2.

DISSERTATIO ACADEMICA

SISTENS COMPARATIONEM DIVERSORUM EXPERIMENTORUM AD DEFINIENDAM DENSITATEM ET VOLUMEN AQUÆ PRO DIVERSA CALORIS TEMPERIE,

CUJUS PARTEM PRIMAM

VENIA AMPLISS. FACULT. PHIL. IN ACAD. IMP. AB.

PUBLICÆ MODESTE SUBJICIUNT CENSURÆ

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h. a. m. s.

ABOÆ, TYPIS FRENCKELLIANIS.

Non in pondere solum specifico corporum investigando, cujus haud semper in una eademque caloris temperatura experiendi data est occasio, unde evenit, ut propter varians volumen liquidi, præsertim aquæ, hanc ob caussam communiter adhibitæ, quo ad normalem quendam calorem fiat reductio, ad hoc respiciendum sit in calculis, sed etiam in multis aliis experimentis physicis necessum est, veram, quantum per sensuum instrumentorumque inevitabiles errores licet, habere mensuram expansionis, quam in diversis subit gradibus caloris aqua. Instituerunt jam talia experimenta Cell. GILPIN, SCHMIDT, DALTON & HÄLLSTRÖM. Quorum vero inventi valores plus an minus

nus a se invicem discrepent, a nemine in his scientiarum partibus versante disquisitum comperimus; quare igitur non plane inutilem nos in hoc argumento tractando posuisse operam putavimus, præsertim cum eo tempore, quo experimenta hæc instituebantur, methodus quam vocant quadratorum minimorum adhuc non fuerit cognita, quæ id præ se habet, ut nisi immane quantum aberrent observationes, veram legem ejus ope ita inveniri censendum sit, ut numeri, qui inde oriantur, longe majorem admittant præcisionis gradum quam singulæ observationes. Quod uberrimæ utilitatis in omnibus, quæ ad Scientiam Naturalem spectant, disquisitionibus inventum, a Cell. Legendre & Gauss publici juris factum, etsi primas ejus lineas jam a Boscovich quodammodo fuisse cognitas notum est, æram quasi affirmandum est constituisse, unde tam in Astronomicis, quam & Physicis, quin etiam Chemicis (usum enim summum habet in numerice definiendis salium proportionibus secundum theoriam Cel. Berzelii) recentior & exactior oriatur naturæ quoad quantitatem numericam definitio, quod, ut apertum est, totius Physicæ experimentalis scopus est summus & fere unicus. Et cum hanc theoriam, ut nobis videtur, minus apud practicos Physicos, quam fas est, cognitam putemus, ab re non esse judicamus, brevem & ad usus practicos accommodatam expositionem for-

mu-

mularum præmittere huc pertinentium, viam sic sternentes ad observationes, quæ ad nostram materiem pertinent, probabiliter corrigendas & emendandas.

§. 1.

Si habentur experimentis datæ sequentes æquationes:

$$\begin{aligned}
o &= n_1 + a_1 p + b_1 q + c_1 r + \ldots \\
o &= n_2 + a_2 p + b_2 q + c_2 r + \ldots \\
o &= n_3 + a_3 p + b_3 q + c_3 r + \ldots \\
&\vdots \\
o &= n_i + a_i p + b_i q + c_i r + \ldots
\end{aligned}$$

quas omnes ex observationibus æquali præcisionis gradu affectis ortas consideramus (in casu enim contrario quævis æquatio, ut apertum est, in suum probabilitatis gradum est ducenda), ubi n, a, b, c, empirice datæ sunt quantitates (a, b, c vero quasi exactas considerare oportet, ut totus error in n_1, n_2, ... jaceat); p, q, r, incognitæ, quarum numerus sit m; perspicuum est, solutionem problematis requiri vel non determinati, vel determinati, vel plus quam determinati, si sit $i < m$, $i = m$, $i > m$; ad hunc vero ultimum casum methodus hæc proprie est accommodata. Cum igitur n_1, n_2, n_3 ... ita debeant esse determinati, ut sint veri æquationum valores = o, hoc vero ob errores in experiendo inevitabiles non habeatur; ponamus singularum æquationum valores veros seu errores esse quantitates exiguas t_1. t_2, t_3 .... Demonstravit summus LAPLACE, probabilissimos erui incognitarum p, q, r.. valores, si ponatur errorum quadratorum summa minima.

