Åbo Akademi University

Xiaolu Wang

Fuzzy Real Option Analysis in Patent Related Decision Making and Patent Valuation

Xiaolu Wang
Born 1983

Bachelor of Science in the University of Edinburgh 2007

Master of Science in Imperial College London 2008

PhD dissertation defence in Åbo Akademi 2015

Fuzzy Real Option Analysis in Patent Related Decision Making and Patent Valuation

Xiaolu Wang

Åbo Akademi University
Doctoral Programme in Systems Analysis, Decision Making and Risk Management
Institute for Advanced Management Systems Research
Joukahaisenkatu 3-5 A, 20520 Turku, Finland

2015

Supervisor

Prof. Christer Carlsson
Institute for Advanced Management Systems Research
Faculty of Social Sciences, Business and Economics
Åbo Akademi University
Joukahaisenkatu 3-5 A, 20520 Turku
Finland

Reviewers

Prof. Kalevi Kyläheiko
School of Business
Lappeenranta University of Technology
Box 20, 53851 Lappeenranta
Finland

Prof. Tuomas Takalo
Department of Economics
Hanken School of Economics
P.O.Box 479, 00101 Helsinki
Finland

Opponent

Prof. Yuri Lawryshyn
Faculty of Applied Science and Engineering
University of Toronto
200 College Street, Toronto, Ontario M5S 3E5
Canada

ISBN 978-952-12-3227-5
Painosalama Oy, Åbo, Finland 2015

Abstract

The shift towards a knowledge-based economy has inevitably prompted the evolution of patent exploitation. Nowadays, patent is more than just a prevention tool for a company to block its competitors from developing rival technologies, but lies at the very heart of its strategy for value creation and is therefore strategically exploited for economic profit and competitive advantage. Along with the evolution of patent exploitation, the demand for reliable and systematic patent valuation has also reached an unprecedented level. However, most of the quantitative approaches in use to assess patent could arguably fall into four categories and they are based solely on the conventional discounted cash flow analysis, whose usability and reliability in the context of patent valuation are greatly limited by five practical issues: the market illiquidity, the poor data availability, discriminatory cash-flow estimations, and its incapability to account for changing risk and managerial flexibility.

This dissertation attempts to overcome these impeding barriers by rationalizing the use of two techniques, namely fuzzy set theory (aiming at the first three issues) and real option analysis (aiming at the last two). It commences with an investigation into the nature of the uncertainties inherent in patent cash flow estimation and claims that two levels of uncertainties must be properly accounted for. Further investigation reveals that both levels of uncertainties fall under the categorization of subjective uncertainty, which differs from objective uncertainty originating from inherent randomness in that uncertainties labelled as subjective are highly related to the behavioural aspects of decision making and are usually witnessed whenever human judgement, evaluation or reasoning is crucial to the system under consideration and there exists a lack of complete knowledge on its variables. Having clarified their nature, the application of fuzzy set theory in modelling patent-related uncertain quantities is effortlessly justified.

The application of real option analysis to patent valuation is prompted by the fact that both patent application process and the subsequent patent exploitation (or commercialization) are subject to a wide range of decisions at multiple successive stages. In other words, both patent applicants and patentees are faced with a large variety of courses of action as to how their

patent applications and granted patents can be managed. Since they have the right to run their projects actively, this flexibility has value and thus must be properly accounted for. Accordingly, an explicit identification of the types of managerial flexibility inherent in patent-related decision making problems and in patent valuation, and a discussion on how they could be interpreted in terms of real options are provided in this dissertation.

Additionally, the use of the proposed techniques in practical applications is demonstrated by three fuzzy real option analysis based models. In particular, the pay-off method and the extended fuzzy Black-Scholes model are employed to investigate the profitability of a patent application project for a new process for the preparation of a gypsum-fibre composite and to justify the subsequent patent commercialization decision, respectively; a fuzzy binomial model is designed to reveal the economic potential of a patent licensing opportunity.

Sammanfattning

Framväxten av en kunskapsbaserad ekonomi har obestridligt drivit utvecklingen av exploatering av patent. Ett patent är för företag inte längre enbart ett sätt att hindra konkurrenterna från att utveckla konkurrerande teknologier utan ligger nuförtiden i kärnan av företagets värdeskapande och utnyttjas därför för ekonomisk vinst och konkurrensfördelar. I samband med exploateringen av patent har också efterfrågan på en pålitlig och systematisk värdering av patent ökat. De kvantitativa metoderna som används för att utvärdera patent kan kategoriseras till fyra klasser men de baserar sig enbart på den konventionella nuvärdesmetoden vars användbarhet och pålitlighet inom patentvärdering begränsas starkt av fem utmaningar: svag likviditet, svår tillgång till data, diskriminerande kassaflödesberäkningar samt dess oförmåga att ta hänsyn till växlande risk och flexibelt beslutsfattande.

Denna avhandling försöker bemästra dessa hinder genom att rationalisera användningen av två tekniker, nämligen fuzzy mängdteori (eng. fuzzy set theory) (relaterat till de tre första utmaningarna) och analys av realoptioner (de två sistnämnda utmaningarna). Avhandlingen börjar med en beskrivning över osäkerheterna i kassaflödesberäkning för patent och påvisar att det finns två nivåer av osäkerheter som man måste reda ut. Vidare utredning avslöjar att båda nivåerna av osäkerhet faller under kategorin subjektiv osäkerhet vilket skiljer sig från objektiv osäkerhet som kommer från inneboende slumpmässighet: de osäkerheter som stämplas som subjektiva är beroende av de beteendemässiga aspekterna av beslutsfattande och kommer vanligen fram alltid när mänskligt omdöme, evaluering eller resonemang är avgörande för systemet som övervägs och det råder brist på fullständig kunskap om dess variabler. Efter att ha förtydligat karaktären hos dessa osäkerheter, är det lätt att motivera tillämpandet av fuzzy mängdteori i modellering av patentrelaterade osäkra kvantiteter.

Analys av realoptioner lämpar sig på patentvärdering för att både ansökningsprocessen för ett patent och efterföljande exploatering av patent (eller kommersialisering) utsätts för väldigt olika typer av beslut under många på varandra följande faser. Med andra ord, det finns en mångfald av tillvägagångssätt både för patentsökande och för patentinnehavare gällande hur deras patentansökningar och beviljade patent kan förvaltas. Eftersom de aktivt

kan föra fram sina projekt, är denna flexibilitet värdefullt och borde därför utredas ordentligt. Därmed innehåller denna avhandling en tydlig identifiering av de typer av flexibelt beslutsfattande som är väsentliga för patentrelaterat beslutsfattande och patentvärdering samt en diskussion över hur dessa typer kunde tolkas i realoptionstermer.

Genom tre modeller som baserar sig på realoptionsanalys med fuzzy tal (eng. fuzzy real option analysis) visar avhandlingen ytterligare hur de föreslagna teknikerna kan användas i praktiska tillämpningar. Specifikt används återbetalningsmetoden (eng. pay-off method) och den utvidgade fuzzy Black-Scholes modellen (eng. extended fuzzy Black-Scholes model) för att undersöka lönsamheten hos ett patentansökningsprojekt gällande en ny process för framställning av gips-fiberkomposit och för att motivera det efterföljande beslutet om kommersialiseringen av patentet; en fuzzy binomialmodell (eng. fuzzy binomial model) konstrueras för att visa ekonomipotentialen hos en patentlicensieringsmöjlighet.

Acknowledgements

I would never have been able to finish my dissertation without the guidance of my supervisor, help from colleagues and friends, and support from my family. It is therefore my great pleasure to thank the many people who made it possible.

My supervisor, **Prof. Christer Carlsson**, not only introduced the topic of real option to me but has taught me how proper research is done. He has been a strong and supportive mentor to me throughout my PhD career, and he has always given me great freedom to pursue independent work, not to mention an excellent atmosphere for doing research. I am forever in debt to him for this unparalleled scientific opportunity and all the economic support.

In the short time I have worked with **Prof. Mikael Collan** in IAMSR, he has had an enormous impact on my work, in particular on the way in which I perform cash flow estimations and treat uncertainties. I am truly grateful for his efforts to improve the quality of my articles, and I have always read his comments with a mixture of gratitude, for their detailed attention to my work, and inspiration, as he always manages to find a way to make it better. Moreover, Prof. Collan has spared no effort to introduce me to people within my discipline, and has always been an advocate of my work.

I have been especially fortunate to know **József Mezei**, who has always been incredibly generous with his time and insights. It is safe to say that I have stolen more ideas from József than from anyone else, and I only wish I had implemented more of them. I would therefore like to express my utmost appreciation on his unconditional support and countless advices, without which my dissertation would not have been accomplished.

I would also like to offer my thanks to **Prof. Pirkko Walden**, the Director of IAMSR, for offering a research environment that promotes good results and encourages doctoral students to move beyond the state-of-the-art.

My special thanks go to **Prof. Yuri Lawryshyn** for all the invested time in assessing my manuscript and acting as my opponent. I am also grateful to

Prof. Kalevi Kyläheiko and **Prof. Tuomas Takalo** who have reviewed this manuscript. Their constructive and inspirational comments have helped me improve my work considerably.

If I were allowed to thoroughly elaborate all the invaluable guidance, boundless generosity and ceaseless trust that have been bestowed upon me by **Matteo Brunelli**, **Henrik Nyman**, **Piia Hirkman** and **Kaj-Mikael Björk**, the acknowledgements would easily reach double digits in number of pages. Matteo, a true friend, sets a wonderful example of professionalism, enthusiasm and good humour, and he taught me how to achieve the perfect balance between my research work and home life. Henrik, quite simply, is someone I have always looked up to as a figure of wisdom, integrity and dedication. The discipline and organization he has demonstrated at work have had a profoundly beneficial effect on the manner I conduct myself. Piia has always been willing to engage ideas and to listen when I needed to talk, and I have lost count of how many times Piia has taken time from her extremely busy schedule to rescue me from impossible situations. I owe a special debt to Kaj-Mikael for his patience, for his warmth and understanding, and for helping me survive the final stage of my thesis writing when the extremely tight time constraints almost got the better of me.

I would also like to thank the good friends that I have made in Åbo Akademi for their support throughout my PhD career. **Prof. Peter Sarlin**, **Samuel Rönnqvist**, **Shahrokh Nikou**, **Robin Wikström**, **Franck Tétard**, **Prof. Yong Liu** and **Eyal Eshet** have all contributed to my dissertation one way or another, and have made the past five years one of the most enjoyable work experiences of my life. My sincere appreciation is extended to my childhood friend **Xiaoning Ma**, who encouraged me to take the challenge of a PhD five years ago, which turned out to be the best decision I have ever made.

Last, but by no means least, I am indebted forever to my **Mum and Dad**, the authors of the author of this dissertation, and the only constant in my life.

I gratefully acknowledge that this work has been supported by:

Stiftelsen för Åbo Akademi
Marcus Wallenbergs Stiftelse för Företagsekonomisk Forskning
Stiftelsen för Handelsutbildniing i Åbo
Jacobssons Fond
Willy Gielens Fond

Xiaolu Wang
Åbo, 25.05.2015

List of original publications

Paper 1 Mikael Collan, Robert Fullér, Xiaolu Wang and József Mezei. Numerical patent analysis with the fuzzy pay-off method: valuing a compound real option, in: Business Intelligence and Financial Engineering (BIFE), 2011 Fourth International Conference on, pp. 405-409. IEEE, 2011.

Paper 2 Mikael Collan, Robert Fullér, Xiaolu Wang and József Mezei. Patent Evaluation with a Numerical Real Option Method, in: Proceeding of The 2nd International Conference on Management Science and Engineering (MSE 2011), Advance in Artificial Intelligence, vol. 1, pp. 685-690. 2011.

Paper 3 Xiaolu Wang and Christer Carlsson. Patent related decision making with fuzzy real option analysis. Accepted to *International Journal of Mathematics in Operational Research*. ISSN: 1757-5869

Paper 4 Xiaolu Wang. Patent valuation with a fuzzy binomial model, in Fuzzy Systems (FUZZ), 2011 IEEE International Conference on, pp. 579-583. IEEE, 2011.

Paper 5 Xiaolu Wang and Christer Carlsson. Discovering the value of a patent licensing opportunity with a fuzzy binomial model. Accepted to Fuzzy Systems (FUZZ), 2015 IEEE International Conference.

Contents

List of Figures

List of Tables

Part I

Research Summary

Chapter 1

Introduction

Intellectual property (IP) is a twofold concept. In a broad sense, it refers to the creations of human mind. The World Intellectual Property Organization (WIPO) defines IP in the following way [3]:

> Intellectual property relates to items of information or knowledge, which can be incorporated in tangible objects at the same time in an unlimited number of copies at different locations anywhere in the world. The property is not in those copies but in the information or knowledge reflected in them.

IP also refers to the legal rights which stem from intellectual activity in the industrial, scientific, literary and artistic fields and are enforced by intellectual property law to safeguard the interests of creators and other producers of the intellectual goods and services [5]. In other words, IP also protects rights to ideas by protecting rights to produce and control physical instantiations of those ideas [62]. It is therefore a summarizing designation for the creations of mind and the exclusive rights which are granted to their creators.

IP is traditionally categorized into two subclasses, namely copyright and industrial property:

- Copyright includes literary and artistic works, films, music and architectural design. Rights related to copyright refer to those of performing artists in their performances, producers of phonograms in their recordings, and broadcasters in their radio and television programs [4].

- Industrial property, very broadly, covers patents for inventions, utility models, industrial designs, trademarks, service marks, commercial names and designations, semiconductor protection and protection against unfair competition. As a reward for creativity, industrial property right confers on creators a monopoly privilege which excludes others.

The oldest evidence of industrial property can be traced back to the 15th century, which is a statute issued by the Republic of Florence in 1421 to architect Brunelleschi who was awarded the exclusive right for three years to build a ship with a lifting apparatus for transporting marble. The first statute of modern copyright is generally considered to be the Statute of Anne issued in 1710 in Great Britain, which granted authors protection on what they produce for fourteen years, with a possible fourteen-year renewal on the condition that the author be still alive. In general, these statutes, together with other parallel legislation, although varying in different geographical regions, have been developed and refined for two major reasons. One is to incentivize further technical developments through publication and to encourage fair trading. The second is to provide those who are intellectually creative with the moral and economic rights in their creations.

1.1 Patent and European patent system

Among the aforementioned instruments and methods existing to appropriate returns from innovation is patent, which has played a very important role in humankind life and, in fact, pervaded every aspect of it, from telephone (patents held by Alexander Graham Bell) and first computer (patents held by Herman Hollerith), to insulin (patents held by the University of Toronto) and holography (patents held by Gábor Dénes). Additionally, as the global economy transitions from a manufacturing-based economy to a more knowledge-based economy, patent is more than just legal instrument for a company, but lies at the very heart of its strategy for value creation and is thus strategically exploited for economic profit and competitive advantage [65]. For example, from 2003 to 2011, Kodak pocketed over $3 billion on licensing revenues from some 20000 patents it owned; in 2011, a consortium mustering Apple, Microsoft and RIM acquired Nortel's portfolio of 6000 wireless communication patents at a staggering price of $4.5 billion [34]. This can also be proved by the fact that the number of patent applications has seen a drastic increase since the 1980s (see Figure 1.1). We shall next discuss the principles concerning patent and the necessity of patent valuation, and brief patent appraisal from a quantitative perspective.

1.1.1 What is a patent?

A patent is an exclusive industrial property right which provides protection for an invention to the owner of the patent for a limited period of time.

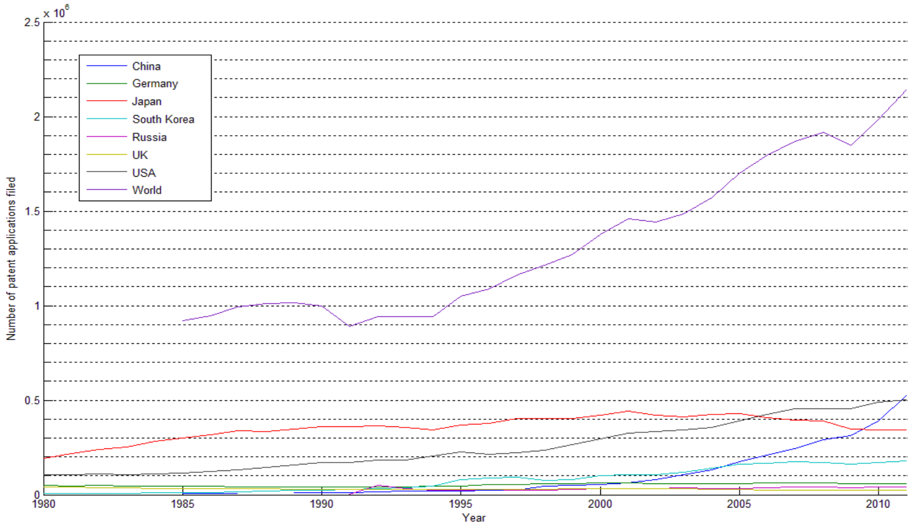


Figure 1.1: Patent application development from 1980 to 2011 (Source: WIPO)

It empowers its owner to exclude others from commercially making, using, distributing or selling the subject matter of the invention in the territory for which it has been granted without the patent owner's consent. In return for the right of exclusion, the patent holder is obliged to publicly disclosure information on the protected invention so that it could act as a viable point of departure for further industrial development. In other words, a patent serves as a contract between the inventor and the public, which exactly reflects the original intention to promote IP.

It should be emphasized, however, that a patent does not automatically entitle its owner to practice the patented invention. Instead, he is given the right to prevent others from commercially exploiting his invention and entitled to sue anyone who does so. For example, it is often the case that the subject matter of a newly granted patent is merely an improvement on earlier inventions which are still under protection of others patents. Therefore, the invention cannot be practised unless a licensing agreement is reached.

1.1.2 Patent application and patent commercialization

Chronologically, a patent is comprised of the patent application process and subsequent patent commercialization which is also referred to as the patent exploitation. We shall next elaborate them individually, with a focus on the key decisions which require attention from the patent applicant and patentee.

