitur series, quo ipso dat integrale absolutum seu Algebraicum.

γ.) Si non abrumpitur series, sistenda erit, quando reliquus terminus summatorius factus fuerit simplicissimus, qui fieri potest, & antequam evanescat forte factor aliquis denominatoris, adeoque terminus infinitus incurrat.

δ) Inepta autem censenda & omnino non adhibenda est series, quoties nec abrumpitur nec ad fluxionem ducit proposita simpliciorem. Ex his principiis facile deduci possunt pro casibus specialibus regulæ jam adferendæ.

§. 6. Primo igitur illos considerabo casus, in quibus uterque indicum *m*, *n* est numerus integer. Quocirca sequentes observentur canones, denotantibus *r*, *q*, *k* numeros quoscunque integros positivos:

Casus I. Si fuerit m (vel n) $= 2r + 1$, & simul 1:o n (vel m) aut $= 2q + 1$, aut $= \pm 2q$: obtinetur fluentis quæsitæ valor absolutus per A & B numero terminorum $= \frac{m+1}{2}$, (vel per 𝔄 & 𝔈, numero terminorum $= \frac{n+1}{2}$). 2:o Si n (vel m) $= -2q - 1$, & quidem α) $m + n = 0$, reducitur integrale quæsitum per A (vel 𝔄) ad $\int \frac{s dZ}{x}$ (vel ad $\int \frac{x dZ}{s}$). β) Si $m + n = \pm 2k$, primo per A (vel 𝔄) reducendum est integrale ad $\int \frac{s^{m+n-1} dZ}{x}$ (vel ad $\int \frac{x^{m+n-1} dZ}{s}$ & hoc ulterius per B (vel C) ad $\int \frac{s dZ}{x}$ (vel $\int \frac{x dZ}{s}$). γ.) Si $m + n = -2k$, obtinetur integrale absolutum commodissime quidem per B (vel C) atque, si $k > 1$, etiam per A (vel 𝔄); quando vero fuerit $k < \frac{m+1}{2}$ (vel $< \frac{n+1}{2}$), aptius id fiet per 𝔅 & 𝔈.

Casus II. Si fuerit m (vel n) $= -2r - 1$, & simul 1:o n (vel m) $= -2q - 1$: continuata serie 𝔅 (vel 𝔈), donec deveniatur ad $\frac{\int x^n dZ}{s}$

(vel $\int \frac{s^m dZ}{x}$), residuum hoc integrale ulterius per 𝔈 (vel 𝔅) reducatur ad $\int \frac{dZ}{sx}$. Per easdem series eademque fere methodo, quando fuerit 2:o n (vel m) $= -2q$, integrale quæsitum reducitur ad $\int \frac{dZ}{s}$ (vel $\int \frac{dZ}{x}$). Sed si 3:o n (vel m) $= 2q$ & quidem α) $m + n = 2k + 1$, adhibenda est series 𝔄 (vel A), donec restet $\int s^{m+n} dZ$ (vel $\int x^{m+n} dZ$) quod per B (vel C) tandem reducitur ad $\int \frac{dZ}{s}$ (vel $\int \frac{dZ}{x}$). Vel etiam producatur series 𝔄 (vel A), usque dum supersit $\int \frac{x^{m+n+1} dZ}{s}$ (vel $\int \frac{s^{m+n+1} dZ}{x}$), quod deinde pariter ad C (vel B) adplicandum erit. Prior methodus eligenda est, quoties $r > q$, posterior contra. β) Si $m + n = -2k - 1$, idem valet canon, ac pro $m + n = 2k + 1$, modo adhibeantur Theor. 𝔅 & 𝔈 respective pro B & C.

Casus III. Quando est m (vel n) $= 2r$ & simul 1:o n (vel m) $= 2q$, reduci potest integrale quæsitum ad $\int dZ$ adhibitis successive seriebus B & C. Si vero 2:o n (vel m) $= -2q$, etiam ad $\int dZ$ reducitur integrale, & quidem α) ope solius seriei A (vel 𝔄), quoties $m + n = 0$; sed β) si $m + n = \pm 2k$, methodo fere eadem utendum est ac pro Casu II. 3. α. β.

Casus IV. Si uterque tam m quam n fuerit numerus negativus par, ad $\int dZ$ reducitur integrale per 𝔅 & ℭ successive.

§. 7. Ad casus autem, in quibus alteruter indicum m & n fuerit non integer, generaliter quidem adplicari nequeunt allata Theoremata. Aliquando tamen etiam in his casibus adhiberi commode possunt: sic 1:o si est m (vel n) numerus qualiscunque non integer, sed n (vel m) fuerit $= 2r+1$, absolute integrabilis est fluxio $s^m x^n dZ$ per Theor A & ℭ, (vel 𝔄 & 𝔅). 2:o Si neuter ipsorum m & n fuerit integer, sed tamen $m+n = -2r$, tum etiam absolute integrari potest $s^m x^n dZ$ per Theor 𝔅 & ℭ Hæc vero specialioribus illustrare exemplis, supervacaneum erit.