Erit ergo

$$
\begin{aligned}
t_1 &= n_1 + a_1 p + b_1 q + c_1 r + \,.. \\
t_2 &= n_2 + a_2 p + b_2 q + c_2 r + \,.. \\
t_3 &= n_3 + a_3 p + b_3 q + c_3 r + \,.. \\
&\ldots \quad \ldots\ldots \\
t^2_1 &= n_1^2 + 2a_1 n_1 p + a_1^2 p^2 + 2a_1 b_1 pq \\
&\quad + 2b_1 n_1 q + b_1^2 q^2 + 2b_1 cqr \\
&\quad + 2c_1 n_1 r + c_1^2 r^2 + 2a_1 cpr
\end{aligned}
$$

& pari modo cum ceteris.

Unde

$$
\begin{aligned}
2t_1\, dt_1 = &\,(2a_1 n_1 + 2a_1 a_1 p + 2a_1 b_1 q + 2a_1 b_1 r \ldots)\, dp \\
&+ (2b_1 n_1 + 2a_1 b_1 p + 2b_1 b_1 q + 2b_1 c_1 r \ldots)\, dq \\
&+ (2c_1 n_1 + 2a_1 c_1 p + 2b_1 c_1 q + 2c_1 c_1 r \ldots)\, dr \\
&\ldots
\end{aligned}
$$

& sic porro. Unde in casu minimi summa omnium $\frac{dt}{dp} = o$, $\frac{dt}{dq} = o$, $\frac{dt}{dr} = o$, ... poni debet quo habeatur:

$$0 = \begin{cases} a_1 n_1 + a_1 a_1 p + a_1 b_1 q + a_1 c_1 r + \ldots \\ a_2 n_2 + a_2 a_2 p + a_2 b_2 q + a_2 c_2 r + \ldots \\ a_3 n_3 + a_3 a_3 p + a_3 b_3 q + a_3 c_3 r + \ldots \\ \ldots \quad \ldots \quad \ldots \quad \ldots \end{cases} \quad a)$$

$$0 = \begin{cases} b_1 n_1 + a_1 b_1 p + b_1 b_1 q + b_1 c_1 r + \ldots \\ b_2 n_2 + a_2 b_2 p + b_2 b_2 q + b_2 c_2 r + \ldots \\ b_3 n_3 + a_3 b_3 p + b_3 b_3 q + b_3 c_3 r + \ldots \\ \ldots \quad \ldots \quad \ldots \end{cases}$$

$$0 = \begin{cases} c_1 n_1 + a_1 c_1 p + b_1 c_1 q + c_1 c_1 r + \ldots \\ c_2 n_2 + a_2 c_2 p + b_2 c_2 q + c_2 c_2 r + \\ c_3 n_3 + a_3 c_3 p + b_3 c_3 q + c_3 c_3 r + \ldots \end{cases}$$

&c;

seu

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a) Hinc sponte sequitur, in determinando uno incognito, ubi plures experimentis inventi sunt hujus valores, (qualis vero multiplex determinatio numquam est negligenda,) medium, ut communiter in usu est, arithmeticum maxime probabilem hujus efficere valorem. Quod si vero diversæ sint probabilitatis hi valores, patet per formulas præcedentes probabilissimum erui valorem si sumatur $p = \dfrac{k_1^2 n_1 + k_2^2 n_2 + k_3^2 n_3 + \ldots}{k_1^2 + k_2^2 + k_3^2 + \ldots}$ denotantibus n_1, n_2, n_3 hos valores, k_1, k_2, k_3 resp. præcisionis gradus. Sed cum sumendo medio arithmetico extimæ determinationes aperte minorem habeant quam mediæ gradum probabilitatis (etsi hic gradus minime theoretice definiri potest), patet $k_1\ k_2\ k_3 \ldots$ posse ita assumi, ut sint in ratione inversa distantiarum cujusvis valoris a medio quodam, pro quo assumi potest medium arithmeticum; igitur æquales $\dfrac{1}{n_1 - \mu}$, $\dfrac{1}{n_2 - \mu}$, $\dfrac{1}{n_3 - \mu}$, ..., denotante μ valorem illum medium.

