25

Dissertatio Mathematica

De

Methodo,

Ex mensuratis duobus Ellipseos arcubus, axes ejus inveniendi,

Quam

Cons. Ampl. Fac. Phil. in Regia Acad. Aboënsi,

Præside

Mag. JOH. HENR. LINDQUIST,

Math. Prof. R. & O. atque R. Acad. Sc. Svec. Membro,

Pro Gradu

Publico Examini submittit

GUSTAVUS GABRIEL HÅLLSTRÖM,

Alumn. Reg. Ostrobotniensis,

In Auditorio Mathem. die XII Junii MDCCXCV,

Horis a. m. solitis.

ABOÆ, TYPIS FRENCKELLIANIS.

§. I.

Notissimum est, quæ ex pluribus institutis observationibus atque mensuris pro determinanda figura & magnitudine telluris deducuntur conclusiones, cum hypothesi, qua assumitur hæc figura ellipsoidica, non satis convenire; quamobrem veritatem hujus hypotheseos in dubium haud pauci vocaverint. Quum tamen eadem hæc hypothesis cum legibus æquilibrii optime conspiret, & de cetero simplicitate sua atque elegantia haud parum sese commendet; istam ob minores aberrationes mox rejiciendam non esse putamus, verum potius mensuras ipsas & observationes ulterius examinandas, immo repetendas, nec non methodos, quibus conclusiones ex his eruuntur, accuratiori censuræ subjiciendas arbitramur. Hanc etiam ob rationem, speciminis Academici loco, brevem exhibere constituimus disquisitionem methodi, qua ex mensuratis duobus arcubus meridiani elliptici investigantur hujus ellipseos axes. Mensuratis scilicet longitudinibus m & μ duorum arcuum meridiani, quorum ille inter Latitudines $L + z$ & $L - z$, hic inter Latitudines $\lambda + \zeta$ & $\lambda - \zeta$ interceptus observatur, adeo ut præter

ho-

horum arcuum longitudines m & μ, cognitæ ſint eorundem amplitudines $2z$ & 2ζ, nec non latitudines mediæ L & λ; quæſtio eo redit, ut ex his datis (poſita figura telluris ellipſoidica compreſſa) inveniatur ſemidiameter æquatoris, quæ dicatur a, & ſemiaxis telluris, qui ſit $= a\sqrt{1-c}$. Hoc problema communiter ita reſolvitur, ut ex data longitudine atque amplitudine utriusque arcus per regulam trium primo quærantur longitudines unius gradus G & g pro latitudinibus iſtis mediis L & λ, inferendo: $2z : 1^{\circ} :: m : G$ & $2\zeta : 1^{\circ} :: \mu : g$; vel quod eodem recidit, (deſignante N arcum circuli radio æqualem) inveniantur pro iisdem latitudinibus radii curvaturæ meridiani R & ϱ, colligendo per eandem regulam: $2z : N :: m : R$ & $2\zeta : N :: \mu : \varrho$; quo facto ex cognitis G, g (vel R, ϱ) atque L, λ inveſtigantur c & a. Quoties minores ſunt $2z$ & 2ζ, hæc quidem ratio computandi ſatis exacta cenſeri poteſt. Quum vero in dimetiendis amplitudinibus arcuum, aliquot ſcrupulorum ſecundorum error vix evitari poſſit, hisque erroribus eo magis afficiantur concluſiones, quo ipſi arcus ſint breviores, manifeſtum eſt, uſum tam exiguorum arcuum in hoc problemate admitti non poſſe. Quaudo autem majores adſumuntur hi arcus, ob inæqualem variationem curvaturæ in ellipſibus non amplius ſine erroris periculo per ſimplicem illam regulam proportionum invenietur utriusque curvatura pro latitudine media. Diſpiciendum igitur erit, quomodo ex menſuratis arcubus utcunque magnis computari possint dimensiones Ellipseos, & quantus pro data magnitudine utriusque arcus, sit error methodi vuglaris.

§. II.