§. 8. Per regulas, quas (§. 6.) exhibui, integrale aut absolutum obtinetur, aut reducitur vel ad $\int dZ = Z$, vel ad simplicissima integralia logarithmica. Horum vero ulterior analysis non est hujus loci. Sufficiat igitur solummodo nominasse, quod sit: $\int \frac{dZ}{s} = \text{Log. Tang.} \tfrac{1}{2} Z$, $\int \frac{dZ}{x} = \text{L. Tang.}$ $(45^\circ + \tfrac{1}{2} Z)$, $\int \frac{s dZ}{x} = -\text{L}\, x = \text{L.Sec.} Z$, $\int \frac{x dZ}{s} = \text{L} s$, & $\int \frac{dZ}{sx} = \text{L. Tang } Z$.

§. 9.

§. 9. Quando m & n sunt numeri positivi integri, etiam alia ratione, quæ ab allatis sex Theorematibus haud pendet, peragi potest integratio, saltem si m & n sint determinati. Nimirum inde quod Sin A. Sin B $= \frac{1}{2}$ Cos(A − B) − $\frac{1}{2}$ Cos (A + B), Cos A. Cos B $= \frac{1}{2}$ Cos (A + B) + $\frac{1}{2}$ Cos (A − B), nec non Sin A. Cos B $= \frac{1}{2}$ Sin (A + B) + $\frac{1}{2}$ Sin (A − B): reperiuntur primum s^m & x^n pro quolibet determinato m & n (*cfr.* EULER. *introd. in Anal. infin. Tom.* 1. §§. 262. 263.) & quidem erit generatim: $x^n = \frac{1}{2^n} \cdot$ [Cos $nZ + \frac{n}{1}$ Cos $\overline{n-2}Z + \frac{n}{1} \cdot \frac{n-1}{2}$ Cos $\overline{n-4}Z + \frac{n}{1} \cdot \frac{n-1}{2} \cdot \frac{n-2}{3} \cdot$ Cos $\overline{n-6}Z + \ldots + \left(\frac{n}{1} \cdot \frac{n-1}{2} \cdot \frac{n-2}{3} \ldots \frac{n-r+1}{r}\right)$ Cos $\overline{n-2r}Z + \ldots$] &, si fuerit m numerus impar, $s^m = \frac{1}{2^m} \cdot$ [$\pm$ Sin $mZ \mp \frac{m}{1}$ Sin $\overline{m-2}Z \pm \frac{m}{1} \cdot \frac{m-1}{2}$ Sin $\overline{m-4}Z \mp \ldots \pm \left(\frac{m}{1} \cdot \frac{m-1}{2} \cdot \frac{m-2}{3} \ldots \frac{m-r+1}{r}\right)$ Sin $\overline{m-2r}Z \mp \ldots$], sed, pro m pari, $s^m = \frac{1}{2^m} \cdot$ [$\mp$ Cos $mZ \pm \frac{m}{1}$ Cos $\overline{m-2}Z \mp \frac{m}{1} \cdot \frac{m-1}{2}$ Cos $\overline{m-4}Z \pm \ldots$

 $\mp$

$+\left(\frac{m}{1}\cdot\frac{m-1}{2}\cdot\frac{m-2}{3}\cdots\frac{m-r+1}{r}\right)\text{Cos}\,\overline{m-2rZ}\pm\ldots]$ (*), in quibus binis postremis formulis signa superiora valent pro m formæ $4q+1$ & $4q+2$, sed inferiora pro m formæ $4q-1$ & $4q$. Hinc facile prodeunt $\int s^m dZ$ & $\int x^n dZ$. Porro per easdem allatas formulas Trigonometricas, pro determinatis saltim m & n, similiter exprimere licet $s^m x^n$ Sinubus vel Cosinubus, simpliciter positis nec in se mutuo ductis, multiplorum ipsius Z, quo facto facile reperitur $\int s^m x^n dZ$. Sic e: g: fit $s^4 x^3 = \frac{1}{64}(\text{Cos}\,7Z - \text{Cos}\,5Z - 3\,\text{Cos}\,3Z + 3\,\text{Cos}\,Z)$ adeoque $\int s^4 x^3 dZ = \frac{1}{2240}(5\,\text{Sin}\,7Z - 7\,\text{Sin}\,5Z - 35\,\text{Sin}\,3Z + 105\,\text{Sin}\,Z)$. Ceterum ex indole & circumstantiis cujusque Problematis, quod integratio hujusmodi fluxionum ingreditur, judicandum erit, an & quatenus postrema hac methodo potius, quam prius tradita, uti præstet.

(*) Ob n (& m) integrum hæ series abrumpuntur facto terminorum numero $= n+1$ (vel $= m+1$). Possunt autem eædem quoque sisti, facto terminorum numero $= \frac{n+1}{2}$ (vel $= \frac{m+1}{2}$) pro n (vel m) impari, & $= \frac{1}{2}n+1$ (vel $= \frac{1}{2}m+1$) pro n (vel m) pari, modo pro 2^n (vel 2^m) sumatur 2^{n-1} (vel 2^{m-1}) respective.