seu, si summæ uniformium productorum uncis distinguantur, habebitur

$$o = (an) + (aa)p + (ab)q + (ac)r + (ad)s + (ae)t + \ldots$$
$$o = (bn) + (ab)p + (bb)q + (bc)r + (bd)s + (be)t + \ldots$$
$$o = (cn) + (ac)p + (bc)q + (cc)r + (cd)s + (ce)t + \ldots$$
$$o = (dn) + (ad)p + (bd)q + (cd)r + (dd)s + (de)t + \ldots$$
$$o = (en) + (ae)p + (be)q + (ce)r + (de)s + (ee)t + \ldots$$

Adsunt igitur tot æquationes, quot incognitæ; quæ igitur simplici eliminatione determinari possunt. Quod si hæc eliminatio recte & commodissime perficiatur, ad eandem pervenitur methodum, quam ad calculum contrahendum, diverso & prolixiori modo, Cel. GAUSS proposuit *b*). Id autem laborem contrahit, quod series æquationum sequentes, eliminatis p, q, r, .. successive, ita symmetricas reddere licet, quam primariæ jam allatæ; eædem nempe coëfficientes in lineis posterioribus repetitæ computum harum reddunt expeditiorem. Ut omnia ad calculum numericum in hac methodo necessaria præsto sint, hanc etiam transformationem commodam sufficienter extensam afferamus.

Si nempe singulæ æquationes respective per coëfficientes τȣ p dividantur, primaque a posterioribus dematur, inde vero orientes æquationes resp. per (ab), (ac), (ad) &c multiplicentur, habebitur ad eliminandam incognitam p

$(bn, 1)$

b) *Commentat. Soc. Reg. Gotting. Recentior. Vol. I.* ubi pag. 23. linn. 13, 14, 15 pro $(an)(bn)$, $(an)(cn)$, $(an)(dn)$ legatur $(an)(ab)$; $(an)(ac)$; $(an)(ad)$.

$$(bn,1) + (bb,1)q + (bc,1)r + (bd,1)s + (be,1)t = o$$
$$(cn,1) + (bc,1)q + (cc,1)r + (cd,1)s + (ce,1)t = o$$
$$(dn,1) + (bd,1)q + (cd,1)r + (dd,1)s + (de,1)t = o$$
$$(en,1) + (be,1)q + (ce,1)r + (de,1)s + (ee,1)t = o$$

positis

$$(bb,1) = (bb) - \frac{(ab)}{(aa)}(ab) \qquad (cc,1) = (cc) - \frac{(ac)}{(aa)}(ac)$$
$$(bc,1) = (bc) - \frac{(ab)}{(aa)}(ac) \qquad (cd,1) = (cd) - \frac{(ac)}{(aa)}(ad)$$
$$(bd,1) = (bd) - \frac{(ab)}{(aa)}(ad) \qquad (ce,1) = (ce) - \frac{(ac)}{(aa)}(ae)$$
$$(be,1) = (be) - \frac{(ab)}{(aa)}(ae) \qquad (cn,1) = (cn) - \frac{(ac)}{(aa)}(an)$$
$$(bn,1) = (bn) - \frac{(ab)}{(aa)}(an)$$

$$(dd,1) = (dd) - \frac{(ad)}{(aa)}(ad) \qquad (ee,1) = (ee) - \frac{(ae)}{(aa)}(ae)$$
$$(de,1) = (de) - \frac{(ad)}{(aa)}(ae) \qquad (en,1) = (en) - \frac{(ae)}{(aa)}(an)$$
$$(dn,1) = (dn) - \frac{(ad)}{(aa)}(an)$$

Analogo modo procedendo per æquationes inventas habebitur ad eliminandam incognitam q &c.