In antecessum adferre juvat methodum, qua ex datis radiis curvaturae meridiani R & ϱ pro Latitudinibus L & λ respective, inveniatur semidiameter aequatoris $= a$ & semiaxis terrae $= a\sqrt{1-c}$. Ex iis quae de radiis osculi Ellipseos traduntur in Doctrina Sectionum Conic. facile demonstrari potest, fore

$$R = \frac{(1-c)a}{(1-c\ Sin\ L^2)^{\frac{3}{2}}}\ \&\ \varrho = \frac{(1-c)a}{(1-c\ Sin\ \lambda^2)^{\frac{3}{2}}},$$ posito

Sinu Toto $= 1$; quibus formulis comparatis, ponendo $\sqrt[3]{\frac{R}{\varrho}} = Tg\ \gamma$, adeoque $R\ Cos\ \gamma^3 = \varrho\ Sin\ \gamma^3$, obtinetur $Tg\ \gamma^2 = \frac{1-c\ Sin\ \lambda^2}{1-c\ Sin\ L^2}$, & hinc facta reductione

$$c = \frac{Cos\ \gamma^2 - Sin\ \gamma^2}{Cos\ \gamma^2\ Sin\ \lambda^2 - Sin\ \gamma^2\ Sin\ L^2}.$$

Hoc vero valore ipsius c in alterutra formularum superiorum substituto, eruitur

$$a = \frac{\varrho\ Sin\ \gamma^3\ (Sin\ \lambda^2 - Sin\ L^2)^{\frac{3}{2}}}{(Sin\gamma^2 Cos L^2 - Cos\gamma^2 Cos\lambda^2)\sqrt{Cos\gamma^2 Sin\lambda^2 - Sin\gamma^2 Sin L^2}},$$

$$\&\ a\sqrt{1-c} = \frac{\varrho\ Sin\ \gamma^3\ (Sin\ \lambda^2 - Sin\ L^2)^{\frac{3}{2}}}{(Cos\gamma^2 Sin\lambda^2 - Sin\gamma^2 Sin L^2)\sqrt{Sin\gamma^2\ Cos L^2 - Cos\gamma^2\ Cos\lambda^2}}$$

Ad has vero formulas calculo logarithmico adaptandas, sumto $L < \lambda$, sequentes adhiberi possunt substitutiones:

$$\frac{Tg\,\gamma\,Sin\,L}{Sin\,\lambda} = Sin\,\phi, \;\&\; \frac{Cotg\,\gamma\,Cos\,\lambda}{Cos\,L} = Sin\,\psi;$$

quarum scilicet ope, debita reductione obtinetur

$$c = \frac{Cos\,2\,\gamma}{Cos\,\gamma^2\,Sin\,\lambda^2\,Cos\,\phi^2};\; a = \frac{\varrho\,Tg\,\gamma\,Sin\,(\lambda+L)^{\frac{3}{2}}\,Sin\,(\lambda-L)^{\frac{1}{2}}}{Sin\,\lambda\,Cos\,L^2\,Cos\,\phi\,Cos\,\psi^2};$$

$$\&\; a\sqrt{1-c} = \frac{\varrho\,Tg\,\gamma^2\,Sin\,(\lambda+L)^{\frac{1}{2}}\,Sin\,(\lambda-L)^{\frac{1}{2}}}{Sin\,\lambda^2\,Cos\,L\,Cos\,\phi^2\,Cos\,\psi}.$$

Cor. 1. Quum sit (§. 1.) $N : 1° :: \varrho : g :: R : G$; facile patet, quomodo ex datis longitudinibus unius gradus G & g pro latitudinibus mediis L & λ, per easdem formulas inveniantur a & c.