European patent application

Unlike copyright, a patent does not come into force instantaneously when the underlying technology asset is created. Every applicant has to go through a sophisticated, sometimes lengthy application process. We devote this subsection to the discussion on the feasible paths a European patent application could follow, which includes the outline of the procedure involved in applying for a European patent through two different filing routes, namely the European route and the Euro-PCT route, and the corresponding costs or fees required at each stage in the application process.

European route As its name implies, the European route is a single European procedure for the grant of patents on the basis of a single application filed in the European Patent Convention (EPC) contracting states.

A European patent is valid for twenty years starting from the date of filing, provided that all the application fees and annual maintenance fees are appropriately paid. However, this does not mean that a European patent would come into force instantaneously at the first filing as some other IPs, for example, copyright. A series of sophisticated examinations concerning the patentability of the patent application and the underlying invention have to be gone through, and the entire European patent grant procedure could take up to five years from when the application is filed.

Generally speaking, the European patent grant procedure consists of two consecutive stages, the first of which "comprises formalities examination, search report preparation and the drafting of an opinion on whether the application and the invention to which it relates seem to meet the requirements of the EPC" [2] and ends with the publication of the European patent application and the search report, which is followed by, at the applicant's request, substantive examination conducted in the second stage. Figure 1.2 summarizes and illustrates the procedure for the grant of a European patent.

Euro-PCT route The other route is designed for those applicants interested in pursuing the procedure under the Patent Cooperation Treaty (PCT) before the European Patent Office (EPO) and seeking patent protection for their inventions simultaneously in each of the Contracting States. In the Euro-PCT route the EPO practically acts in its capacity as a patent application receiving office, an International Searching Authority (ISA), a Supplementary International Searching Authority (SISA), an International Preliminary Examining Authority (IPEA) and a designated office simultaneously. The Euro-PCT route could accordingly be broken up into two phases, namely the international phase and the subsequent European phase. Please refer to Figure 1.3 for a detailed illustration of the Euro-PCT route and the major costs required.

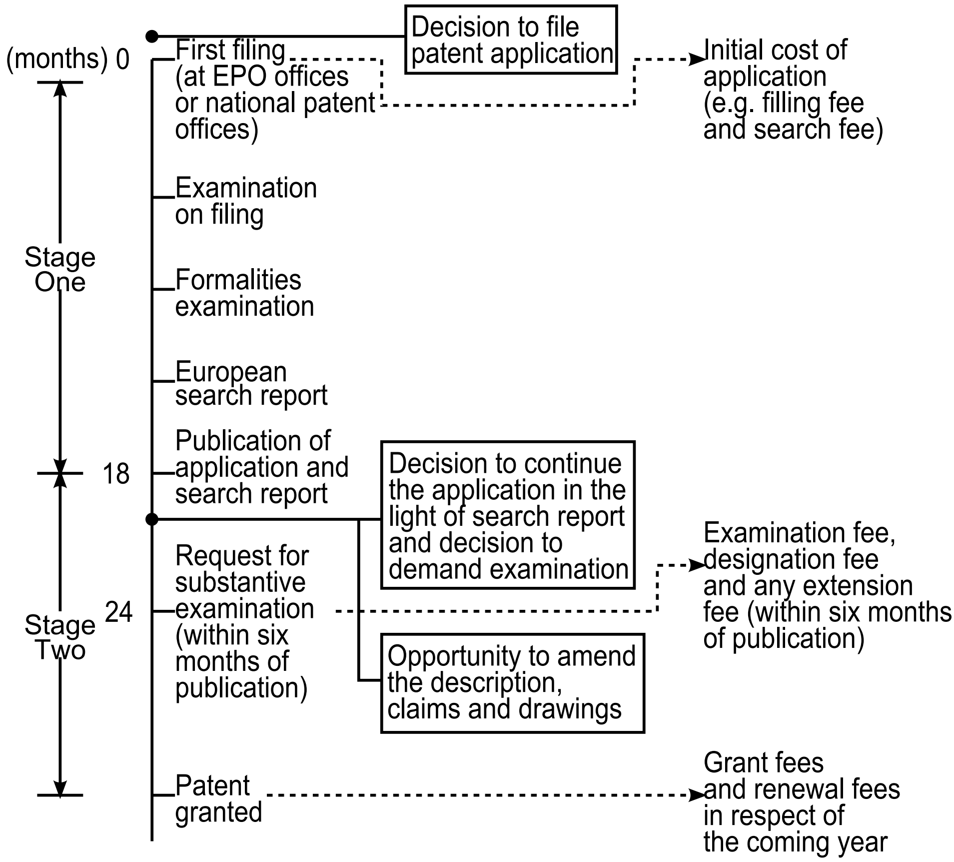


Figure 1.2: European route for patent application

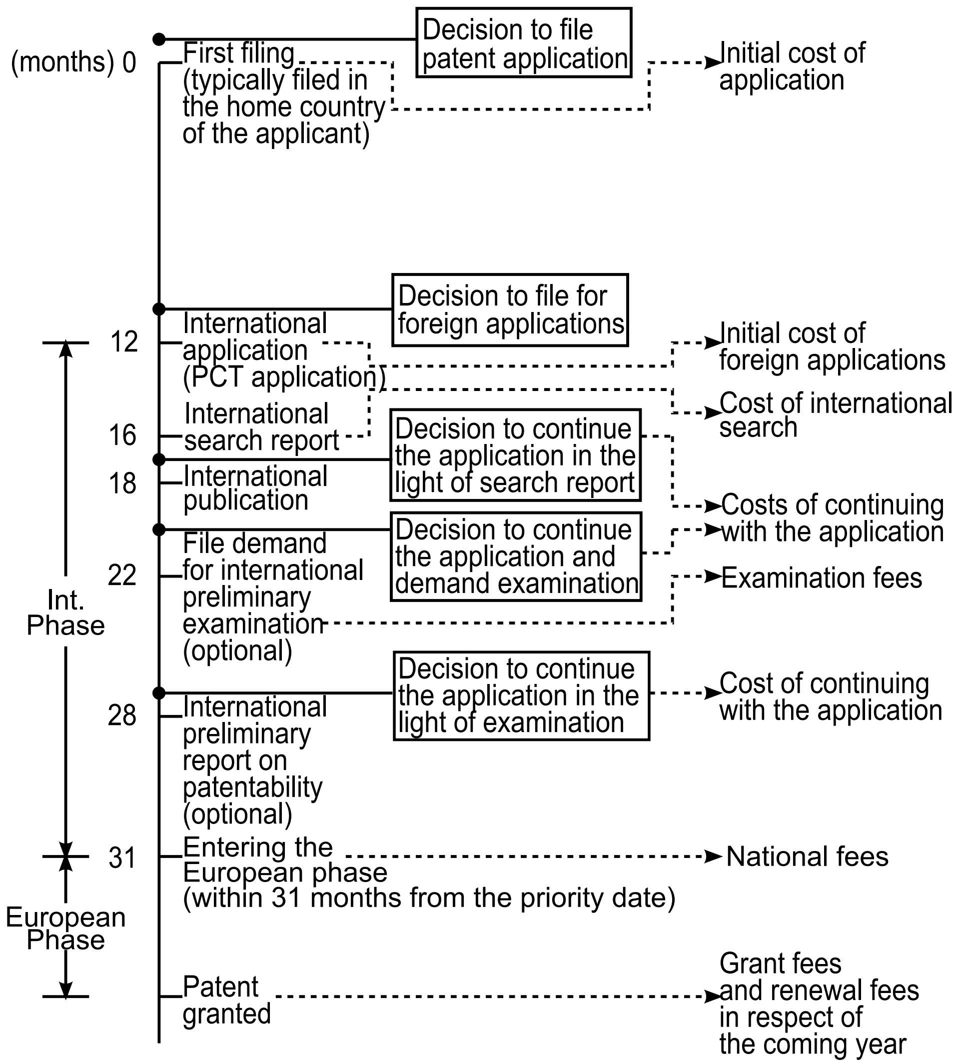


Figure 1.3: Euro-PCT route for patent application

Within 12 months from the filing date of the first patent application, which is typically a national patent application in the home country of the applicant, an international application (the PCT application) could be filed with the EPO if the applicant is a national or resident of a Contracting State of the PCT which is also party to the regional office. The international application is then subjected to an international search, the results of which are issued in the form of an "international search report", consisting of a listing of the citations of published documents that might affect the patentability of the claimed invention in the international application. The international search report, together with a written opinion on patentability conducted by an International Searching Authority (ISA), will be then communicated by the ISA to the applicant, who now has the option to withdraw his or her application, in particular when the said report or opinion suggests the application be rejected. If the applicant chooses to proceed, both the application and the international search report excluding the written opinion on patentability would be published, after which the applicant may choose to order or request for an international preliminary examination of the underlying patent application.

Within a time frame of 31 months from the priority date, the applicant would have to decide the countries where he or she wishes to enter the national phase before the EPO, after which the international application is said to be in the European phase [1] and the EPO will carry out the examination of the patent and determine whether it could be granted.

Patent commercialization

Once a patent has been granted, its holder could immediately proceed to the commercialization phase. Generally speaking, patent exploitation could be classified into five categories according to the level of intensity (see Figure 1.4), which are patent exploitation for defence, patent exploitation for securing superiority, patent exploitation as business strategy, patent exploitation as management strategy and patent exploitation as financial assets [74].

- Patent exploitation as defence, as its name implies, refers to the passive strategy adopted by a company to use patent as prevention tool to block its competitors from developing rival technologies.

- Patent exploitation for securing superiority includes such measures as taking legal action against infringer, and expanding upon existing knowledge, which is frequently referred to as inventing around.

- The next level of exploitation could be seen as the extension of the one of securing superiority as it formalizes the realization of patent as a legal weapon and regards it as a source of profit.

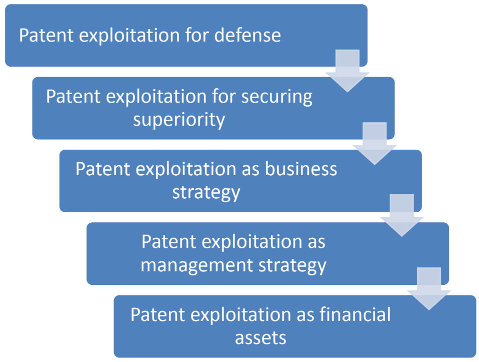


Figure 1.4: Evolution of patent exploitation

- The use of patent has also evolved to exploitation as management strategy and business asset. For example, there is a common tendency among firms to license out their patents to exact as much profits as possible from their protected innovation, and to gain external knowledge through inward licenses; research further suggests that a strong patent portfolio can boost the bargaining position of technology-intensive small firms and start-ups in negotiation with larger firms.

- In recent years, the new trend of patent as financial asset has gained great popularity among not only inventors and science and innovation-based business but "slick investors, from hedge funds and private-equity firms to venture capitalists and even distressed-debt funds" [89]. The emergence of patent as alternative investment, however, came as no surprise. On one hand investors from financial industry seek to profit from arbitrage trading by taking advantages of the pricing inefficiencies in illiquid patent market and the practical difficulties in valuing patent; those on the sellers' side, such as individual inventors and universities that lack the required resources to market their innovation, firms filing for bankruptcy and technology providers with a large amount of unutilized patents, are increasingly looking to cash in their patents as external sources of finance.

1.2 Patent valuation

1.2.1 The need for patent valuation

As was noted in the preceding work, the shift towards a knowledge-based economy has prompted the evolution of patent exploitation, which has been witnessed in the form of heightened propensity to patent, or in other words

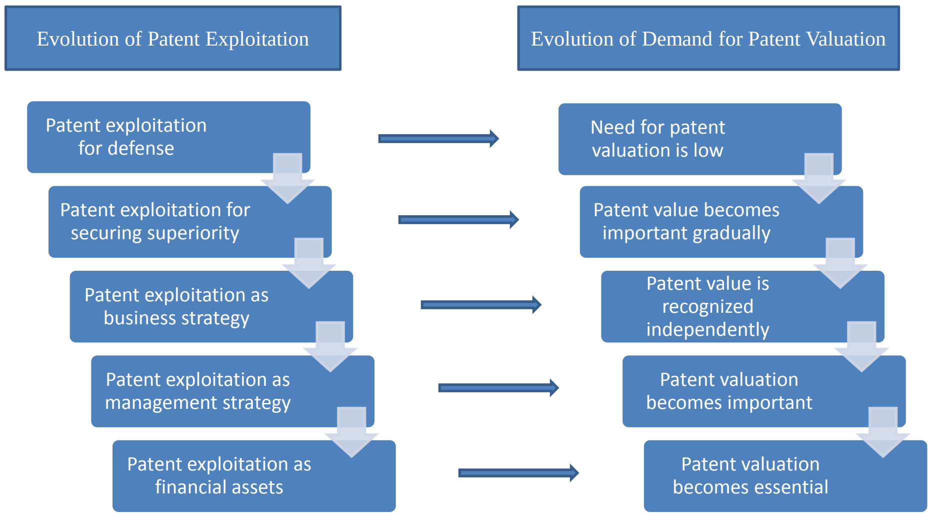


Figure 1.5: Evolution of patent exploitation and implication for patent valuation at each stage

a surge in the number of patent applications filed, over the last two decades. Along with the evolution of patent exploitation, the demand for systematic patent valuation has arisen. As can be seen in Figure 1.5 [74] although such need does not seem to be persuasive in the situation when patent is merely adopted as a defensive tool, it is undeniably evident in the cases of patent exploitation as management strategy and the financialization of patent. For example, it is crucial for IP managers to know the value of their patents when determining the royalty rates for licensing agreements. It has also been proved that patent has a positive impact on acquisition premium when estimating the value of a merge and acquisition deal. For example, out of the total purchase price of 5.44 billion euros for the Microsoft-Nokia deal, 1.65 billion relates to the mutual patent agreement and future option granting Microsoft the right to extend this mutual patent agreement in perpetuity. Those in pursuit of external sources of financing, whether they are new technology-based firms collateralizing their patents for bank loans or securitizing them to raise capital, or start-up firms seeking venture capital, are also in great need of rigorous techniques for determining patent value, preferably in monetary terms, to facilitate their exploitation.

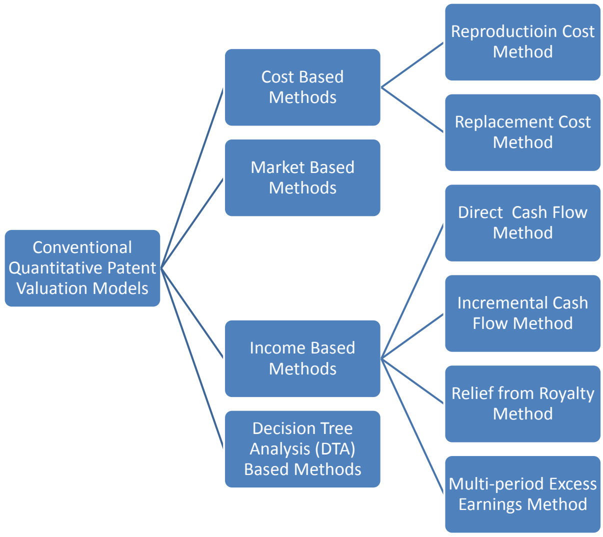


Figure 1.6: Classic quantitative patent valuation models

1.2.2 A review of the conventional quantitative models for patent valuation

As a consequence of the increased awareness of patents as means to appropriate innovation rents and growing enthusiasm for reliable quantitative patent valuation methods witnessed within not only practitioners such as IP managers, accountants, consultants, financial analysts and venture capitalists, but also the academic, a variety of approaches have been proposed for different evaluation circumstances. We shall in this subsection provide a brief introduction to the four prevailing methodologies as shown in Figure 1.6, namely the income based methods, cost based methods, market based methods and decision tree analysis (DTA) based methods, with a specific aim to pinpoint their limitations. Before proceeding with the elaboration of the major quantitative patent valuation methods, it is important to point out that such methods are primarily used in accounting, management science and operations research and the following work would by no means serve as a comprehensive survey of all patent valuation methods that exist. Methods based on a qualitative approach have been briefed in, for example, Lagrost et al. [51], Munari and Oriani [65] and Bessen and Meurer [10].

Cost based methods

The cost based methods are derived on the principle that an investor would not pay more for an asset than the cost to acquire similar benefits from another asset. They therefore determine the value of a patent by measuring either the reproduction (internal) cost or the replacement (external) cost. The cost of reproduction refers to the consumption of economic assets (tangible and intangible) needed in developing the exact replica of the invention protected by the patent being valued, while the cost of replacement equals to the expenditures required to obtain an invention which offers equivalent functionality to the patented invention subject to valuation but may fulfil the required tasks in a different way [27, 88].

Despite its ease of use, the cost approach is rarely adopted in patent appraisal. The crucial weakness impeding its application is that it does not account for any potential economic benefits associated with the subject patent, which limits their usability to such circumstances as early stage technologies from which little or no revenue could possibly be generated. Harald Wirtz [94] pointed out that, in practice, the cost approach normally only serves as a plausibility check of values calculated by other approaches, for example, the income approach. He also argued that the value determined by the cost approach could be used as the minimum since no rational investor would pay more for a patent than the price of an asset with the same utility.

Market based methods

The market based methods, which are also known under the names of transactional approach and comparison approach, use the price paid for a comparable asset in a very recent commercial transaction as a basis to determine the value of the subject asset being valued.