$$(cn,2) + (cc,2)r + (cd,2)s + (ce,2)t = 0$$
$$(dn,2) + (cd,2)r + (dd,2)s + (de,2)t = 0$$
$$(en,2) + (ce,2)r + (de,2)s + (ee,2)t = 0$$
$$(dn,3) + (dd,3)s + (de,3)t = 0$$
$$(en,3) + (de,3)s + (ee,3)t = 0$$
$$(en,4) + (ee,4)t = 0$$

factis ut antea

$$(cc,2) = (cc,1) - \frac{(bc,1)}{(bb,1)}(bc,1) \quad (dd,2) = (dd,1) - \frac{(bd,1)}{(bb,1)}(bd,1)$$
$$(cd,2) = (cd,1) - \frac{(bc,1)}{(bb,1)}(bd,1) \quad (de,2) = (de,1) - \frac{(bd,1)}{(bb,1)}(be,1)$$
$$(ce,2) = (ce,1) - \frac{(bc,1)}{(bb,1)}(be,1) \quad (dn,2) = (dn,1) - \frac{(bd,1)}{(bb,1)}(bn,1)$$
$$(cn,2) = (cn,1) - \frac{(bc,1)}{(bb,1)}(bn,1)$$

$$(ee,2) = (ee,1) - \frac{(be,1)}{(bb,1)}(be,1)$$
$$(en,2) = (en,1) - \frac{(be,1)}{(bb,1)}(bn,1)$$

$$(dd,3) = (dd,2) - \frac{(cd,2)}{(cc,2)}(cd,2) \quad (ee,3) = (ee,2) - \frac{(ce,2)}{(cc,2)}(ce,2)$$
$$(de,3) = (de,2) - \frac{(cd,2)}{(cc,2)}(ce,2) \quad (en,3) = (en,2) - \frac{(ce,2)}{(cc,2)}(cn,2)$$
$$(dn,3) = (dn,2) - \frac{(cd,2)}{(cc,2)}(cn,2)$$

$$(ee,4) = (ee,3) - \frac{(de,3)}{(dd,3)}(de,3)$$
$$(en,4) = (en,3) - \frac{(de,3)}{(dd,3)}(dn,3)$$

Casus

Casus, qui sæpe occurrit, v. gr. in formulis pro longitudine penduli vel gradus meridiani, & in genere, in definienda quavis functione uniformiter respectu argumenti crescente, hic est, ubi

$$t_1 = n_1 + p + b_1 q$$
$$t_2 = n_2 + p + b_2 q$$
$$t_i = n_i + p + b_i q, \text{ ubi igitur}$$

$$o = (n) + ip + (b) q$$
$$o = (bn) + (b) p + (bb) q$$

$$p = -\frac{(n) + (b) q}{i}; \; q = \frac{i(bn) - (b)(n)}{i(bb) - (b)(b)}.$$

Sed etiam in hoc casu præstat operam in eo contrahere, quod addantur æquationes ad inveniendum medium arithmeticum, illaque media æquatio a quavis reliquarum dematur. Sit nempe media æquatio $= o = \nu + p + \beta q$, & si nominetur $n_1 - \nu = r_1$; $n_2 - \nu = r_2 \ldots . b_1 - \beta = k_1$; $b_2 - \beta = k_2 \ldots .$; habebitur $q = \frac{k_1 r_1 + k_2 r_2 + k_3 r_3 + \ldots}{k_1^2 + k_2^2 + k_3^2 + \ldots}$, $p = -\nu - \beta q$.

Apertum est, valores incognitarum, in genere, eo certiores inveniri, quo plura adhibeantur experimenta & plures inde deductæ æquationes.