Cor. 2. Si manentibus L & λ varientur R & ϱ, ita tamen ut admodum parvæ sint eorum variationes, atque da, dc, dR & $d\varrho$ designent augmenta simultanea ipsarum quantitatum a, c, R & ϱ respective; erit

$$dc = \frac{2}{3}\left(\frac{d\varrho}{\varrho} - \frac{dR}{R}\right)\frac{Tg\,\gamma^2\,Sin\,(\lambda+L)\,Sin\,(\lambda-L)}{Sin\,\lambda^4\,Cos\,\phi^4} \;\&$$

$$\frac{da}{a} = \frac{dR}{R\,Cos^2} - \frac{d\varrho}{\varrho}\,Tg\,\phi^2 + \frac{2}{3}\left(\frac{d\varrho}{\varrho} - \frac{dR}{\varrho}\right)\frac{Sin(\lambda+L)Sin(\lambda-L)}{Sin\,\lambda^2 Cos\,L^2\,Cos\,\phi^2 Cos\,\psi^2}.$$

Existente igitur (§. 1.) $R = \frac{Nm}{2z}$ & $\varrho = \frac{N\mu}{2\zeta}$, si manentibus z & ζ varientur m & μ, sintque horum aug-

 men-

menta dm & $d\mu$ respective, erit $\frac{dR}{R} = \frac{dm}{m}$ & $\frac{d\varrho}{\varrho} = \frac{d\mu}{\mu}$, quos valores pro $\frac{dR}{R}$ & $\frac{d\varrho}{\varrho}$ in allatis formulis substituendo, obtinentur regulae pro supputandis variationibus ipsorum c & a ex datis vel suppositis erroribus ipsorum m & μ.

§. III.

Quando majores sunt arcus illi mensurati, quam ut sine errore assumi poterit $R = \frac{Nm}{2z}$ & $\varrho = \frac{N\mu}{2\zeta}$, si exacti desiderentur valores ipsorum a & c, ad rectificationem Ellipseos recurrendum erit. Si igitur generatim sit arcus meridiani elliptici, inter aequatorem & latitudinem quamvis $= v$ intercepti, longitudo $= s$, posita ut prius semidiametro aequatoris $= a$ & semiaxe telluris $= a\sqrt{1-c}$, erit ad latitudinem v radius curvaturae meridiani

$$= \frac{(1-c)\,a}{(1-c\ Sin\ v^2)^{\frac{3}{2}}} = \frac{ds}{dv};\ \text{adeoque}\ ds = \frac{(1-c)\,a\,dv}{(1-c\ Sin\,v^2)^{\frac{3}{2}}}.$$

Haec formula secundum Theorema binomiale Newtonianum in seriem evoluta dat:

$$\frac{ds}{(1-c)\,a} = dv\left[1 + \tfrac{3}{2}\,c\ Sin\ v^2 + \frac{3.\,5}{2.\,4}.\ c^2\ Sin\ v^4 + \frac{3.\,5.\,7}{2.\,4.\,6}c^3\ Sin\,v^6 + \ldots + \frac{3.\,5.\,7\ldots(2n+1)}{2.\,4.\,6\ldots 2n}c^n v^{2n} + \ldots\right].$$

In

In qua ſerie porro evolvantur ſinguli termini ope ſequentis formulæ:

$$Sin\, v^{2n} = \frac{1.3.5.\ldots(2n-1)}{2.4.6.\ldots 2n}\Big[1 - \frac{2n}{n+1} Cof\, 2v + \frac{2n(n-1)}{(n+1)(n+2)} Cof\, 4v - \frac{2n(n-1)(n-2)}{(n+1)(n+2)(n+3)} Cof\, 6v + \ldots \pm \frac{2n(n-1)(n-2)\ldots(n-i+1)}{(n+1)(n+2)(n+3)\ldots(n+i)} Cof\, 2iv \mp \ldots\Big];$$

cujus veritas ex iis, quæ demonſtrat Cel. EULER *Introd. in Anal. infin. Tom. I* §. 262 & *Inſtit. Calc. integr. Tom. 1.* §. 272, facile evincitur. Hoc vero facto, obtinetur