Compared with the income and cost based methods, the market based methods seem to be relatively easy to use, but much work is still required in adapting the transaction prices collected. As was noted above, the applicability of such methods hinges upon two prerequisites, namely a transparent market for or the availability of transactional data on the comparable asset and the comparability between the “outside evidence” [27] and the asset subject to valuation, neither of which, however, could be easily satisfied in the context of patent. First, the fact that there is no one well-established intermediary which facilitates transactions between patent creators and patent users in the similar manner as London Stock Exchange or New York Stock Exchange in equity market, and that the market for patent has long lacked liquidity and transparency due to, for example, the portfolio effects, high search costs on both sides of the patent market and risk of litigation make it difficult to find comparative assets and accessible transaction prices. Second, every patent is unique by nature as it or rather the innovation it protects must exhibit an element of novelty, that is, some new characteristic which is not already covered in the prior art, which, to a certain extent, limits its comparability with others and further exacerbates the difficulty of patent transaction. It is therefore important to make necessary adjustments for comparability between the subject patent and its surrogate when the difference between them is not negligible. A number of factors affecting the comparability should be taken into consideration, among which are the lifespan, inventive step/distinctive feature, scope and geographic coverage of the subject patent and the comparable patent, and the comparability of the markets they will respectively be exploited in. Besides, due to the fact that confidentiality agreement surrounds most of the patent transactions, one has to bear the risk that the transaction price used does not fully reflect the true value of the comparable asset as the result of, for example, a forced agreement. In addition, quantifying all the aforementioned information, which is normally realized in practice by crumbing everything into one single number referred to as the relevant multiple, can by no means be said to be easy.

Income based methods

The income based methods are based on the discounted cash flow (DCF) theory, which, loosely speaking, defines the asset value as the present value of the anticipated future incomes to be generated from the subject patent throughout its expected economic life. Among all the derived methods that reside under the umbrella of income based approach, the direct cash flow method, incremental cash flow method, relief from royalty method and multi-period excess earnings method are the most widely accepted and used ones.

- The direct cash flow method, as its name suggest, uses the income flow which is directly attributable to the underlying asset. It is therefore applicable to the case where the subject innovation is not implemented in production process by its owner but is licensed out to a third party. The licensing fee can then serve as the direct cash flow in the valuation.

- Incremental cash flow method compares the estimated future cash flows from the target company holding the technology asset which is under the protection of the subject patent being valued, with the cash flows from a comparable company, ideally in the same sector, which has no possession of such asset. The difference in the cash flows per period between them is then discounted to its present value with a discount factor which is appropriate with respect to risk and time before added up together. The cumulative sum plus the present value of the tax amortization benefit (TAB) serve as the value of the subject patent.

- Relief from royalty method, as its name implies, suggests that the fair value of a patent be determined upon the royalty income attributable to it. The principle behind this method is that the value of a patent equals to the cost its proprietor saves on owning it instead of leasing it. It is therefore important to emphasize that the royalty income in this context does not refer to the license fees receivable but rather the cost saving on royalties which the target company would have had to pay to a third party licensor if it did not own the subject patent. Accordingly, it requires the estimation of revenues attributable to the technology asset over a certain period and the appropriate royalty rate the company would have had to pay in exchange for an inward license, which could be determined based on available market data for licensing agreements involving similar assets, industries and other characteristics. The present value of the projected royalty payment for each year, capitalized at an appropriate discount rate, is then calculated before added up together with any tax benefits. Since the relief from royalty method bears a striking resemblance to the market approach as it also requires the access to information on historical transactions of reference value, it unfortunately has to share the same problem of poor data availability as the market approach.

- Multi-period excess earnings method (MPEEM), in general terms, looks at the present value of the net cash flows stemming from the asset being valued for its remaining expected lifespan. These net cash flows are those in excess of fair returns on all the contributing assets which are necessary to the realization of the cash flows. In other words, the MPEEM isolates the cash flows attributable exclusively to the subject patent from the overall cash-flow stream. To this end, a contributory asset charge, which is also known as the fictive fee and could be considered as "rents" paid to a hypothetical third party for borrowing all the contributory assets, is made against the overall cash amount. The resulting "excess-earnings stream" [66] plus any tax benefit then comprise the patent value. The multi-period excess earnings method is similar to the relief from royalty method in that the latter requires the fictive royalty for the subject patent while the former finds the fictive fee for all other contributory assets. As a result of its specific usage, the multi-period excess earnings method is recommended only if the patent under valuation contributes the most to the cash flow, otherwise the bandwidth of assessment would be too large and the estimation accuracy too low [94].

As was noted, the reliability of the aforementioned techniques rests with two major factors, which are the selection of discount factor and the estimation accuracy of the overall future cash-flow stream which consists of the total inward cash flows attributable to the subject patent and all direct and indirect costs associated with it. This, however, can by no means be said to be an easy task. First, the discount rate applied is affected by several factors including inflation rate, asset liquidity, real interest and risk premium. Second, the cash-flow estimation demands not only a comprehensive grasp of the attributes of the specific property but substantial knowledge of "the competitive and economic environment in place during the appropriate time frame for the valuation" [57]. Finally, the data of patent-related transactions is rarely published.

One major drawback of the income-based approaches is the usage of one single discount factor, Ideally, the discount factor is supposed to reflect the risk of the cash flow concerned. Therefore, the fact that the any patent or patent application involves cash flows occurring at multiple stages, each of which is associated with unique risk characteristic, hampers the use of a constant discount factor which implicitly assumes that the risk associated with the patent cash flows increases at a constant rate over time.

Decision tree analysis (DTA) based methods

In addition to the difficulties of selecting appropriate discount rates that reflect the risk inherent in the various stages of a patent's lifespan and estimating the corresponding future cash flows, the income based methods, or rather the DCF methods fail to account for the various possibilities available to patent managers. For example, a patent could be abandoned at any stage into the post-grant phase, and even an ongoing patent application at its final phase could be withdrawn. Additionally, any patent manager has to face the decision as to whether or not to file for foreign applications.

The decision tree analysis (DTA) based methods attempt to take into consideration the possibility of later decisions by "mapping out all feasible alternatives managerial actions contingent on the possible states of nature (chance events) in a hierarchical manner" [92] and assigning each possible future event a probability of occurrence. Management can then select the alternative which is consistent with the objective of value maximization. Despite its clear theoretical advantage over the DCF analysis, the DTA would still require, for each stage of a patent, a discount rate which is appropriate to the risk level in that stage. As was discussed in the previous subsection, the risk level and hence the corresponding discount rate is most likely to vary through time.

1.2.3 Problems of the conventional quantitative models for patent valuation

The scope of the revision would not be restricted to the characteristics of their weaknesses or utilities. What really interest me here are the fundamental causes behind them. Accordingly, I claim that there are five major issues that limit the usability and reliability of the classic approaches for patent appraisal. Although these causes have been partially disposed in our preceding discussion, we shall next re-emphasize them from a more theoretical perspective which we hope could also serve as the formal foundation for our future work.

Illiquid market for patents

Compared to patents which have been around for hundreds of years, the patent market, in which patents are isolated from the innovation they protect and considered as tradable assets and which emerged along with the evolution of the knowledge-based economy and the accompanying recognition of patents as alternative investments, is still in its infancy. This is best exemplified by the comparison given by an IP lawyer based in Silicon Valley whom Monk interviewed for his work [61]. In the interview, this IP lawyer noted "the market for patents is real and it is growing, but describing it to outsiders is like trying to describe financial derivatives in 1982". Indeed, while growing market activity has been reported and data collected [32, 34], there is still no one well-established and documented market for patents which bears any resemblance or comparability to the financial markets in London, New York and Chicago, and the existing patent market is arguable one of the highly inefficient and illiquid markets in the economy.

Studies that provide supporting evidence for this claim include [31, 44, 61], in which the following fundamental factors have been proven to contribute to the illiquidity of the patent market:

- The infungibility or rather the lack of comparables of a patent impedes transactions between potential patent owners and patent buyers. On one hand, it could be extremely time-consuming for patent owners to find all the potential applications for their patents. On the other hand, with millions of granted patents in circulation, it requires not only time but also industry specific knowledge and technical expertise from patent buyers or users to conduct patent search and discover those that might cover the technology embedded in their products. What is worse, the fact that patent applicants tend to disclose only the minimal information demanded by the patent-issuing authorities only exacerbates these difficulties. As a result, the high search costs on both sides of the market greatly limit the usability of any comparable transaction based analysis, such as the market approach.

- The value of a patent is subject to strong complementarities and portfolio effects. This is particularly evident in the technology intensive industries as most of their products, whether it is a particular semiconductor chip or a smart phone, are covered by dozens or even hundreds of "interdependent patents". For example, the curvy corners of an iPhone, its mail push function, the "gesture entry techniques" [16] which enable device unlocking by swiping fingers across its screen and the word-recommendation feature are all under protection of patents. The complementarities and portfolio effects would therefore dampen the incentive of potential buyers to purchase or cross-license an individual patent and undermine its value unless "it complements a portfolio that the potential buyers already own" [31], which, however, is rarely the case for a given patent and hence reduces its liquidity as an asset.

- The risk of litigation has pervaded and affected every economic aspect of patents, from transaction to valuation, and has in fact aggravated all the aforementioned issues. For example, the fear of infringement suits from large operation companies among individual inventors and small firms would further jeopardize their enthusiasm in monetizing their patents due to the lack of bargaining leverage in negotiating, for example, a cross-licensing agreement with the well-established firms. This is because individual inventors and small-sized establishment usually lack a large portfolio and possess very limited financial resource and legal expertise. Their counterparties would therefore seek to require large royalties for the deal or simply threat to sue for infringement, none of which they could possibility afford [40]. All in all, the prevalence of patent litigation makes potential market participants very cautious.

Poor data availability

Strictly speaking, the poor availability of quantitative data regarding patent transactions is another major contributory factor to the illiquidity and inefficiency of the patent market. Nonetheless, the author has extracted and devoted a whole subsection to it as a traceable and transparent transaction history of the activities on the market is of fundamental importance in valuing any tradable assets, especially patents. Indeed, each of the abovementioned patent valuation methods to some extent requires the projection of future cash flows of a target patent or the transaction data of a comparable asset. For example, both the direct cash flow method and incremental cash flow method would need a complete breakdown of the future income to be generated from the patent from its grant until lapse; an accurate prediction on the total costs required in both the patent application process and the post-grant phase would also be a prerequisite in some cases; the reliability of the market based approaches rely heavily on the availability of the transaction price of its reference object, which is usually just one of the many elements of a large transaction and hence not individually appraised.

The biggest obstacle that impedes the acquisition of such data is the confidentiality agreements surrounding patent-related transactions. For example, the seller of a patent could be constrained from disclosing the licensees or any details of the existing licensing agreement; the buyer sometimes choose to remain anonymous in order not to expose what they are after to other industry participants [9, 61].

Furthermore, the grant or acquisition of patents may influence the end-use market in non-stochastic ways and have a remarkable impact upon, for example, market strategies, technology strategies, competitive positions and business models [15]. Therefore, the simplifying assumption that the patent-related future cash flows are purely stochastic does not seem well-founded.

Estimation biases

As a result of the poor data availability of patents, the accuracy of the estimation on their future cash flows usually lies heavily on the judgements from patent experts, which include the inventor who usually has a clear idea of how significant and advanced the invention is compared to other technologies in the same field, the patent agent who is responsible for filing the patent application and usually has "a view of the scope and quality of patent protection that might be obtained" [79], and those responsible for marketing the underlying invention who are capable of assessing its market share, the monopoly benefits from patent protection whether directly through sales or indirectly through licensing and other economic benefits resulting from the exploitation of technology including greater efficiency in production, improved quality, lessened environmental and safety hazards, etc. It is very likely that there would be great discrepancies in their cash flow estimations.

Lack of accounting for changing risk

Both the conventional DCF based patent valuation methods such as the income approach and the DTA based methods suffer from the rule of thumb of "one single constant discount factor". In fact, the same flawed assumption could also be witnessed in the cost and market based approaches as some sort of discounting would be inevitable in either case.

Generally speaking, conventional DCF based methods for patent appraisal implicitly assume the existence of an equivalent investment alternative, such as the stock of a publicly traded company engaged exclusively in the same type of business as the patent proprietor, which has a similar (i.e., perfectly or highly correlated) risk profile, compared to the market, as the patent under consideration. The expected rate of return demanded by an investor in such a "twin security", which can be determined by the after-tax weighted average cost of capital (WACC), is then used as the opportunity cost of capital at which the future cash flows of the subject patent would be discounted [87]. The after-tax WACC is given by

$$\text{WACC} = r_E \times \frac{E}{V} + (1-t) \times r_D \times \frac{D}{V}$$

with

$$\begin{aligned} r_E &= \text{cost of equity} \\ r_D &= \text{cost of debt} \\ t &= \text{tax rate} \\ E &= \text{market value of equity} \\ D &= \text{market value of debt} \\ V &= \text{market value of total capital.} \end{aligned}$$

The cost of equity r_E is usually obtained on the basis of Capital Asset Pricing Model (CAPM), which is given as follows:

$$r_E = r + \beta \times (r_M - r)$$

with

$$\begin{aligned} r &= \text{risk-free rate} \\ r_M &= \text{return on the market portfolio} \\ \beta &= \text{the asset's systematic risk.} \end{aligned}$$

The fact that the CAPM shown above follows logically from the Markowitz model which is designed to find a minimum-variance portfolio for a single-period investment makes its use in a multi-period scenario questionable [94]. In particular, by applying the WACC for the evaluation of a patent, the manager is implicitly assuming that the β of the underlying patent is identical to the one of the twin security. Myers and Turnbill [71] pointed out that the β of a project could be influenced by various factors, including its lifespan, the pattern of its expected cash flows and "the characteristics of any individual underlying components of these cash flows" [92]. A patent as well as a patent application usually involves a multi-stage cash flow and the risk associated with the cash flow is very likely to vary through the lifespan. Therefore, for the sake of rigour, the β and hence the discount rate of a patent cannot be assumed to be constant throughout its entire life cycle. This claim is also supported by Pitkethly [79], who compared a newly granted patent which is about to face its first litigation with a veteran which is 15 years into the post-grant phase and has survived multiple attempts to invalidate it, and claimed that the use of a single constant discount rate actually violates the fact that the former patent would be much riskier than the latter.

Ideally, a sequence of distinct risk phases together with a sequence of discount rates, each of which reflects the risk level associated with the corresponding phase, need to be worked out. That is to say, the valuation of a patent should be split into at least the application phase and exploitation phase, each having a unique discount rate. Intuitively, a more sophisticated breakdown would be preferred. For example, the application phase could be further split into the phase from filing the patent application to the publication of search results, and the phase from the decision to continue and demand substantive examination to its grant. The split of the exploitation phase depends on the commercialization strategy and is hence not exclusive. For example, it could be resolved into the phase from grant to the beginning of commercialization, the period until the intellectual asset becomes well established and the remaining life of this patent until expiry; or it could simply be resolved into a sequence of annual phases. Choosing an appropriate future scenario for a patent, however, can never be said to be an easy task as it might requires a projection of as much as 20 years into the future. Alternatively, some methods that are capable of dealing with changing risk, such as means of risk neutral valuation would be highly appreciated in this context. I shall come back to this point in the following work.

Neglected managerial flexibility

Managerial flexibility refers to the discretion available to management to adjust its operating strategy in response to the ever-changing market conditions. Such flexibility permeates capital investment-projects and could also be witnessed in patents. Recall from Section 1.1.2, both the patent application process and post-grant patents are subject to a wide range of decisions at multiple stages. For example, an applicant is required to decide whether to file for a foreign application after initial filing and whether to continue application after the search report is published; once the patent is granted, its proprietor would be confronted with the decision as to whether to license, sell or abandon the patent on a regular basis until its expiry. Therefore, any IPR manager is faced with a large variety of courses of action as to how their patents can be managed and thus has to accept a high degree of uncertainty concerning their eventual value.

However, except the DTA based methods, none of the aforementioned patent valuation methodologies manages to build in the value of such flexibility frequently encountered in a patent:

- The cost based methods completely overlook the economic potential of a patent, let alone any managerial discretion.

- The market based methods rely heavily on the exterior evidence, which is the transaction data of a similar intellectual asset recently traded. Even if the value of managerial flexibility is reflected in such data, it is hardly possible to know how much of the transaction price is contributory to it or if the managerial strategy of the comparable asset is applicable to the patent subject to valuation.

- In the standard DCF analysis on which the income based methods are based, an implicit assumption concerning the expected scenario of cash flows is made and an irreversible managerial commitment to a certain operating strategy is presumed in the first instance [92]. The underlying project, such as a patent application, would be immediately accepted on the condition that the resulting net present value (NPV) proves to be positive. That is to say, the decision could only be made now or never. However, it is very likely that the realization of cash flows would differ from the estimated scenario as a result of the possible highly volatile market environment and unexpected competitive interactions, which forces the management to actively adjust its original operating strategy in the light of market evolution and gradual resolution of uncertainty in order to "capitalize on favourable future opportunities" [92] or avoiding potential losses. Obviously the standard DCF analysis is vulnerable to such unexpected market movements and hence incapable of accounting for any embedded managerial flexibility, which greatly limits its feasibility in valuing a patent and a patent application. For example, an IPR manager needs to actively review the technology areas where similar innovation to his own could be developed and at the same time look for any possible conflict with the existing IPs of others, and justify the continuance of his ongoing patent application accordingly.

Among all the improvements made on the standard DCF approach is the DTA, which, as was discussed earlier, allows some account to be taken of the possibilities available to decision makers by assigning each alternative (or branch) a probability of occurrence and processing each branch with the DCF analysis. However, the DTA still does not solve the issue of changing risk and the selection of probabilities of occurrence has proved to be difficult.

1.2.4 Discussion

As was noted above, the major challenge faced by us is to develop additional methods to improve on the standard DCF approaches, which on one hand are able to provide us with more numerically sophisticated analyses for patenting decision making and patent valuation, and on the other hand retain both academic rigour and practical relevance.

Chapter 2

Research problems and methodology

2.1 Research problems and the structure of the thesis

As was revealed in the introductory chapter, there exist five major issues that hamper the usability and reliability of the conventional DCF based patent valuation methods, which are

1. the illiquidity of the patent market,
2. the poor availability of quantitative data regarding patent transactions,
3. estimation biases among patent experts,
4. lack of accounting for changing risk,
5. neglected managerial flexibility permeating both patent application and patent exploitation.

This dissertation attempts to overcome the aforementioned practical issues by rationalizing the use of two techniques, namely fuzzy set theory (aiming at problems (1) to (3)) and real option analysis (aiming at problems (4) and (5)). Additionally, the author is enthusiastic about applying a new methodology combining both techniques, namely the fuzzy real option analysis, to patent-related decision making problems and patent valuation.