Ubi numeri a sunt rationales integri, naturali ordine progredientes, quod sæpe in disquisitionibus physicis occurrit, & simul $b = a^2$, $c = a^3$, observandum est, quod ad calculi compendium heic affertur, esse:

$$
\begin{aligned}
(a) &= \tfrac{1}{2}a^2 + \tfrac{1}{2}a \\
(a^2) &= \tfrac{1}{3}a^3 + \tfrac{1}{2}a^2 + \tfrac{1}{6}a \\
(a^3) &= \tfrac{1}{4}a^4 + \tfrac{1}{2}a^3 + \tfrac{1}{2}a^2 \\
(a^4) &= \tfrac{1}{5}a^5 + \tfrac{1}{2}a^4 + \tfrac{1}{3}a^3 - \tfrac{1}{30}a \\
(a^5) &= \tfrac{1}{6}a^6 + \tfrac{1}{2}a^5 + \tfrac{5}{12}a^4 - \tfrac{1}{12}a^2 \\
(a^6) &= \tfrac{1}{7}a^7 + \tfrac{1}{2}a^6 + \tfrac{1}{2}a^5 - \tfrac{1}{6}a^3 + \tfrac{1}{42}a.
\end{aligned}
$$

Observandum est, formulas jam allatas valere tam in eo casu, ubi p, q, r, sunt elementa incognita, quam eorum correctiones: quod posterius præsertim in eo casu obtinet, ubi æquatio determinanda non linearis est, sed ubi ita appropinquate jam supponi possunt cognita esse elementa, ut quadrata correctionum possint negligi; quod semper, si calculus denuo instituatur, in potestate nostra est facile dijudicare.

Absolutis his, jam ad materiam nostram specialius tractandam nos accingimus.

§. 2.

Experimenta Cel. GILPIN occurrunt in Philosophical Transactions for the Year 1792 p. 428 &

& 1794 p. 382. Cum vero non pro singulis, sed pro quovis quinto gradu a 30 ad 100 scalæ Fahrenheitianæ pondus specificum aquæ determinaverit; cetera vero interpolatione in tabella sua adhibita attulerit; patet, illa tantum in formula construenda adhibenda esse. Cum differentiæ secundæ, exceptis primis decem gradibus, vel circa maximam aquæ densitatem, sint proxime æquales, ita ut certus valor tertiæ differentiæ ex his non definiri possit; patet, pro pondere specifico aquæ in diversis caloris gradibus th. F. adhiberi posse formulam algebraicam secundi gradus, $P = x + yf + zf^2$, vel denotando ordinem experimentorum per m, $P = p + qm + rm^2$, ad quas incognitas eruendas ex observationibus ejus sequuntur

$$12{,}97902 = 13p + 78q + 650r$$
$$77{,}76787 = 78p + 650q + 6084r$$
$$647{,}59471 = 650p + 6084q + 60710r;$$

unde habetur aquæ pondus specificum

$$P = 1{,}001025 - 0{,}0001129m - 0{,}000039233\,m^2;$$

quæ formula ita cum experimentis consentit:

m.	f.	Pond Specif. Formula	Aquæ Exper.	Diff.
0	40°	1,00102	1,00094	+ 0,00008
1	45	1,00087	1,00086	+ 1
2	50	1,00064	1,00068	— 4
3	55	1,00033	1,00038	— 5
4	60	0,99995	1,00000	— 5
5	65	0,99948	0,99950	— 2
6	70	0,99893	0,99894	— 1
7	75	0,99831	0,99830	+ 1
8	80	0,99761	0,99759	+ 2
9	85	0,99683	0,99681	+ 2
10	90	0,99597	0,99598	— 1
11	95	0,99503	0,99502	+ 1
12	100	0,99402	0,99402	0

Cum sit $f = 5m + 40 = \frac{9}{5}c - 32 = \frac{9}{4}r - 32$, designantibus f, c, r, gradus th. sc. Fahrenheitianæ, Cent. & Réaum., erit igitur secundum experimenta Gilpiniana densitas seu pondus specificum aquæ hac formula definita: $P = 0{,}999417 + 0{,}00010296\,f - 0{,}0000015693\,f^2$; existente $P = 1$ in calore fere 60° Fahr. Etsi aptissimum videatur, maximam aquæ densitatem ut unitatem assumere, attamen cum hoc arbitrio sit relictum, nos commodius quidem judicamus esse gravitatem specificam ad illam pro $f = 50°$ reducere, quod in ceteris scalis etiam numero rotundo definitur. Habetur igitur grav. aquæ (= 1 in cal. 50° f) $P = 0{,}998776 + 0{,}000010289\,f - 0{,}0000015683\,f^2$, vel etiam $P =$