$$\frac{ds}{(1-c)a} = dv\Big[1 + \tfrac{3}{2}.\tfrac{1}{2}\, c\,(1 - Cof\, 2v) + \frac{3.5}{2.4}.\frac{1.3}{2.4}.c^2\Big(1 - \frac{2.2}{3} Cof\, 2v + \frac{2.2.1}{3.4} Cof\, 4v\Big) + \frac{3.5.7}{2.4.6}.\frac{1.3.5}{2.4.6}c^3\Big(1 - \frac{2.3}{4} Cof\, 2v + \frac{2.3.2}{4.5} Cof\, 4v - \frac{2.3.2.1}{4.5.6} Cof\, 6v\Big) + \frac{3.5.7.9}{2.4.6.8}.\frac{1.3.5.7}{2.4.6.8}c^4\Big(1 - \frac{2.4}{5} Cof\, 2v + \frac{2.4.3}{5.6} Cof\, 4v - \frac{2.4.3.2}{5.6.7} Cof\, 6v - \frac{2.4.3.2.1}{5.6.7.8} Cof\, 8v\Big) + \&c\Big].$$

Integralibus itaque ſumtis, & quidem ita ut pro $v = 0$ ſingula evaneſcant, ob $\int dv\, Cof\, 2iv = \frac{Sin\, 2iv}{2i}$, erit 𝔄)

$$\frac{s}{(1-c)a} = v + \tfrac{3}{2}.\tfrac{1}{2}\, c\,(v - \tfrac{1}{2} Sin\, 2v) + \frac{3.5}{2.4}.\frac{1.3}{2.4}c^2\Big(v - \tfrac{2}{3} Sin\, 2v + \frac{2.1}{3.4}.\frac{Sin\, 4v}{2}\Big) + \frac{3.5.7}{2.4.6}.\frac{1.3.5}{2.4.6}c^3\Big(v - \tfrac{3}{4} Sin\, 2v + \frac{3.2}{4.5}\frac{Sin\, 4v}{2} - \frac{3.2.1}{4.5.6}\frac{Sin\, 6v}{3}\Big) + \frac{3.5.7.9}{2.4.6.8}.\frac{1.3.5.7}{2.4.6.8}c^4\Big(v - \tfrac{4}{5} Sin\, 2v + \frac{4.3}{5.6}.\frac{Sin\, 4v}{2} - \frac{4.3.2}{5.6.7}\frac{Sin\, 6v}{3} + \frac{4.3.2.1}{5.6.7.8}\frac{Sin\, 8v}{4}\Big) + \&c.$$

in qua ſerie lex progreſſio-

nis

nis manifesta est. Per se etiam intelligitur, ob assumtum radium seu Sinum totum $=1$, ipsum angulum v in partibus radii exprimendum esse.

§. IV.

Si igitur existente $v = L + z$ sit $s = S$, & pro $v = L - z$ sit $s = S'$, atque per formulam 𝔄 (§. 3.) quærantur S & S', horumque sumatur differentia $S - S' = m$, facta reductione secundum regulam Trigonometricam: $Sin\,(p+q) - Sin\,(p-q) = 2\, Cos\, p\, Sin\, q$; prodit æquatio:

$$\text{𝔅}).\ \frac{m}{2a(1-c)} = z + \tfrac{3}{2}.\tfrac{1}{2}\,c\left(z - \tfrac{1}{2}.\,Cos\,2L\,Sin\,2z\right) +$$

$$+ \frac{3.5}{2.4}.\frac{1.3}{2.4}c^2\left(z - \tfrac{2}{3}\,Cos\,2L\,Sin\,2z + \frac{2.1}{3.4}.\frac{Cos\,4L\,Sin\,4z}{2}\right)$$

$$+ \frac{3.5.7}{2.4.6}.\frac{1.3.5}{2.4.6}c^3\left(z = \tfrac{3}{4}\,Cos\,2L\,Sin\,2z + \frac{3.2}{4.5}\frac{Cos\,4L\,Sin\,4z}{2}\right.$$

$$\left. - \frac{3.2.1}{4.5.6}.\frac{Cos\,6L\,Sin\,6z}{3}\right) + \frac{3.5.7.9}{2.4.6.8}.\frac{1.3.5.7}{2.4.6.8}c^4\left(z - \tfrac{4}{5}\,Cos\,2L\,Sin\,2z\right.$$

$$+ \frac{4.3}{5.6}\frac{Cos\,4L\,Sin\,4z}{2} - \frac{4.3.2}{5.6.7}.\frac{Cos\,6L\,Sin\,6z}{3}$$

$$\left. + \frac{4.3.2.1}{5.6.7.8}.\frac{Cos\,8L\,Sin\,8z}{4}\right) + \text{\&c.}$$