Accordingly, the dissertation strives to answer the following questions first:

RQ1 What is the nature of the uncertainties inherent in patent cash flow estimation?

RQ2 Is fuzzy set theory a feasible uncertainty theory to capture the patent-related uncertainties?

RQ3 How to conceptualize and quantify the embedded managerial flexibility in a patent with real option analysis?

In addition, it attempts to investigate how the proposed techniques could be implemented in practice, that is, to exhibit:

- the application of fuzzy real option analysis in patent-related decision making problems,
- and the application of fuzzy real option analysis in valuing a patent licensing opportunity.

It should be noted that the research questions are not arranged arbitrarily. This is because the understanding of a given problem requires the understanding of the previous problems. Consequently, for the sake of logical coherence, this dissertation has been organized in such a way that each chapter is devoted to the investigation of one particular research question or problem in the same order as they are listed above. Dependencies, in terms of chapters, are illustrated in Figure 2.1. Additionally, the structure of the rest of this dissertation and the contribution of each subsequent chapter are summarized as follows:

- In Chapter 3, a close examination of the consequences caused by the first three aforementioned issues is provided, from which it will be concluded that there exist two levels of uncertainties which must be accounted for when estimating future cash flows of patent (*i.e.*, in response to **RQ1**). To select an appropriate uncertainty theory for their modelling, a discussion of the classification standards for uncertainty, in particular the Knightian distinction and the objective/subjective distinction, is first exhibited, which is followed by an explanation of why both levels of uncertainties encountered in patent cash-flow estimation fall under the categorization of subjective uncertainty. Having revealed their nature, the application of fuzzy set theory in modelling patent-related uncertain quantities can be convincingly justified (*i.e.*, in response to **RQ2**). The basic definitions concerning fuzzy set theory, fuzzy numbers and fuzzy arithmetic operations on fuzzy numbers are also provided.
- To answer the question of how to conceptualize and quantify the embedded managerial flexibility in patents with real option analysis (*i.e.*, **RQ3**), Chapter 4 first introduces real option analysis as an extension of financial option pricing theory to the evaluation of real asset investment, and familiarises the reader with this technique by demonstrating

its utility in capturing the management discretion conferred upon the holder of a construction warrant and the managerial flexibility inherent in a pharmaceutical research and development (R&D) project. It then proceeds to rationalize its application in the context of patent valuation. In particular, the analysis is first applied to patent application and patent exploitation which are treated as a sequential process before the focus is limited to the application process and post-grant phase of a patent, respectively. For example, the analysis suggests that a patenting decision making problem could be conceptualized as pricing a compound real option, the decision to commercialize a patent hinges on the value of an European-styled option, the opportunity to out-licence a patent in the post-grant phase resembles an American-styled switch option, just to name a few.

- Patent-related decision making problems are essentially equivalent to those of evaluating investment opportunities which involve initial costs and potential future benefits unfolding over time. Therefore, the real option values of a patent application and a patent commercialization project are what the applicant and patentee, respectively, would pay for the right to undertake the investment project with its inherent decision points [91]. It will be demonstrated in Chapter 5 how two fuzzy real option models, which are real option analysis based and are designed to work under fuzzy environment owing to vague and imprecise concepts frequently represented in decision data, could be employed to assist in patent-related decision making. The following original papers contribute to this chapter:

  Paper 1 Mikael Collan, Robert Fullér, Xiaolu Wang and József Mezei. Numerical patent analysis with the fuzzy pay-off method: valuing a compound real option, in: Business Intelligence and Financial Engineering (BIFE), 2011 Fourth International Conference on, pp. 405-409. IEEE, 2011.

  Paper 2 Mikael Collan, Robert Fullér, Xiaolu Wang and József Mezei. Patent Evaluation with a Numerical Real Option Method, in: Proceeding of The 2nd International Conference on Management Science and Engineering (MSE 2011), Advance in Artificial Intelligence, vol. 1, pp. 685-690. 2011.

  Paper 3 Xiaolu Wang and Christer Carlsson. Patent related decision making with fuzzy real option analysis. Accepted to *International Journal of Mathematics in Operational Research.*

- A patent licensing agreement is an important instrument used by firms to generate revenues and to stabilize the existing market structure.

From a real option analysis standpoint, the opportunity to out licence at any time in the post-grant phase is likened to an American-styled option. That is to say, the patent proprietor has the flexibility to choose between the self-generated profit without licensing and weakened monopoly benefits due to licensing plus the compensation generated through royalties. In Chapter 6, a fuzzy binomial model is proposed to determine the inherent option to license. This chapter is an extension of the following original article:

Paper 4 Xiaolu Wang. Patent valuation with a fuzzy binomial model, in Fuzzy Systems (FUZZ), 2011 IEEE International Conference on, pp. 579-583. IEEE, 2011.

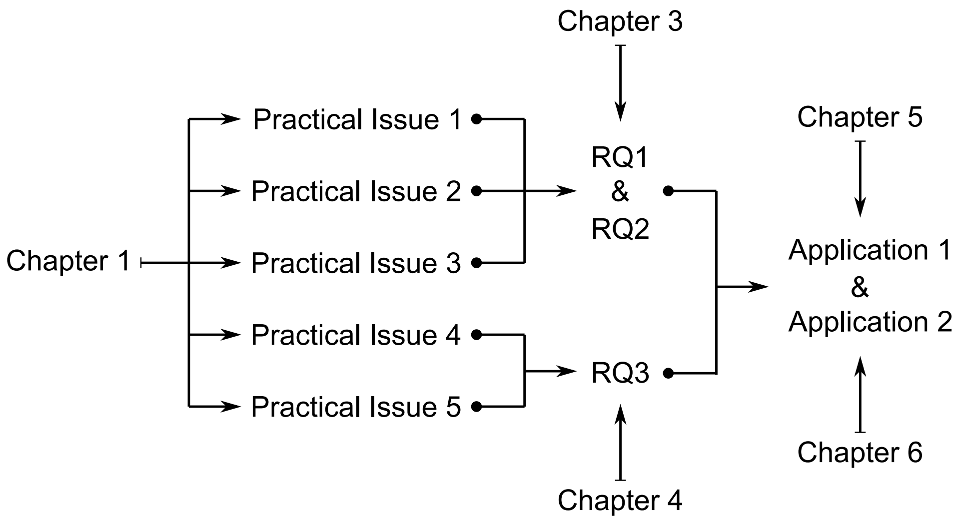
Paper 5 Xiaolu Wang. Discovering the value of a patent licensing opportunity with a fuzzy binomial model. Accepted to Fuzzy Systems (FUZZ), 2015 IEEE International Conference.

2.2 Methodology

This dissertation presents the results of interdisciplinary research, which borrows the skills and knowledge from both mathematical finance and operations research. In this section, I seek to give a flavour of the fundamentals underlying and the scope of each discipline first before incorporating them and thereby accentuating the theoretical positioning of this dissertation.

2.2.1 Mathematical finance

Fundamentally, mathematical finance is nothing but the modelling of risky asset prices; hence, it is generally agreed that mathematical finance was born at the outset of the twentieth century when Louis Bachelier presented his doctoral thesis named *Théorie de la spéculation* [7] in 1900, in which he introduced Brownian motion as a model for stock prices [41]. However, it was not until 1965 that Samuelson [83, 82] recounted and improved upon Bachelier's work by first pointing out that his model fails to ensure that stock prices always be positive and then introducing what has now become the standard model, the geometric Brownian motion. It is important to note that a large number of ground-breaking results with regard to mathematical and economic theories were also published within the same period, such as (see, *e.g.*, Jarrow and Protter [41] for a complete chronicle of mathematical finance and stochastic integration), which together formed the foundation and paved the way for the discoveries of the most famous Black-Scholes formula and the binomial model.



Practical Issue 1: the illiquidity of the patent market

Practical Issue 2: the poor availability of quantitative data

Practical Issue 3: estimation biases among patent experts

Practical Issue 4: lack of accounting for changing risk

Practical Issue 5: neglected managerial flexibility

RQ1: What is the nature of the uncertainties inherent in patent cash flow estimation?

RQ2: Is fuzzy set theory a feasible uncertainty theory to capture the patent-related uncertainties?

RQ3: How to conceptualize and quantify the embedded managerial flexibility in a patent with real option analysis?

Application 1: fuzzy real option analysis in patent-related decision making problems

Application 2: fuzzy real option analysis in valuing a patent licensing opportunity

Figure 2.1: Structure of the dissertation

It is largely accepted that the opening of the world's first option exchange (*i.e.*, Chicago Board Options Exchange) in 1973 and the publication of the eponymous formula by Black, Scholes and Merton [11] in the same year marked the birth of the financial derivatives market, which has now evolved into an industry that encompasses a bewildering variety of complex financial instruments and stands at the centre of modern global economic development. Our focus, however, is limited on the basic options that lie at the heart of any other exotic derivatives. An option, which is one of the most popular derivatives contracts deriving their value from an underlying asset or index such as a stock or interest rate, endows its holder with the right, but not the obligation, to buy or sell a certain amount of underlying assets at or before a specific date for a predetermined price. In particular, a *call* option gives its owner the right to *buy*, and a *put* option gives its owner the right to *sell*, at the fixed *strike price*; an option is labelled as *European* if it can only be exercised at *maturity*, while an option is *American* if it can be exercised at any time up to the maturity date.

The problem of option pricing is to determine the fair price to which both the option buyer and seller (*i.e.*, the writer) would logically agree at a given time (*e.g.*, at time zero). In other words, the option price could be considered as the premium, or loosely speaking, the "entry fee" [25], the buyer must pay to the seller in exchange for the right granted by the option; on the other hand, the writer of the option trades in the market, by using the premium as capital, so as to fulfil the later payment obligation and at the same time mitigate the exposure to fluctuation. In practice, a mathematical model is not a requisite in determining the prices of *exchange-traded* options as they are publicly quoted. *Over the counter* (OTC) options, however, are not closely regulated and are negotiated on a case-to-case basis. As a result of the varying features of OTC options, their prices are not publicly quoted and a model is required for their pricing.

A simple yet powerful option pricing model is the binomial model, which was coined by Cox, Ross and Rubinstein [19] in 1979 and has played "a decisive role in the development of the derivative industry and its easy implementation has given analysts the ability to price a huge range of financial derivatives in an almost routine way" [21]. Briefly speaking, the binomial model assumes the asset price $S(t)$ follows a random walk. That is, after one period of time Δt, the asset price $S(t+\Delta t)$ will either move up to $uS(t)$ or move down to $dS(t)$ with u and d being the jump factors. For a multiplicative n-period binomial process, the value of a European call option is given as

$$C = R^{-n} \sum_{j=0}^{n} \binom{n}{j} p^j (1-p)^{n-j} \max(u^j d^{n-j} S - X,\, 0),$$

where $\binom{n}{j} = \frac{n!}{j!(n-j)!}$ is the binomial coefficient, $R = \mathrm{e}^{r\Delta t}$, r is the risk-free interest rate, p and $1 - p$ are the risk-neutral probabilities and X is the strike price. If we define k to be the smallest non-negative integer such that $u^k d^{n-k} S \geq X$, the above equation could be simplified as

$$c = S\Psi(n, k, p') - XR^{-n}\Psi(n, k, p),$$

where $p' = \frac{up}{R}$ and Ψ is the complementary binomial distribution function defined as

$$\Psi(n, k, p) = \sum_{j=k}^{n} \binom{n}{j} p^j (1-p)^{n-j}. \tag{2.1}$$

As the period of time Δt approaches to zero, the price of the n-period European call option given in 2.1 tends to the classic Black-Scholes formula

$$C = S\Phi(d_1) - X\mathrm{e}^{-rT}\Phi(d_2)$$

where

$$\begin{aligned} d_1 &= \frac{\ln\left(\frac{S}{X}\right) + \left(r + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}, \\ d_2 &= d_1 - \sigma\sqrt{T}. \end{aligned}$$

It is important to point out that a lot more has been left out of this subsection than has been included. For example, much detailed work is skipped in the above derivation and, as a matter of fact, the Black-Scholes formula could be derived by multiple methods (see, *e.g.*, Dana and Jeanblanc [20]).

As was noted, an option, or any derivative security, is essentially a contract which pays tomorrow an amount that depends only on the asset price tomorrow. Towards the end of the last century, the traditional contingent claim analysis (as a generalization of the option pricing theory) started to be applied to non-traditional underlying assets (also known as the real assets), which gradually formed a new research discipline named real option analysis. Dixit and Pindyck [23] brought real options to the attention of nearly every economist and set the scientific foundations of this field. Ever since then, real option analysis has been popularized by business publications and valuation texts (see, *e.g.*, Copeland and Tufano [18], Smit and Trigeorgis [87] and Trigeorgis [92]). For example, the theory of real option has been applied to corporate decision making concerning R&D investments, Venture Capital (VC) funding, real estate development, facilities planning and construction in cyclical industries, operations in natural resource industries, just to name a few. From a mathematical finance standpoint, most of the aforementioned fields have incomplete markets, in which hardly any market prices are available for the underlying assets, such prices do not necessarily follow geometric Brownian motion, and investments exercised by one firm will affect the market value of the option for other firms (also known as the feedback effect [22]). As a result, the application of probability and stochastic process theory which proves efficient in making choices and decision under uncertainty in financial market trading could not be simply replicated in pricing real options. The area of mathematics that finds its natural application in real option analysis is fuzzy set theory, which is capable of modelling the effects of not only monetary factors but also non-monetary (qualitative) aspects of a project, such as legal, political and human factors which are left out of the traditional option pricing models.

2.2.2 Operations research

Briefly speaking, operations research (or operational research) is a discipline that helps executives make better decisions and build more productive systems by using analytical techniques such as mathematical modelling. A formal yet concise definition is given by the Operational Research Society of Britain (as cited in [33]) as follows:

> Operational research is the application of the methods of science to complex problems in the direction and management of large systems of men, machine, materials and money in industry, business, government and defence.

As was discussed in the previous chapter, the need for patent valuation is prompted by the ever-broadening set of strategies for patent exploitation. For example, IPR managers need to know the value of their patents when determining royalty rates for licensing agreements, patents and other technology assets play an important role in decision makings about potential merges and acquisitions, and financial institutions value patents when they are used as collateral for loans [42]. Therefore, the essential purpose of patent valuation is to facilitate decision making, which corresponds to the aim of operations research.

The root of operations research could be traced back to the Second World War, and its practice was initially seen when the Allies' engineers were deployed to study the effects of the location of radar stations on detection efficiency and to optimize the distribution of maintenance personnel [80]. The success of operations research in the war, which was also witnessed in the warfare against German U-Boat fleet and in strategic bombing, effortlessly spurred intense interest in introducing it to a variety of non-military organizations in business, industry and government in the post-war phase, and the developed mathematical models accompanying its evolution have been applied to a wide range of problems ever since. Applications of these mathematical models are to the transportation and assignment problems, inventory analysis, queuing systems, capital budgeting, financial engineering, just to name a few.

It is worth noting that the scope of modern operations research has reached and overlapped with a number of other disciplines and that applications may be found for its techniques and models in almost every branch of human knowledge [37]. It therefore would be very difficult to justify if a specific problem lies in the domain of operations research or others. Among the disciplines infiltrated by operations research (or subdisciplines of it) is probabilistic modelling, which has been prevailing in dealing with decision making problems in the face of uncertainty, such as in pricing financial derivatives (see *e.g.*, [25, 56]). As a result of the increased awareness that probability theory is less than adequate to model uncertainty caused by incomplete states of knowledge (or partial ignorance), a number of non-probabilistic models of uncertainty have been introduced, among which are fuzzy set theory and possibility theory, which have played a fundamental role in decision analysis and have been successfully applied to a variety of operations research topics, such as fuzzy optimization, preference modelling and fuzzy game theory (see *e.g.*, [95, 37, 86]). Another popular research discipline which is also related to operations research is that of real option analysis. Indeed, real option analysis, which is an application and extension of financial option pricing theory to the valuation of investment in real assets (*i.e.*, permanent, fixed or immovable assets) in the face of uncertainty, conceptualizes and quantifies the economic potential (*i.e.*, the value of real options) derived from active management, assists value-maximizing firms in making the best decisions, and hence serves as a "mental model" [17] for strategic and operational decision-making.

2.2.3 Theoretical positioning of this dissertation

It should be clear by now that this dissertation could be placed at the intersection of mathematical finance and operations research (see Figure 2.2). As its name implies, operations research adopts a procedure which bears a striking resemblance to the way "research" is conduced in established scientific fields. An additional characteristic of operations research, as was explicitly implied in its definition given above, is its organizational point of view. Therefore, the process of problem solving through the means of operations research ought to be interactive-oriented, that is to say, whatever model is adopted must provide understandable and verifiable solutions to decision makers. Accordingly, the process could be broken down into the following phases [38, 13, 60]:

1. Observing and investigating the real world situation, and collecting all relevant information and data.

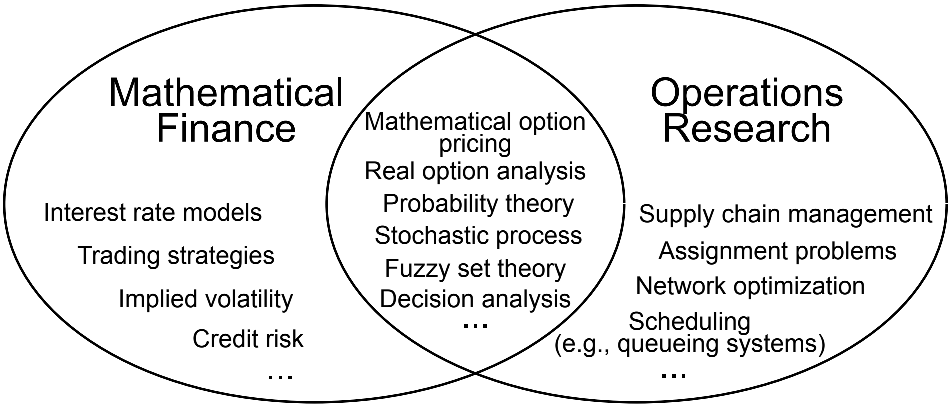


Figure 2.2: Problems addressed by disciplines related to my dissertation

2. Abstracting the essence of the problem of concern; constructing a mathematical model which ought to be capable of addressing it in a clear and concise fashion, that is, a model being a sufficiently precise representation of the essential features of the situation (including the consideration of the preferences of the decision maker), taking in and returning all the necessary information to solve the problem.

3. Solution of the model and identification of other approaches that address the same problem.

4. Validation of the obtained solution and modification of the proposed model if decision makers are not satisfied with, for example, the results under real data or internal workings of the mathematical model (see *e.g.*, [26, 52]).

5. Implementation of the solution.

Within the framework given above, this dissertation involves the first three points of activities. In particular, the barriers impeding the adoption of conventional patent valuation methods are unveiled in the introductory chapter; Chapters 3 and 4 abstract the essence of the real obstacles and propose two techniques as solutions; Chapter 5 and 6 construct the mathematical models and test them with real world cases.

Chapter 3

Fuzzy set theory in patent valuation

As was discussed in the introductory chapter, the illiquidity and inefficiency of the patent market, together with the rarely published quantitative data regarding patent transactions, urge the IPR managers to consult and rely heavily on other patent practitioners such as inventors, patent engineers, patent attorneys and marketing personnel when faced with patent valuation or patenting decision making tasks. The above patent experts will correspondingly leverage their expertise and give their own estimate on the future economical benefits the subject patent could generate, ideally in monetary terms. For example, inventors are capable of providing an incomparable insight into the technological superiority of the innovation asset under protection over similar technologies. Patent engineers, who are typically responsible of conducting patent search, overseeing patent landscape, filing for patent application and maintaining patent database, possess a better awareness of the possible patent scope and its complementarity with the existing patent portfolio. Patent attorneys legally represent patent applicants, proprietor and opponents before the official patent-issuing authorities such as EPO and WIPO, and therefore have better foresight of possible patent infringement and its effects. Market consultants and business experts leverage their knowledge and experience to perform a market due diligence, which includes identifying the market and predicting the market share of the technology asset. Such consultants and business experts also assess the potential benefits from IPR-oriented business such as licensing and selling the patent, and other economic benefits resulting from the exploitation of the technology, which include greater efficiency in production, improved quality, and lessened environmental and safety hazards. Patent brokers work closely with the above professionals and take care of identifying the appropriate patent licensees.

Although it is arguably true that the wider the net of professionals, the more reliable the result of the valuation would be [9], one of the intrinsic factors behind a reliable and accurate valuation of the subject patent is the manager's ability to properly interpret, or rather quantify and aggregate those individual evaluations. This is because it is inevitable that there would exist significant perception gaps with respect to the potential economical benefits of the subject patent within such a sophisticated collaboration network that combines expertises in the technical, business and legal areas. For example, inventors normally underestimate the cost required to turn their patents into marketable product due to their lack of business insight [9] while patent attorneys tend to be more conservative on the potential of their patents.

In addition to the substantial discrepancy among the aforementioned professionals, there could exist great uncertainty within any individual evaluation. For example, in spite of his expertise, it would still be virtually impossible for a well experienced patent engineer to give an absolutely accurate and correct estimation on the future cash flows of any of his patents. In most cases, he would be able to provide only an approximation, which, based on his degree of confidence, could be given in the form of a single number such as "the monopoly benefit of next year would be €500,000" or a range of possible values such as "the monopoly benefit of next year would be between €450,000 and €550,000". Sometimes, the estimation is only available in linguistic terms such as "the patent protected product would be *successful*" or a linguistic value combined with describing modifiers such as "the share of sales that is attributable to the patent protected technical part is *very significant*".

As was noted, there exist two levels of uncertainties which must be taken into consideration when valuing a patent, or in other words, a trustworthy patent valuation method must be capable of properly accounting for them. Intuitively, to find such a tool requires first a closer examination on the characteristics of the aforementioned types of uncertainties and a review of the relevant uncertainty theory.

3.1 The nature of the uncertainty encountered in patent valuation

3.1.1 A brief chronicle of the evolution of uncertainty theory

The work on the formalization of uncertainty has witnessed substantial developments since the beginning of the 20th century. They could be best characterized as a transition from the traditional view of uncertainty, which claims that "uncertainty is undesirable in science and should be avoided by all possible means" [46] and had prevailed prior to the 20th century, to the modern view of uncertainty in which it is accepted that uncertainty is an "unavoidable plague" [46] but has significant utility, and such developments could be summarized into the following two major phases.

The first phase began in the late 19th century with the emergence of statistical methods based upon probability theory, which were originally developed to study physical processes at the molecular level, such as the motions of gas molecules which the laws of classical mechanics (or Newtonian mechanics) fail to account for, but also found utility in other fields including actuarial science, econometrics, engineering, social science and so on (see e.g., [47]). Generally speaking, contrary to classical mechanics which studies the motion of an object characterized by a small number of interrelated parameters, statistical methods have proved to be applicable to problems involving a large number of variables each of which behaves in a highly random manner, and the precision of statistical methods improves along with the number of variables and their degree of randomness; probability theory, which plays the same role as the calculus in classical mechanics assuming definite and knowable attributes to subject matters, applies to the analysis of the concomitant random phenomena or rather captures the uncertainty.

Ever since the acknowledgement of statistical mechanics as a legitimate scientific area and the accompanying recognition of uncertainty as a valuable commodity in systems modelling at the beginning of the 20th century, probability theory had been considered as the only feasible theory to capture any type of uncertainty until its dominance was challenged by several new uncertainty theories which emerged at the second half of the 20th century. Their emergence also marked the commencement of the second phase of the transition from the traditional view of uncertainty to the modern view. Among all those new theories is fuzzy set theory introduced by Lotfi Zadeh [98], whose subjects, namely fuzzy sets, are sets with blurred boundaries in contrast to classical sets the belonging of an object to which is a precise concept. In other words, a given object does not necessarily belong to or reside outside a fuzzy set strictly, but is allowed to belong to it to a degree. Fuzzy set theory therefore serves as a generalization of classical set theory. The most profound contribution of such a generalization to the theory of uncertainty is that it challenged probability theory as the "sole agent" [46] for any type of uncertainty, or rather it enlarged significantly the framework of formalizing uncertainty and manifested that probability theory is only applicable to one of several distinct types of uncertainty. For example, the capability of fuzzy sets to convey gradual, rather than abrupt, transitions from membership to non-membership and vice versa enables us to scientifically depict the vagueness prevailing in natural languages, which classical two-value logic upon which probability theory is based is inadequate for.

3.1.2 Selecting the appropriate uncertainty theory for patent valuation

Ever since the emergence of fuzzy set theory, there have been such heated debates as if fuzzy set theory is subsumed under probability theory or vice versa, and if any task that could be done with fuzzy set theory could be equally well or better performed with probability theory. In this thesis, however, it will be argued that this is not a matter of one theory being superior to the other, and that the choice of the appropriate uncertainty theory should always be context dependent.

However, the direct comparison between fuzzy set theory and probability theory has proved to be difficult, and such difficulty could be imputed to the lack of unique definitions of fuzziness and the selection of aspects (mathematical, semantic, linguistic, etc) with respect to which they shall be compared [102]. Therefore, I have chosen to examine the two theories from a usability perspective, that is, to look into the cause of each distinct type of uncertainty and define the set of problems to which fuzzy set theory and probability theory are applicable, respectively. It is hoped that such a clarification could also shed some light on the optimal uncertainty theory for patent valuation.

Types of uncertainty

As was discussed earlier, in light of the recognition of uncertainty as being ineluctable but useful, there have been tremendous research efforts devoted to the development of tools for modelling uncertainty. However, a general definition for uncertainty with unanimous approval is still missing, which could be partially imputed to the fact that uncertainty itself is an exceedingly broad concept and a somewhat vague term. Correspondingly, there is not a unique way of classifying uncertainty either. Among the various classification standards for uncertainty, I have chosen to present in the following discussion the Knightian distinction and the objective/subjective distinction for their generalities.

In his renowned and influential work, Frank Knight [49] categorized future outcomes into the following three groups:

- Outcomes to which mathematical probability could be assigned *a priori*, such as the coin-toss game whose outcome could be modelled by the Bernoulli distribution.

- Outcomes which could be grouped and the expected outcome of this group as a whole could be determined through sufficient historical data. For example, while the probability that an arbitrary male aged between 20 and 30 in a specified population is of a particular height could not be determined *a priori*, it is possible to estimate the probability that he is at least 6 feet tall with, for instance, a log-normal distribution.

- Outcomes which could not be grouped and whose likelihood could not be determined from historical data. In other words, a full-fledged probabilistic modelling is not possible.

Knight categorized the first two groups as *a priori probability situations* and *statistical probability situations*, respectively, and named them *risk* as a whole; to contrast with the characteristics of the first two groups, he also labelled the last group as *uncertainty*. Knight further claimed that the instances falling under *uncertainty* are "so entirely unique that there are no others or not a sufficient number to make it possible to tabulate enough like it to form a basis for any inference of value about any real probability" [49]. Clearly, the uncertainties encountered in valuing a patent fall under the categorization of "uncertainty" according to the Knightian distinction.

A great deal of effort has been devoted to studying the nature of uncertainties and the manner in which they should be dealt with in the post-Knight phase (see e.g., [76, 63]). Despite the great discrepancies among the definitions of uncertainty in the literature, it has been generally agreed that uncertainty could be classified into at least two major categories based on their causes (see e.g., [24, 8, 39, 36, 45]). They are:

- objective (or aleatory) uncertainties which are subject to natural variability of observations, or rather inherent randomness,

- human-related, subjective interpretation of uncertainties (or epistemic uncertainties) which depend on the quantity and quality of information which is available to a human being about a system or its behaviour that the human being want to describe, predict or prescribe [101].

The uncertainties pertaining to the first class are perceived to be nondeterministic in nature since they are attributed to the physical real systems. It is therefore hardly possible to prevent or reduce them by enhancing the underlying knowledge base [6]. A simple yet remarkable example of this uncertainty type is again the coin-toss game, with the outcome of each toss being strictly stochastic and unpredictable. Other problems or scenarios involving randomness, which might not be explicitly recognizable, include the measurement of the height of an adult male in a country, the grades of high school students in a university entrance exam, the occurrence of hurricanes in China South Sea at a future date and weather forecasting in which the randomness stems from the nonlinearity of the flow interactions in atmosphere and ocean [75].

It is believed that uncertainties of the second class are by far more prevalent in real-world phenomena (see e.g., [93, 46, 48, 102]) and one would expect to find an appropriate definition of this type. Nevertheless, I have failed to unearth any well-established or accustomed definition, but rather discovered inconsistent and even contrary views towards their characteristics in the literature. For example, Dubois [24] describes them as being "totally *deterministic* but anyway ill-known", while Weaver [93] argues that they are "usually *nondeterministic*, but not as a result of randomness that could yield meaningful statistical averages". Despite the disagreement between determinist and indeterminist, they all tend to support the claim that epistemic uncertainties are highly related to the behavioural aspects of decision making and are usually witnessed whenever human judgement, evaluation or reasoning is crucial to the system under consideration and there exists a lack of complete knowledge on its variables. A more intuitive, although not rigorous, explanation for the emergence of epistemic uncertainties is, when the limit in data or understanding precludes the calculation of the statistical properties of an uncertain quantity, it is logical to turn to experts for their professional opinions, which will inevitably involve inaccuracies, omissions or even biased judgements. Such imprecision is usually referred to as epistemic uncertainty and has also been labelled as subjective uncertainty [6]. Kiureghian and Ditlevsen [45] also distinguish epistemic uncertainties from aleatory uncertainties by arguing that it is possible to reduce the former by gathering more data or by refining models, while it is not for the latter.

Subjective uncertainties could be further categorized into a few subclasses according to the causes they arise from:

- Lack of knowledge is one of the most frequent causes of subjective uncertainty. Such uncertainties are usually inherent in situations in which one does not have any information regarding which one of the possible states of world will happen [101]. As a result of the knowledge deficiency, one might be able to acquire only interval-valued information or an approximation, that is to say, some sort of measurement uncertainties (*i.e.*, imprecision) would be inevitable. It is important to emphasize that uncertainties caused by lack of knowledge are different from those stemming from the circumstances in which one is equipped with a complete probability distribution of the possible outcomes but does not know which one is correct, such as the coin-toss game discussed earlier in which the probability of a head and the probability of a tail are known to be identical assuming the coin is fair. Such distinguishment coincides with the Knightian distinction discussed above, which differentiates between *risk* and *uncertainty* by arguing that it is possible to derive a distribution of future outcomes when the former is present, while it is not when the latter is reigning.

- Subjective uncertainties also stem from the limited ability of human being in perceiving, processing and developing on the available (objective) information. Such vulnerability is best manifested by the vagueness inherent in natural languages, which is usually witnessed in the descriptions of the semantic meaning of phenomena, events, belief and so forth, and is pervading our daily communications. For example, such vagueness would occur when linguistic variables are involved. In contrary to numerical variables which take numerical values, the values of a linguistic variable are words, phrases or sentences in natural or artificial languages [12]. The word *age* is a typical linguistic variable whose linguistic values include young, adult, middle-aged, aged, old and so on. However, such values (or labels) cannot be characterized precisely; no sharp boundaries exist to distinguish between them; the context (observer's educational and culture backgrounds, social environment, etc) has great influence on their meanings. The resulting vagueness is also referred to as "intrinsic fuzziness" [81], and such vague proposition as "John is a *young* man" is known as a fuzzy event. Similar vagueness is also visible in other circumstances, such as in linguistic modifiers (fairly, very, extremely, etc), and in fuzzy probabilities conveyed through, for instance, likely, unlikely, probably, possibly, and so forth.

It needs to be clarified that the classification demonstrated above is by no means intended as exhaustive or exclusive, and the embedded uncertainty found in a real-world phenomenon could be derived from more than one source or even interpreted as heterogeneous uncertainty (*i.e.*, a mixture of objective and subjective uncertainties). Additionally, it is of little practical significance to determine whether a particular uncertainty belongs to the objective uncertainty category or the subjective uncertainty category without defining the context, and these concepts make unambiguous sense only if the system (*i.e.*, physical reality, socioeconomic systems, man-made systems, etc [101]), whose behaviour human being wants to understand and predict, and the quality and quantity of whose emitted information or data determine the characterization of the embedded uncertainty, is made explicit [45]. Once the object and purpose of study are ascertained, it rests with the observer to make the pragmatic choice of uncertainty class for the addressed uncertainty, and his decision should always be conditional on the degree of scientific knowledge he possessed and on the quality of quantity of the information he has access to.

It should be clear by now that at least one of the two levels of uncertainties encountered in patent valuation fall under the categorization of subjective uncertainty, that is, the uncertainties embedded in expert judgements at the individual level which are incurred by the scarce quantitative data regarding patents and patent transactions, and patent experts' limited ability to process and develop on the available scientific evidence. The other level of uncertainties, arising from the interpersonal disagreements, arguably fall into the subjective category too. As was noted, such disagreements mainly result from the experts' different technical interpretations of the same phenomenon. It is noticeable that poor data availability and their motivational biases could also influence the existing discrepancies. For instance, the limit in data would prolong the procedure for experts to reconcile or preclude them from reaching a consensus at all; they might have direct or indirect stakes in the outcome (e.g., the outcome could influence their income, reputation or even career) and hence their judgements might reflect their motivational biases [63].

The choice of uncertainty theory

The taxonomy of uncertainty reached above would logically suggest the uncertainty theory chosen for the depiction of an uncertain phenomenon be homogeneous as to the features of the phenomenon, that is to say, the choice of uncertainty theory be dependent upon the characteristics of the inherent uncertainty. However, this seemingly intuitive judgement did not receive much support until the 1960s, before which the view that probability theory is the only feasible theory to capture any type of uncertainty had been prevailing. Warren Weaver's well-known paper [93] is largely believed to be one of the pioneering works which addressed this issue. In his work, he initiated a trichotomy of scientific problems and argued they could be classified into one of the following categories:

- Problems of (organized) simplicity: mostly physical science related problems which have a very small number of variables that are interrelated in a predictable manner, and which could be sufficiently dealt with by the calculus.

- Problems of disorganized complexity: such problems are the other extreme compared to the problems of simplicity, that is to say, they usually involve a large number of variables, each of which behaves in an erratic or totally random manner; despite the unknown behaviour of individual variables, the average properties of the system as a whole could be obtained with the methods of statistical mechanics.

- Problems of organized complexity: the vast majority of real-world problems fall under this middle region, whose members are typically nonlinear systems with a considerable number of variables; such problems are "organized" in the sense that, in contrast to the "disorganized" situations whose statistical averages are attainable, their variables share rich interactions which are not random in nature and hence additional methods to statistical techniques are required.

It could be inferred from Weaver's categorization that objective uncertainties as depicted earlier could be effectively measured by probability theory, which is also supported by substantive literature that followed Weaver's work (see e.g., [46, 102]). The applicability of probability theory to model subjective (*i.e.*, epistemic) uncertainties, however, remains vague and thus requires further examination. As was claimed by Paté-Cornell [76], an unavoidable and difficult problem in dealing with epistemic uncertainties is "the encoding of probability distributions based on scientific evidence and expert judgements".

As was discussed earlier, subjective uncertainties could be caused by knowledge deficiency, and when lack of knowledge is at stake, an exact description of the system under consideration is usually not possible and sometimes only interval-information is available. For example, with probability theory or rather the Aristotelian two-value (classical) logic upon which probability theory is based, the earlier estimation on the monopoly benefits of a patent could only be manifested as a bounded and closed (or open) interval of $[450,000,\ 550,000]$ (or $(450,000,\ 550,000)$), which strictly rules out any other possibilities. However, does this mean that $449,000$ or $550,001$ is not a feasible estimate? This is clearly unrealistic as 1 Euro hardly seems to be a distinguishing quantity. The vulnerability of probability theory to subjective uncertainties could be further exemplified by its inefficiency in measuring linguistic values, not to mention linguistic values combined with describing modifiers which prevail in human language. As was discussed at the beginning of this chapter, linguistic values are commonly adopted by patent agents to reflect their imprecise estimation on the future cash flows of their patents, such as "the share of sales that is attributable to the patent protected technical part is *significant*". In order for the term *significant* to covey the desired introduction of vagueness, it is clearly not rational to assign 100% to its contribution to the overall sales. It is also safe to rule out any percentage that is smaller than 40%. Some intermediate states such as those between 50% and 70% are acceptable. But again, where should we draw the line? If 70% could be accepted, is 71% not *significant*? If 71% was considered *significant* since 1% hardly makes an essential difference, could the same logic be applied to all the numbers that follow? If it could, it would eventually result in the acceptance of all the percentages larger than 50% as *significant*. Then how could we discriminate *significant* from *very significant* in percentage terms?