Pari modo, denotante μ longitudinem arcus meridiani inter latitudines $\lambda + \zeta$ & $\lambda - \zeta$ intercepti, obtinetur æquatio:

$$\text{ℭ}).\ \frac{\mu}{2a(1-c)} = \zeta + \tfrac{3}{2}.\tfrac{1}{2}\,c\left(\zeta - \tfrac{1}{2}\,Cos\,2\lambda\,Sin\,2\zeta\right)$$

$$+ \frac{3.5}{2.4}.\frac{1.3}{2.4}c^2\left(\zeta - \tfrac{2}{3}\,Cos\,2\lambda\,Sin\,2\zeta + \frac{2.1}{3.4}.\frac{Cos\,4\lambda\,Sin\,4\zeta}{2}\right)$$

$+$ &c. &c. Per has æquationes 𝔅 & ℭ ex da-

tis

tis m, μ, z, ζ, L & λ determinantur quantitates incognitæ c & a. Et quidem mox exterminari poteſt a multiplicando æqv. 𝔅 per μ & æqv. ℭ per m, atque æquationes productas ſubtrahendo. Ita videlicet prodit æquatio hujus formæ:

$$\mathfrak{D})\ o = A - Bc - Cc^2 - Dc^3 - Ec^4 - \&c.$$

in qua A, B, C, D, E &c. ſunt quantitates cognitæ, ex coëfficientibus æquationum 𝔅 & ℭ dependentes, adeo ut ſit $\mu z - m\zeta = A$; $\frac{3}{2}.\frac{1}{2}(\mu z - m\zeta - \frac{1}{2}\mu\, Cos\, 2L\, Sin\, 2z + \frac{1}{2}\, m\, Cos\, 2\lambda\, Sin\, 2\zeta) = -B$, & ſic porro. Ex æquatione 𝔇 valor ipſius c inveſtigatur, quod quidem ob c admodum exiguum per vulgarem approximationis methodum facillime fiet. Quum videlicet ex 𝔇 ſequatur eſſe:

$$\mathfrak{E}).\ c = \frac{A}{B + Cc + Dc^2 + Ec^3 + \&c.};$$

ſi detur ipſius c valor quivis approximatus $= k$, facto ulterius $\frac{A}{B + Ck + Dk^2 + Ek^3 + \&c.} = h$; manifeſtum eſt, h propius quam k ad verum valorem c accedere. Hac igitur ratione calculum repetendo, donec perveniatur ad duos valores k & h, quorum evaneſcit differentia, habetur valor exactus ipſius c. Quo denique in alterutra æqv. 𝔅 vel ℭ ſubſtituto, invenitur a.

§. V.

Ut jam pro dato quovis caſu dijudicari poterit, utrum error methodi vulgaris ſenſibilis fiat, ſit fit $\frac{m}{2z} = R$ & $\frac{\mu}{2z} = \varrho$, (amplitudinibus fcil. $2z$ & 2ζ brevitatis cauffa in partibus radii expreffis), veri autem valores radiorum curvaturæ meridiani pro latitudinibus L & λ fint $R + dR$ & $\varrho + d\varrho$ refpective. His pofitis, retentisque de cetero prioribus denominationibus, ex formula 𝔅 (§. 4.) deducitur:

$$\mathfrak{F}).\ \frac{R}{(1-c)a} = 1 + \tfrac{3}{2}.\tfrac{1}{2}c\left(1 - \frac{2.1}{2}\frac{Cos\,2L\,Sin\,2z}{2z}\right)$$