From our previous discussion, we could argue that classical set theory and two-value logic upon which probability theory is based seem to be less efficient in measuring subjective uncertainties, or at least in measuring patent valuation-related uncertainties. What is needed instead is a generalization of the classical set theory which abolishes the sharp and unambiguous boundaries, allows degree (or grade) of belonging and hence facilitates a gradual transition from membership to non-membership. This is, as a matter of fact, precisely the fundamental of fuzzy set theory. As was claimed by Zadeh, fuzzy set theory illustrates obvious superiority in dealing with vague conceptual phenomena as "such a framework provides a natural way of dealing with problems in which the source of imprecision is the absence of sharply defined criteria of class membership rather than the presence of random variables" [98]. The applicability of fuzzy logic, as contrasted with classical (two-valued) logic, could be arguably extended to the formalization of all subjective uncertainties. Recall that it is claimed earlier in this chapter that subjective uncertainties could be categorized into at least two subclasses according to their causes, which are lack of knowledge and the limited ability of human being in perceiving, processing and developing on the available information. Fuzzy logic, if viewed as a formal mechanization of human capabilities, is capable of formalizing, first, "our capability to converse, reason and make rational decisions" in an environment of imperfect information, and second, "our capability to perform a wide variety of physical and mental tasks without any measurements and any computations" [100].

3.2 Fuzzy sets, fuzzy numbers and arithmetic operations on fuzzy numbers

Fuzzy set theory, which was invented as an extension of the classical set theory and dual logic, weakens their yes-or-no or rather precise and crisp restriction. As was discussed earlier, it has proven to be useful in the modelling of real systems, which are usually so complex that it is not possible for a human being to give a complete description of them or make any precise and yet significant statement about their behaviours [99]. For example, the applications of fuzzy set theory have been witnessed in control engineering, expert systems, artificial intelligence, robotics, management science, etc [102]. In the rest of this chapter, the elements of fuzzy set theory will be introduced.

3.2.1 Basic definitions

Definition 3.2.1 *A fuzzy set $\widetilde{A}$ defined in a crisp set X is a set of ordered pairs, in which the first element is taken from X and the second element is a value in $[0,1]$. That is,*

$$\widetilde{A} = \left\{ \left(x, \mu_{\widetilde{A}}(x) \,|x \in X \right) \right\}.$$

$\mu_{\widetilde{A}}$ *is the membership function (generalised characteristic function) which maps* X *to the membership space* $[0,1]$*. That is,*

$$\mu_{\widetilde{A}} : X \to [0,1] .$$

It is important to point out that the membership function of a fuzzy set $\widetilde{A}$ could also be denoted by the symbol of the fuzzy set, namely $\widetilde{A}$. The double use of the symbol, however, should result in no ambiguity as each fuzzy set is "completely and uniquely defined by one particular membership function" [46]. In this thesis, each fuzzy set and the associated membership function will be represented by the same capital letter with tilde.

Definition 3.2.2 *A fuzzy set* $\widetilde{A}$ *is convex if*

$$\widetilde{A}(\lambda x_1 + (1-\lambda) x_2) \geq \min\left\{\widetilde{A}(x_1), \widetilde{A}(x_2)\right\},$$

where $x_1, x_2 \in X$ *and* $\lambda \in [0,1]$*.*

In many real-life scenarios people inevitably use linguistic terms representing approximate values of a variable to characterize imprecise or incomplete numeric information. For example, it is very common to see such terms as "about 100 kilos", "approximately 5 minutes", "1.5% more or less" and "significantly more than 1 million". All these ambiguous expressions are examples of what are called fuzzy numbers, *i.e.*, numbers representing imprecision. Formally speaking:

Definition 3.2.3 *A fuzzy number* $\widetilde{A}$ *is a convex fuzzy set defined on the real line* $\mathbb{R}$ *such that*

- *There exists an* $x \in X$ *such that* $\widetilde{A}(x) = 1$ *(i.e.,* $\widetilde{A}$ *is a normalized fuzzy set).*
- $\widetilde{A}(x)$ *is piecewise continuous.*

Two of the most commonly used fuzzy numbers, namely triangular fuzzy number and trapezoidal fuzzy number, are defined as follows:

Definition 3.2.4 *A fuzzy set* $\widetilde{A}$ *is called triangular fuzzy number with peak* a*, left width* $\alpha > 0$ *and right width* $\beta > 0$ *if its membership function has the following form*

$$\widetilde{A}(x) = \begin{cases} 1 - \dfrac{a-x}{\alpha} & \text{if } a-\alpha \leq x \leq a \\ 1 - \dfrac{x-a}{\beta} & \text{if } a < x \leq a+\beta \\ 0 & \text{otherwise.} \end{cases}$$

It will be denoted by $\widetilde{A} = (a, \alpha, \beta)$*.*

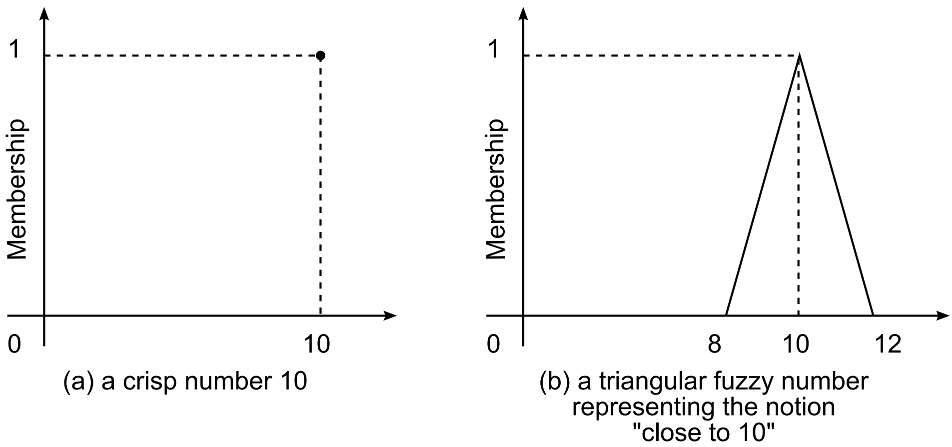


Figure 3.1: A comparison of a crisp number with a triangular fuzzy number

Definition 3.2.5 *A fuzzy set* $\widetilde{A}$ *is called trapezoidal fuzzy number with tolerance interval (or core)* $[a, b]$, *left width* $\alpha > 0$ *and right width* $\beta > 0$ *if its membership function has the following form*

$$\widetilde{A}(x) = \begin{cases} 1 - \dfrac{a-x}{\alpha} & \textit{if } a - \alpha \leq x < a \\ 1 & \textit{if } a \leq x \leq b \\ 1 - \dfrac{x-b}{\beta} & \textit{if } b < x \leq b + \beta \\ 0 & \textit{otherwise.} \end{cases}$$

It will be denoted by $\widetilde{A} = (a, b, \alpha, \beta)$.

Intuitively, a triangular fuzzy number with peak a could be used to represent such a vague concept as "x is approximately equal to a". In a similar manner, a trapezoidal fuzzy number with tolerance interval $[a, b]$ could be seen as a vague concept "x is approximately in the interval $[a, b]$" [28]. Figures 3.1 and 3.2 illustrate a triangular fuzzy number $\widetilde{A} = (10, 2, 2)$ and a trapezoidal fuzzy number $\widetilde{B} = (9.5, 10.5, 1.5, 1.5)$, respectively, and their comparison with a crisp number 10 and a crisp interval $[9.5, 10.5]$.

Some other important definitions are given as follows:

Definition 3.2.6 *The support of a fuzzy set* $\widetilde{A}$ *is the crisp set of all* $x \in X$ *such that* $\widetilde{A}(x) > 0$.

Definition 3.2.7 *A* γ*-cut (or a* γ*-level set) of a fuzzy set* $\widetilde{A}$ *defined in* X *is the crisp set of elements in* X *which belong to the fuzzy set* $\widetilde{A}$ *at least to*

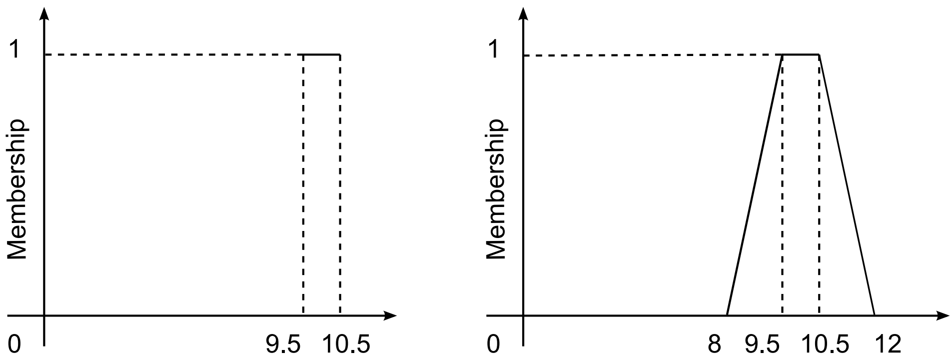


Figure 3.2: A comparison of a crisp interval with a trapezoidal fuzzy number

the degree of γ*. That is*

$$[\widetilde{A}]^{\gamma} = \begin{cases} \left\{x \in X | \widetilde{A}\,(x) \geq \gamma\right\} & \textit{if } \gamma > 0 \\ \mathrm{cl}(\mathrm{supp}(\widetilde{\mathrm{A}})) & \textit{if } \gamma = 0, \end{cases}$$

where $\mathrm{cl}(\mathrm{supp}(\widetilde{A}))$ *denotes the closure of the support of* $\widetilde{A}$*.*

It is easy to verify that the γ-cuts of a triangular fuzzy number $\widetilde{A} = (a_{\widetilde{A}}, \alpha_{\widetilde{A}}, \beta_{\widetilde{A}})$ are

$$[\widetilde{A}]^{\gamma} = \left[a_{\widetilde{A}} - (1-\gamma)\,\alpha_{\widetilde{A}},\ a_{\widetilde{A}} + (1-\gamma)\,\beta_{\widetilde{A}}\right] \text{ for all } \gamma \in [0,1],$$

and the γ-cuts of a trapezoidal fuzzy number $\widetilde{B} = (a_{\widetilde{B}}, b_{\widetilde{B}}, \alpha_{\widetilde{B}}, \beta_{\widetilde{B}})$ are

$$[\widetilde{B}]^{\gamma} = \left[a_{\widetilde{B}} - (1-\gamma)\,\alpha_{\widetilde{B}},\ b_{\widetilde{B}} + (1-\gamma)\,\beta_{\widetilde{B}}\right] \text{ for all } \gamma \in [0,1].$$

3.2.2 Arithmetic operations on fuzzy numbers

A variety of methods exist for arithmetic operations on fuzzy numbers. In this thesis, they are defined in terms of arithmetic operations on their γ-cuts. Such a choice is prompted by the fact that a fuzzy number could be completely represented by its γ-cuts and that arithmetic operations on γ-cuts are essentially arithmetic operations on closed intervals which are well defined and easy to implement. In other words, fuzzy arithmetics in terms of γ-cuts is applicable to all types of fuzzy numbers and the corresponding results are also fuzzy numbers (see e.g., [46]) for more details on and other methods of arithmetic operations on fuzzy numbers).

Let $\widetilde{A}$ and $\widetilde{B}$ be two fuzzy numbers. Without loss of generality, their γ-cuts could be simplified as

$$\begin{aligned} [\widetilde{A}]^\gamma &= [a,\ b]\,, \\ [\widetilde{B}]^\gamma &= [c,\ d]\,. \end{aligned}$$

The four basic arithmetic operations on $\widetilde{A}$ and $\widetilde{B}$ are defined in terms of $[\widetilde{A}+\widetilde{B}]^\gamma$, $[\widetilde{A}-\widetilde{B}]^\gamma$, $[\widetilde{A}\cdot\widetilde{B}]^\gamma$ and $[\widetilde{A}/\widetilde{B}]^\gamma$, which are:

$$\begin{aligned} [\widetilde{A}+\widetilde{B}]^\gamma &= [\widetilde{A}]^\gamma + [\widetilde{B}]^\gamma \\ &= [a+c,\ b+d]\,, \\ [\widetilde{A}-\widetilde{B}]^\gamma &= [\widetilde{A}]^\gamma - [\widetilde{B}]^\gamma \\ &= [a-d,\ b-c]\,, \\ [\widetilde{A}\cdot\widetilde{B}]^\gamma &= [\widetilde{A}]^\gamma \cdot [\widetilde{B}]^\gamma \\ &= \big[\min(ac, ad, bc, bd)\,,\ \max(ac, ad, bc, bd)\,\big], \end{aligned}$$

and, provided that $0 \notin [c,\ d]$ for all $\gamma \in [0,1]$,

$$\begin{aligned} [\widetilde{A}/\widetilde{B}]^\gamma &= [\widetilde{A}]^\gamma / [\widetilde{B}]^\gamma \\ &= [a,\ b]\cdot[1/d,\ 1/c] \\ &= \big[\min(a/c, a/d, b/c, b/d)\,,\ \max(a/c, a/d, b/c, b/d)\,\big]. \end{aligned}$$

It is worth pointing out that the addition and subtraction of trapezoidal fuzzy numbers could be equivalently formulated as follows:

$$\widetilde{A}+\widetilde{B} = \big(a_{\widetilde{A}}+a_{\widetilde{B}}, b_{\widetilde{A}}+b_{\widetilde{B}}, \alpha_{\widetilde{A}}+\alpha_{\widetilde{B}}, \beta_{\widetilde{A}}+\beta_{\widetilde{B}}\big)\,, \tag{3.1}$$

$$\widetilde{A}-\widetilde{B} = \big(a_{\widetilde{A}}-b_{\widetilde{B}}, b_{\widetilde{A}}-a_{\widetilde{B}}, \alpha_{\widetilde{A}}+\beta_{\widetilde{B}}, \beta_{\widetilde{A}}+\alpha_{\widetilde{B}}\big)\,. \tag{3.2}$$

In a similar manner, the scalar multiplication of trapezoidal fuzzy numbers is defined as follows:

$$\lambda\widetilde{A} = \begin{cases} \big(\lambda a_{\widetilde{A}}, \lambda b_{\widetilde{A}}, \lambda\alpha_{\widetilde{A}}, \lambda\beta_{\widetilde{A}}\big) & \text{if } \lambda \geq 0 \\ \big(\lambda b_{\widetilde{A}}, \lambda a_{\widetilde{A}}, |\lambda|\beta_{\widetilde{A}}, |\lambda|\alpha_{\widetilde{A}}\big) & \text{if } \lambda < 0, \end{cases} \tag{3.3}$$

provided that $\lambda \in \mathbb{R}$. Note that Equations 3.1 - 3.3 also apply to triangular fuzzy numbers. In fact, a triangular fuzzy number, say $\widetilde{A} = (a, \alpha, \beta)$, could be legitimately considered as a trapezoidal fuzzy number of the form $\widetilde{A} = (a, a, \alpha, \beta)$.

3.2.3 Weighted mean value and weighted variance of fuzzy numbers

Like real-valued random variables, the summary statistics including the mean value, variance, covariance and correlation of fuzzy numbers could also be defined. In order to properly present these definitions, it is necessary to introduce the possibility distribution. Very loosely speaking, a possibility distribution could be interpreted by a fuzzy number in very much the same manner as how a random variable is associated with a probability distribution. In other words, a fuzzy number could be treated as "a fuzzy value that we assign to a variable, viewed as a possibility distribution" [69]. Formally speaking, given a fuzzy number $\widetilde{A}$, a variable X taking values in $\mathbb{R}$ and the proposition "X is $\widetilde{A}$", the possibility of $X = x$ for each $x \in \mathbb{R}$ is numerically equal to the degree to which x belongs to $\widetilde{A}$ (*i.e.*, the compatibility of x with $\widetilde{A}$), which is

$$\pi_X(x) \triangleq \mu_{\widetilde{A}}(x)$$

for all $x \in \mathbb{R}$, where π_X is the possibility distribution function associated with X and $\mu_{\widetilde{A}}$ is the membership function of $\widetilde{A}$ [72]. Therefore, in the following work, fuzzy numbers and possibility distributions will be used interchangeably.

Interpreting fuzzy numbers as possibility distributions enables us to define the summary statistics of fuzzy numbers in terms of their counterparts of the corresponding possibility distributions, which are defined on a uniform distribution on the γ-cuts of the possibility distributions, weighted by an appropriately chosen real-valued function (*i.e.*, a weighting function) [29, 30, 60]. Recall that the probability distribution function of a continuous uniform distribution with support $[a, b]$ is

$$f(x) = \begin{cases} \dfrac{1}{b-a} & \text{if } a \leq x \leq b, \\ 0 & \text{otherwise,} \end{cases}$$

and its mean value and variance are $(a + b)/2$ and $(b - a)^2/12$, respectively.