$$+ \frac{3.5}{2.4}.\frac{1.3}{2.4}c^2\left(1 - \frac{2.2}{3}\frac{Cos\,2L\,Sin\,2z}{2z} + \frac{2.2.1}{3.4}.\frac{Cos\,4L\,Sin\,4z}{4z}\right)$$

$$+ \frac{3.5.7}{2.4.6}.\frac{1.3.5}{2.4.6}c^3\left(1 - \frac{2.3}{4}.\frac{Cos\,2L\,Sin\,2z}{2z} + \frac{2.3.2}{4.5}.\frac{Cos\,4L\,Sin\,4z}{4z}\right.$$

$$\left. - \frac{2.3.2.1}{4.5.6}.\frac{Cos\,6L\,Sin\,6z}{6z}\right) + \&c.$$

Quumque fit ad latitudinem L radius ofculi feu $R + dR = \frac{(1-c)a}{(1-c\,Sin\,L^2)^{\frac{3}{2}}}$, hanc formulam fecundum methodum (§. 3.) adhibitam evolvendo obtinetur:

$$\mathfrak{G}).\ \frac{R + dR}{(1-c)a} = 1 + \tfrac{3}{2}.\tfrac{1}{2}c\left(1 - \frac{2.1}{2}Cos\,2L\right)$$

$$+ \frac{3.5}{2.4}.\frac{1.3}{2.4}c^2\left(1 - \frac{2.2}{3}Cos\,2L + \frac{2.2.1}{3.4}Cos\,4L\right)$$

$$+ \frac{3.5.7}{2.4.6}.\frac{1.3.5}{2.4.6}c^3\left(1 - \frac{2.3}{4}Cos\,2L + \frac{2.3.2}{4.5}Cos\,4L\right.$$

$$\left. - \frac{2.3.2.1}{4.5.6}Cos\,6L\right) + \&c.$$ Æquatione igitur 𝔉 a 𝔊 fubtra-

cta

cta, positis $\frac{Sin\,2z}{2z} = 1 - z'$, $\frac{Sin\,4z}{4z} = 1 - z''$, $\frac{Sin\,6z}{6z} = 1 - z'''$, &c. eruitur:

$$\mathfrak{H}).\ \frac{dR}{(1-c)a} = -\frac{3}{2}.\frac{1}{2}cz'\,Cos\,2L - \frac{3.5}{2.4}\,\frac{1.3}{2.4}c^2\left(\frac{2.2}{2}z'\,Cos\,2L - \frac{2.2.1}{3.4}z''\,Cos\,4L\right) - \frac{3.5.7}{2.4.6}\,\frac{1.3.5}{2.4.6}c^3\left(\frac{3.5}{4}z'\,Cos\,2L - \frac{2.3.2}{4.5}z''\,Cos\,4L + \frac{2.3.2}{4.5}z'''\,Cos\,6L\right) + \&c.$$

Valores ipsorum z', z'', z''' &c. generatim computari possunt secundum formulam:

$$\frac{Sin\,nz}{nz} = 1 - \frac{n^2z^2}{1.2.3} + \frac{n^4z^4}{1.2.3.4.5} + \frac{n^6z^6}{1.2.3.4.5.6.7} + \&c.$$

Quoties vero amplitudo arcus mensurati paucis constat gradibus, ob quantitatem c admodum exiguam, quippe quae ex comparatis pluribus mensurationibus invenitur $< 0{,}01$, sine errore assumi poterit

$$\frac{Sin\,nz}{nz} = 1 - \frac{n^2z^2}{2.3}.$$

Si porro in aeqv. $\mathfrak{H}$ pro dR, z, & L respective substituantur $d\varrho$, ζ & λ, obtinetur etiam valor ipsius $d\varrho$. Datis vero variationibus dR, & $d\varrho$, secundum regulas supra (§. 2. Cor. 2) traditas inveniuntur dc & da.