In [29], the mean value of a possibility distribution $\widetilde{A}$ is defined as the weighted average of the probabilistic mean values of the respective uniform distributions on the γ-cuts of $\widetilde{A}$, which is $[\widetilde{A}]^\gamma = [a_1(\gamma), a_2(\gamma)]$ for all $\gamma \in [0, 1]$. Formally,

Definition 3.2.8 *The f-weighted possibilistic mean value of $\widetilde{A}$ is*

$$\mathrm{E}_f(\widetilde{A}) = \int_0^1 \frac{a_1(\gamma) + a_2(\gamma)}{2} f(\gamma)\, \mathrm{d}\gamma. \tag{3.4}$$

In [30], the variance of a possibility distribution $\widetilde{A}$ is defined as the weighted average of the probabilistic variances of the respective uniform distributions on the γ-cuts of $\widetilde{A}$. That is,

Definition 3.2.9 *The f-weighted possibilistic variance of $\widetilde{A}$ is*

$$\mathrm{Var}_f(\widetilde{A}) = \int_0^1 \frac{(a_2(\gamma) - a_1(\gamma))^2}{12} f(\gamma)\, \mathrm{d}\gamma. \tag{3.5}$$

Note that $f(\gamma)$ in Equations 3.4 and 3.5 denotes the weighting function. Formally speaking,

Definition 3.2.10 *A function $f : [0,1] \to \mathbb{R}$ is called a weighting function if*

- *f is non-negative and monotone increasing,*
- *and $\int_0^1 f(x)\,\mathrm{d}x = 1$.*

It is important to point out that the weight assigned to each level of the γ-cuts of the fuzzy number varies along the weighting function used, and, as was imposed in the above definition, f must be monotone increasing because the higher the level of γ-cut (*i.e.*, the larger the γ), the more the weight.

It is easy to see that $f(x) = 2x$ is a natural candidate for the weighting function as $\int_0^1 2x\,\mathrm{d}x = 1$ and it is monotone increasing. The corresponding f-weighted possibilistic mean value and variance of an arbitrary triangular fuzzy number $\widetilde{A} = (a, \alpha, \beta)$ have the following forms:

$$
\begin{aligned}
\mathrm{E}_f(\widetilde{A}) &= \int_0^1 \frac{a_1(\gamma) + a_2(\gamma)}{2} 2\gamma\,\mathrm{d}\gamma \\
&= \int_0^1 \frac{[(a - (1-\gamma)\alpha) + (a + (1-\gamma)\beta)]}{2} 2\gamma\,\mathrm{d}\gamma \\
&= a + \frac{\beta - \alpha}{6}, \qquad (3.6) \\
\mathrm{Var}_f(\widetilde{A}) &= \int_0^1 \frac{(a_2(\gamma) - a_1(\gamma))^2}{12} 2\gamma\,\mathrm{d}\gamma \\
&= \int_0^1 \frac{[(a + (1-\gamma)\beta) - (a - (1-\gamma)\alpha)]^2}{12} 2\gamma\,\mathrm{d}\gamma \\
&= \frac{(\alpha + \beta)^2}{72}.
\end{aligned}
$$

Similarly, given $f(x) = 2x$, the f-weighted possibilistic mean value and variance of an arbitrary trapezoidal fuzzy number $\widetilde{A} = (a, b, \alpha, \beta)$ have the following forms:

$$
\begin{aligned}
\mathrm{E}_f(\widetilde{A}) &= \int_0^1 \frac{a_1(\gamma)+a_2(\gamma)}{2} 2\gamma\,\mathrm{d}\gamma \\
&= \int_0^1 \frac{[(a-(1-\gamma)\,\alpha)+(b+(1-\gamma)\,\beta)]}{2} 2\gamma\,\mathrm{d}\gamma \\
&= \frac{a+b}{2}+\frac{\beta-\alpha}{6}, \qquad (3.7)\\
\mathrm{Var}_f(\widetilde{A}) &= \int_0^1 \frac{(a_2(\gamma)-a_1(\gamma))^2}{12} 2\gamma\,\mathrm{d}\gamma \\
&= \int_0^1 \frac{[(b+(1-\gamma)\,\beta)-(a-(1-\gamma)\,\alpha)]^2}{12} 2\gamma\,\mathrm{d}\gamma \\
&= \frac{(b-a)^2}{12}+\frac{(b-a)(\alpha+\beta)}{18}+\frac{(\alpha+\beta)^2}{72}.
\end{aligned}
$$

3.3 Discussion

It has been explicated in the first half of this chapter how the uncertainties underlying patent cash flow estimation differ from those subject to randomness. That is, they are derived from subjective judgements which are affected by the completeness of information and human being's limited ability to perceive, process and develop on the available (objective) information, and hence could also be labelled as subjective uncertainties. Since probability theory proves to be inefficient in modelling such human-related and subjectively-interpreted uncertainties, the justification of fuzzy set theory in particular fuzzy numbers for patent cash flow estimation has also been provided. In the rest of this chapter, fundamental concepts regarding fuzzy numbers and arithmetic operations on fuzzy numbers are elaborated. Particularly, a great emphasis has been put upon the summary statistics of fuzzy numbers (*i.e.*, weighted mean value and weighted variance of fuzzy numbers). This is because, in our experience, most industrial practitioners prefer single crisp numbers, and fuzzy mean value and variance are capable of aggregating the imprecision inherent in results in a rigorous yet straightforward way.

Chapter 4

Real option analysis in patent valuation

It has been clearly demonstrated in our earlier discussion Chapter 1 that the conventional discounted cash flow based patent valuation methods, such as the income based approach, suffer from the rule of thumb of “one single discount rate” for all the future cash flows and from their neglect of managerial flexibility inherent in both the patent application process and the patent exploitation. The decision tree analysis based methods, which build in some of the value of flexibility encountered in a patent, still require the appropriate depiction of the risk involved at each stage and accompanying each “branch” of the decision tree (*i.e.*, each type of alternatives). In practice, however, a constant discount factor is usually applied for the sake of simplicity [79]. Additionally, the construction of a decision tree for a patent which is subject to various decisions through its lifespan that could be up to 20 years is by no means easy. It is those aforementioned obstructions that provide a motive for and fund research into applying real option analysis to patent valuation.

4.1 Real option analysis as a means to account for managerial flexibility

The field of real options originated from the realization that the high level of discretion in strategic decision making (or managerial flexibility) endowed with management, which allows them to discretionarily make necessary adjustments to adapt to any unexpected market condition, bears a striking resemblance to financial options, and conventional valuation tools such as DCF based techniques discussed earlier are vulnerable to such unexpected market conditions and hence incapable of measuring those inherent *real* options.

In contrast to a financial option which endues its owner with the right, but not the obligation, to purchase or sell a predetermined amount of the underlying financial assets at or before a specified future date for a specified price, a real option allows a manager to undertake certain business initiatives, such as expanding, contracting, deferring or abandoning an underlying capital investment project, and such managerial flexibilities could be measured mathematically in the same way as their financial counterparts. Accordingly, a investment opportunity could be considered as one or a series of coexistent or successive real options on real assets. For instance, a lease on a valuable land which grants its owner the right to construct a residential lot on it resembles an American call option sharing the same term of validity as the lease, with the corresponding underlying asset and strike price being the present value of the potential revenues that could be generated either from property rentals or outright sales, and the present value of the overall investment outlay, respectively. In other words, the real option captures the managerial flexibility available to the lease holder to wait until more information arrives and uncertainty resolves, and make the investment on the premise that the housing market is at its height, without bearing any obligation to invest and incur losses under unfavourable market condition.

Real option analysis could also be applied to more complex cases, such as a pharmaceutical research and development (R&D) project which usually consists of several consecutive development phases, during which "the firm gathers evidence to convince government regulators that it can consistently manufacture a safe and efficacious form of the compound for the medical condition it is intended to treat" [43]. Pennings and Sereno [77] studied the compoundness of a drug R&D project available to a pharmaceutical firm and argued that such an investment project could be considered as nested compound (or growth) real options. That is to say, by committing to the initial drug discovery the firm is implicitly granted an option to conduct the subsequent pre-clinical testing (the testing would be approved only on the premise that the drug discovery succeeds), its commitment to the second phase would open up the option to enter Phase 1 clinical trials, the Phase 1 trails then act as an option on Phase 2 trails and so on until the ultimate market launch.

As was discussed in Chapter 1, both the patent application process and the subsequent patent exploitation (or commercialization) are subject to a wide range of alternatives. To justify the feasibility of real option analysis in the context of patent valuation, we shall first treat patent application and patent exploitation as a unity, and then limit our focus to the application process and post-grant phase of a patent, respectively.

4.1.1 Patent application and patent exploitation

Recall that a patent, in contrast to a copyright, does not come into force instantaneously when the underlying technology asset is created. A sophisticated, sometimes time-consuming, application process must be gone through in exchange for its grant and any potential economical benefit that could be generated by exploiting the patent. It therefore makes perfect sense to apply real option thinking to this sequential procedure. This is because, by committing oneself to the lengthy patent application, the applicant has in fact acquired a right, but not an obligation, to make discretionary investment in patent commercialization projects once the application is approved. As was discussed in Section 1.1.2, the investor (e.g., the patentee and Technology Transfer Office) could either choose to pursue the economic dividends by, for instance, licensing the intellectual property or implementing the innovation it protects, and incur any necessary investment cost and maintenance fee; or simply let it lapse if the expected economic benefits do not seem to be enough to compensate for the accompanying costs. Since the investor has the right to run the project actively, this flexibility has value [77]. From a real option analysis standpoint, such flexibility exhibits the similar compoundness as witnessed in the pharmaceutical R&D project discussed above. That is, the commitment to one phase of the project offers an opportunity to proceed to the next one and the advance is possible only if the preceding phase succeeds. Therefore, a compound real option model could also be applied to a patenting decision making problem, with the underlying value of the patent application project being the subsequent commercialization option (the second option). Correspondingly, the exercise price of the option to file for the application (the first option) is the present value of the patent application fees.

The above interpretation is clearly not unique. In certain circumstances, an one-off investment is required at the beginning of the commercialization phase and the size of the investment depends on the nature of the underlying patent. For example, if a patent is granted on a revolutionary car manufacturing process, the implementation of such a patented innovation would inevitably cause considerable capital outlay, and the commercialization of the resulting product could be undertaken only if the implementation has been completed. It is therefore a legitimate question whether it is profitable to invest in the commercial project. From a real option analysis standpoint, the opportunity to invest resembles a European call option, with the underlying asset and exercise price being the present value of a claim on the expected commercial profits and the commercialization costs, respectively, and the patent application fees can be likened to dividend-like expenses before the option is exercised. Clearly, in the most general context, the task of assessing a patent commercialization project boils down to the pricing of

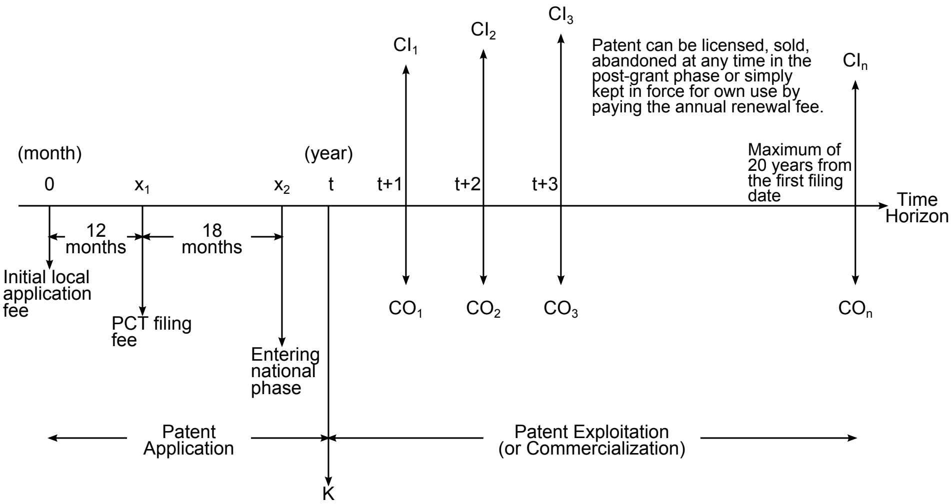


Figure 4.1: Capital outlays for patent application program and expected cash flows from the follow-on commercial project (or patent exploitation)

a European option, in other words, the decision on investing in commercialization simply rests with the value of this option.

Figure 4.1 exhibits the major capital outlays for a patent application project and the expected cash flows from the follow-on patent commercialization, in which CI_i and CO_i stand for the expected cash inflows and cash outflows at the i^{th} year into the post-grant phase, respectively, assuming the commercialization phase starts from year t, and K stands for the expected one-off investment at the beginning of the commercialization.

4.1.2 Patent application as an asset

As was noted in Section 1.1.2, a patent applicant will be confronted with the decision as to whether to continue or forego the application at several stages in the process, regardless of which filing route he follows. Let us take the EURO-PCT route as an example, an applicant has up to 12 months after the priority date to file for foreign applications with the EPO for the purpose of business expansion, he will subsequently face the decision on whether to proceed with or withdraw the application in the light of the international search report, the applicant may also request for an international preliminary examination of the underlying application afterwards and decide on the continuation according to the examination report. From a real option analysis standpoint, each of these opportunities to proceed resembles a call option, whose underlying asset and strike price are the option to continue the application to the next stage (except the last one) and the corresponding cost of continuance (e.g., foreign application fees, cost of international search and examination fees), respectively. Therefore, a patent application project, with each stage contingent on those that precede it, could be valued as a chain of call options (*i.e.*, nested compound options), each of which is exercisable by payment of the next requested fee. It should be noted that the underlying asset of the final link (*i.e.*, option) in the chain is the present value of the expected future monopoly benefits from the underlying patent.

In addition to the options to proceed, an applicant could also abandon the ongoing application at any aforementioned decision point and the right to abandon resembles a put option with the exercising price being the as yet unspent future application fees.

4.1.3 Post-grant phase of a patent

As was discussed in Section 1.1.2: Patent commercialization, the realization of patents as financial assets has helped accelerate the transition of patent exploitation strategy from the passive (*i.e.*, defensive) one of using patents merely as a prevention tool to obstruct competitors from developing rival technologies or as a legal weapon against infringement, to the proactive use of patents as alternative investment. For example, there is a common tendency among firms to license out their patents to entitle others to do their part in commercializing the benefits of the underlying technology asset and collect license fees and royalties in return. Additionally, a patent could be sold if the proprietor is pessimistic about its commercial promise and convinced that the future monopoly benefits would not be adequate to meet the maintenance costs. A patent could also be abandoned which allows its holder to avoid incurring the fixed costs of continuance. Given the wide variety of courses of action available to patent holders in the post-grant phase, it would be of great interest to identify the types of options inherent in a granted patent.

- As was noted above, patent licensing has been widely acknowledged and adopted by firms performing in a innovation-driven industry as a more effective means to reap economic profits from their intangible assets than other types of mechanism including trade secret, increasing complexity of product design and faster market introduction [64]. Accordingly, it has been argued that licensing outcomes have become not only an important indicator for a firm's (or a university's) innovation performance, but a measure for knowledge diffusion in society [53, 64, 73]. From an economic perspective, it is therefore in every patent applicant's and patentee's interest to realize, even before their patents are granted, the enormous economic potential inherent in a licensing opportunity. Real option based analysis serves as a natural candidate for measuring such economic potential. Since the opportunity to license out a patent endows its proprietor a right to sell the actual value of the patented innovation in exchange for certain (*i.e.*, licensing fee) and uncertain (*i.e.*, royalties) cash flows, it could be likened to an option to switch use [92]. To be more specific, the option to license translates into the patent proprietor's flexibility to choose the maximum of self-generated revenues without licensing (*i.e.*, income generated exclusively through patent monopoly), and licensing revenues plus the lessened monopoly benefits as a result of the "rent dissipation effect" [64] of licensing.

- Similar to the abandonment options embedded in a patent application project, a patentee is endowed with the flexibility to permanently abandon his patent at any stage into the post-grant phase before expiry in exchange for its salvage value, which is equivalent to the as yet unspent patent maintenance fees. From a real option analysis standpoint, the opportunity to let a patent lapse resembles an American put option which is exercisable up till the payday of the last maintenance fee.

- As was depicted in the introductory chapter, even though a patent could be kept in force for as long as 20 years (after the priority date) in theory, it is bound to face a wide range of challenges to both its validity and sustainability while it is being applied for and following grant. Assuming an European patent application manages to survive several rounds of sophisticated scrutiny and revisions and receives EPO approval, the patentee still has to investigate its future profitability and decide whether to keep it in force for another year on an annual basis. By now, it should have become intuitive to you that whenever one has the right to run a project actively, this flexibility has value, and this rule also applies to the patent renewal decision. That is to say, payment of a renewal fee for a granted patent buys not only the monopoly profits of the coming year but an option on renewing the patent at the end of the coming year which is exercisable by the next renewal fee. That being the case, a granted patent could be treated as a sequence of European call options on the following years benefits including an option to renew.

4.2 Real option analysis as a means to account for changing risk

We have seen in the introductory chapter that, in addition to its inefficiency in accounting for managerial flexibility inherent in either patent application or patent exploitation, the adaptability of DCF based valuation approaches to the context of patent is further hampered, or at least challenged, by its limited aptitude in accounting for the changing risk associated with the future cash flows expected throughout a patent's life.

Real option analysis is capable of bypassing the problem of changing risk with the construction of a replicating portfolio, which is comprised of risk-free bonds, and a publicly traded security exhibiting the same risk characteristics as the cash flows of the patent under consideration (*i.e.*, the twin asset in standard NPV analysis). By purchasing a certain number of shares of this twin asset while financing the purchase by selling risk-free bonds or borrowing at risk-free rate, a (replicating) portfolio could be constructed so that it yields the same payoff as the patent at all times, and thus has the same price. Accordingly, the same framework could also be used to replicate any embedded real option in the patent.

A real option could also be determined with a risk-neutral valuation, which is induced from the above replicating portfolio framework. Generally speaking, option replication is capable of implicitly transforming the actual scenario probabilities as seen in DTA which reflect investor's attitude to risk, to risk-neutral probabilities enabling all assets to be assessed in a risk-neutral world in which risk is irrelevant, or in other words, investors are indifferent to risk. Since investors' attitudes to risk do not matter in such a setup, all assets would earn the risk-free return and thus all expected cash flows could be discounted at the risk-free rate. It therefore successfully evades the intractable task of determining the appropriate discount rates proportional to risks.

4.3 Discussion

It is hoped that the previous discussion has convincingly justified the application of real option pricing methods to patent valuation and patenting decision making. However, it is important to point out that the above exemplification and interpretation of managerial flexibility in terms of real options can by no means be said to be exclusive or complete, and there does not exist a unique collection of real option based valuation methodologies that could be applied to patents. Such a realization could be generally attributed to two basic facts. First, as contrasted to financial options and other financial derivatives, the flexibility inherent in patents is rarely well defined and the variables needed for a full-fledged option pricing are not always intuitively identifiable. Second, the comprehension of latent value from flexibility might vary among managers. As was argued by Koller *et al.* [50], "a lot depends on management's ability to recognize, structure, and manage opportunities to create value from operating and strategic flexibility"; when experts' opinions are at stake, it is always the case that someone excels others in terms of knowledge and insight.

Another important issue requiring attention is that, in most capital investment projects, the inherent real options are not always directly measurable (see e.g., [87, 92, 23, 50]), but are embedded in the so-called "Expanded NPV" [92, 87] or "Contingent NPV" [50] which are normally more explicitly recognizable. The difference between expanded NPV and conventional NPV could be highlighted as follows:

- In contrast to conventional NPV based analysis which forces a decision according to "today's expectation of future information" [50], that is to say managerial choices are implicitly limited to the initial decision, expanded NPV also captures the value of active management in response to unexpected events and market movements in the future (*i.e.*, the value of making decision when more information arrives), which consists of two major components [92]:

$$\text{Expanded NPV} = \text{Static NPV} + \text{Option Premium}.$$

  Intuitively, the component of "Option Premium" above represents the value of the real option embedded. It is noticeable that static NPV, which is determined under the expected scenario of future cash flows, also forms a crucial component of the expanded NPV framework.

- The distinction between expanded NPV and conventional NPV could also been depicted in a less abstract manner. Recall that

$$\text{Static NPV} = \frac{\sum_{t=1}^{n} \text{Expected future cash flow at } t}{\text{Cost of Capital}}.$$

  Since expanded NPV is endowed with the merit of option thinking, it practically leads to skewness in the outcome distribution of the underlying project. This is because option analysis captures the benefits of forward looking judgments (*i.e.*, the right to adapt), but does not incur any symmetric obligation. Therefore, expanded NPV could be mathematically described as [50]

$$\boldsymbol{E}_{t=0}\left[\text{Max}\left(\frac{\text{Cash Flows Contingent on Information}}{\text{Cost of Capital}},\ 0\right)\right].$$

Chapter 5

Patenting Decision Making with Fuzzy Numbers and Real Option Analysis

Fundamentally, the problem of patenting decision making shares a similar principle with capital budgeting of investment projects, as they are both concerned with the allocation of resources among knowledge, innovation and other forms of intangible assets for the purpose of value maximization. The primary financial objective of a patent applicant is therefore to maximize his "satisfaction or utility of consumption across time" [92], and such satisfaction or utility is preferably manifested in monetary terms. Based on such an analogy, a patent application could be generally considered as an investment project which requires initial costs and has the potential to collect future economical benefits which unfold through time, and it is the value of this investment project that serves as the determinant of the patenting decision under consideration.

The feasibilities of fuzzy set theory in patent cash flows estimation and real option analysis in exploring managerial flexibility inherent in a patent have been elaborated in Chapters 3 and 4, respectively. It will be demonstrated in the rest of this chapter how a combination of real option analysis and fuzzy set theory could facilitate patenting decision making. In particular, two distinct pricing models, namely the pay-off method and extended fuzzy Black-Scholes model, have been selected for this purpose.

5.1 Facilitating patenting decision making with the pay-off method

The pay-off method, which was first coined by [17] in 2011, is a fuzzy logic based approach to assessing the profitability of capital investments. Briefly speaking, it takes three or four cash-flow scenarios for possible project costs and revenues as input, which normally include the optimistic (*i.e.*, maximum possible), best guess (or, in the case of four scenarios, maximum best guess and minimum best guess) and pessimistic (*i.e.*, minimum possible) scenarios; the scenarios then result in three or four cost (or revenue) estimates for each time period, which are fuzzified into an appropriate fuzzy number, depending on the number of scenarios adopted (*i.e.*, triangular-shaped for three scenarios or trapezoidal-shaped for four scenarios); these fuzzy numbers are subsequently aggregated which gives a fuzzy pay-off distribution (as of time zero) from the project (also a fuzzy number). The pay-off method implies that the negative outcomes (subject to terminating the project) be truncated into one chunk that will cause a zero payoff and the area-weighted average value of the resulting pay-off distribution be the value of the project. That is to say, the profitability of the underlying project is determined by the area-weighted average of the fuzzy mean value of the positive outcomes of the distribution and zero (because the negative outcomes result in a zero payoff). Mathematically speaking, the project value is determined as

$$\text{Profitability of project} = \frac{\text{Area of positive outcomes}}{\text{Total area under the pay-off distribution}} \times \text{Fuzzy mean value of the positive side.} \quad (5.1)$$

It is easy to see from Equation 5.1 that if the entire pay-off distribution is above zero, the fuzzy mean value of this distribution (as a fuzzy number) is the project's value, and if the entire pay-off distribution is below zero, the project is unprofitable.

Figure 5.1 illustrates a pay-off distribution depicted as a triangular fuzzy number $\widetilde{A}$. According to Equation 5.1, the profitability of the corresponding project represented by $\widetilde{A}$ could be determined as follows [17]:

$$\text{Profitability of project} = \frac{\int_0^\infty \widetilde{A}(x)\,\mathrm{d}x}{\int_{-\infty}^\infty \widetilde{A}(x)\,\mathrm{d}x} \times \boldsymbol{E}(\widetilde{A}_+), \quad (5.2)$$

where $\boldsymbol{E}(\widetilde{A}_+)$ stands for the fuzzy mean value of the positive side the fuzzy number $\widetilde{A}$, and $\int_0^\infty \widetilde{A}(x)\,\mathrm{d}x$ and $\int_{-\infty}^\infty \widetilde{A}(x)\,\mathrm{d}x$ measure the area of the positive side of $\widetilde{A}$ and the total area under $\widetilde{A}$, respectively.

The pay-off method could arguably be applied to real option valuation too [17]. First, it shares the same fundamental behind real option thinking,

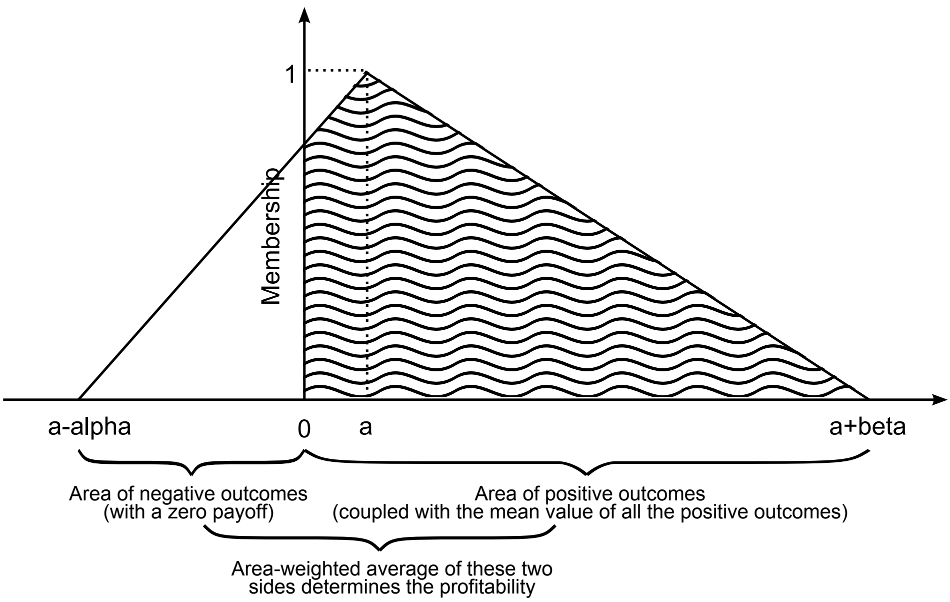


Figure 5.1: A triangular fuzzy number (a pay-off distribution) $\widetilde{A} = (a, \alpha, \beta)$ depicting the NPV of a prospective project

which is to capture the asymmetry or rather the upward skewness in the probability distribution of NPV through actively managing an investment project and hence improve its upside potential while limit its downside losses [92]. The pay-off method seeks to achieve this by pruning all the negative outcomes in the pay-off distribution from a project and focusing on those economically promising (*i.e.*, profitable) opportunities. Second, by taking cash-flow scenarios as input, the pay-off method implicitly translates the strategic intelligence of experts from different backgrounds into a business plan with flexibility. For example, the truncation of all the negative outcomes coincides with the real option analysis intuition that the project is terminated if a loss is forecast. Third, the manifestation of cash-flow scenarios at each time point as an appropriate fuzzy number properly reflects the uncertainty inherent in the corresponding cash-flow estimation (see Chapter 3).

5.1.1 The application of the pay-off method with two types of fuzzy numbers

This subsection will be devoted to the derivation of Equation 5.2 when the input pay-off distribution, namely $\widetilde{A}$, is a triangular-shaped and a trapezoidal-shaped fuzzy number, respectively.

If the pay-off distribution is a triangular fuzzy number, that is to say $\widetilde{A} = (a, \alpha, \beta)$, Equation 5.2 could take one of the following four forms:

- In the case of $0 \leq a - \alpha$, the entire distribution is above zero and thus the fuzzy mean value of the positive part of $\widetilde{A}$ is the fuzzy mean value of $\widetilde{A}$ (*i.e.*, $\boldsymbol{E}(\widetilde{A}_{+}) = \boldsymbol{E}(\widetilde{A})$). Recall that the fuzzy mean value of a triangular fuzzy number is (see Equation 6 in Chapter 3)

$$\begin{aligned}\boldsymbol{E}(\widetilde{A}) &= \int_0^1 [(a - (1-\gamma)\,\alpha) + (a + (1-\gamma)\,\beta)]\,\gamma\,\mathrm{d}\gamma \\ &= a + \frac{\beta - \alpha}{6}.\end{aligned}$$

  Therefore Equation 5.2 could be written as

$$\begin{aligned}\text{Profitability of project} &= \frac{\int_0^\infty \widetilde{A}(x)\,\mathrm{d}x}{\int_{-\infty}^\infty \widetilde{A}(x)\,\mathrm{d}x} \times \boldsymbol{E}(\widetilde{A}) \\ &= a + \frac{\beta - \alpha}{6}\end{aligned}$$

- In the case of $a - \alpha < 0 \leq a$, the fuzzy mean value of the positive part of $\widetilde{A}$ equals to

$$\boldsymbol{E}(\widetilde{A}_{+}) = \frac{(\alpha - a)^3}{6\alpha^2} + a + \frac{\beta - \alpha}{6}.$$

  Equation 5.2 then becomes

$$\begin{aligned}\text{Profitability of project} &= \frac{\int_{-\infty}^\infty \widetilde{A}(x)\,\mathrm{d}x - \int_{-\infty}^0 \widetilde{A}(x)\,\mathrm{d}x}{\int_{-\infty}^\infty \widetilde{A}(x)\,\mathrm{d}x} \times \boldsymbol{E}(\widetilde{A}_{+}) \\ &= \left(1 - \frac{\int_{-\infty}^0 \widetilde{A}(x)\,\mathrm{d}x}{\int_{-\infty}^\infty \widetilde{A}(x)\,\mathrm{d}x}\right) \times \boldsymbol{E}(\widetilde{A}_{+}) \\ &= \left(1 - \frac{(\alpha - a)^2}{\alpha\,(\alpha + \beta)}\right) \times \left(\frac{(\alpha - a)^3}{6\alpha^2} + a + \frac{\beta - \alpha}{6}\right).\end{aligned}$$

- In the case of $a < 0 < a + \beta$, the fuzzy mean value of the positive part of $\widetilde{A}$ equals

$$\boldsymbol{E}(\widetilde{A}_{+}) = \frac{(a + \beta)^3}{6\beta^2}$$

  Equation 5.2 then becomes

$$\begin{aligned}\text{Profitability of project} &= \frac{(a + \beta)^2}{\beta\,(\alpha + \beta)} \times \frac{(a + \beta)^3}{6\beta^2} \\ &= \frac{(a + \beta)^5}{6\beta^3\,(\alpha + \beta)}.\end{aligned}$$

- In the case of $a + \beta \leq 0$, the entire distribution is below zero and thus $\boldsymbol{E}(\widetilde{A}_+) = 0$.

If the pay-off distribution is a trapezoidal fuzzy number, that is to say $\widetilde{A} = (a, b, \alpha, \beta)$, Equation 5.2 could take one of the following five forms:

- In the case of $0 \leq a - \alpha$, since $\widetilde{A}$ is entirely above zero, the fuzzy mean value of its positive area equals to the fuzzy mean value of itself, which could be determined as (see Equation 7 in Chapter 3)

$$\begin{aligned}\boldsymbol{E}(\widetilde{A}) &= \int_0^1 [(a - (1-\gamma)\,\alpha) + (b + (1-\gamma)\,\beta)]\,\gamma\,\mathrm{d}\gamma \\ &= \frac{a+b}{2} + \frac{\beta - \alpha}{6}.\end{aligned}$$

Equation 5.2 could therefore be written as

$$\begin{aligned}\text{Profitability of project} &= \frac{\int_0^\infty \widetilde{A}(x)\,\mathrm{d}x}{\int_{-\infty}^\infty \widetilde{A}(x)\,\mathrm{d}x} \times \boldsymbol{E}(\widetilde{A}) \\ &= \frac{a+b}{2} + \frac{\beta - \alpha}{6}.\end{aligned}$$

- In the case of $a - \alpha < 0 < a$, the fuzzy mean value of the positive area of $\widetilde{A}$ is

$$\boldsymbol{E}(\widetilde{A}_+) = \frac{a+b}{2} + \frac{\beta - \alpha}{6} + \frac{(\alpha - a)^3}{6\alpha^3}.$$

Substituting the above $\boldsymbol{E}(\widetilde{A}_+)$ into Equation 5.2 gives

$$\begin{aligned}\text{Profitability of project} &= \frac{\int_0^\infty \widetilde{A}(x)\,\mathrm{d}x}{\int_{-\infty}^\infty \widetilde{A}(x)\,\mathrm{d}x} \times \left(\frac{a+b}{2} + \frac{\beta - \alpha}{6} + \frac{(\alpha - a)^3}{6\alpha^3}\right) \\ &= \left(1 - \frac{\int_{-\infty}^0 \widetilde{A}(x)\,\mathrm{d}x}{\int_{-\infty}^\infty \widetilde{A}(x)\,\mathrm{d}x}\right) \\ &\quad \times \left(\frac{a+b}{2} + \frac{\beta - \alpha}{6} + \frac{(\alpha - a)^3}{6\alpha^3}\right) \\ &= \left(1 - \frac{(\alpha - a)^2}{\alpha\,(2b - 2a + \alpha + \beta)}\right) \\ &\quad \times \left(\frac{a+b}{2} + \frac{\beta - \alpha}{6} + \frac{(\alpha - a)^3}{6\alpha^3}\right).\end{aligned} \tag{5.3}$$

- In the case of $a \leq 0 \leq b$, the fuzzy mean value of the positive area of $\widetilde{A}$ becomes

$$\boldsymbol{E}(\widetilde{A}_+) = \frac{b}{2} + \frac{\beta}{6}.$$

Substituting the above $\boldsymbol{E}(\widetilde{A}_+)$ into Equation 5.2 gives

$$\begin{aligned}\text{Profitability of project} &= \frac{\int_0^\infty \widetilde{A}(x)\,\mathrm{d}x}{\int_{-\infty}^\infty \widetilde{A}(x)\,\mathrm{d}x} \times \left(\frac{b}{2} + \frac{\beta}{6}\right) \\ &= \frac{2b+\beta}{2b-2a+\alpha+\beta} \times \left(\frac{b}{2} + \frac{\beta}{6}\right).\end{aligned}$$

- In the case of $b < 0 \leq b + \beta$, $\boldsymbol{E}(\widetilde{A}_+)$ is given as

$$\boldsymbol{E}(\widetilde{A}_+) = \frac{(b+\beta)^3}{6\beta^2}.$$

Equation 5.2 then becomes

$$\begin{aligned}\text{Profitability of project} &= \frac{\int_0^\infty \widetilde{A}(x)\,\mathrm{d}x}{\int_{-\infty}^\infty \widetilde{A}(x)\,\mathrm{d}x} \times \frac{(b+\beta)^3}{6\beta^2} \\ &= \frac{(b+\beta)^2}{\beta\,(2b-2a+\alpha+\beta)} \times \frac{(b+\beta)^3}{6\beta^2} \\ &= \frac{(b+\beta)^5}{6\beta^3\,(2b-2a+\alpha+\beta)}.\end{aligned}$$

- In the case of $b + \beta < 0$, Equation 5.2 leads to zero as $\boldsymbol{E}(\widetilde{A}_+)$ is zero.

5.1.2 Patenting decision making with the pay-off method

As was discussed in Chapter 4 Section 1.1: Patent application and patent exploitation, the sequential structure of patent application and patent exploitation manifests a high degree of compoundness which has also been witnessed in pharmaceutical R&D project. That is to say, the commitment to investing in patent application endows the applicant with an opportunity (*i.e.*, a right), but not an obligation, to proceed to patent commercialization once the patent is granted. In other words, the patent applicant is not required to make a full commitment to both patent application and commercialization at the outset, but is allowed to justify his decision to commercialize at a later time. This sequential setup and the inherent managerial flexibility can translate into a compound option from a real option analysis standpoint, that is:

- The underlying value of the patent application investment is the subsequent commercialization project with the corresponding exercise price being the present value (as of time zero) of the overall application fees. That is to say, the patent application project should be undertaken only if the value of the commercialization project surpasses the overall costs. Recall Figure 1 in Chapter 4, this exercise price which is labeled as I_0 could be calculated as

$$I_0 = \text{Initial local application fee} + \frac{\text{PCT filing fee}}{(1+r)^{\frac{x_1}{12}}} + \frac{\text{Major costs required in the national phase}}{(1+r)^{\frac{x_2}{12}}}, \qquad (5.4)$$

assuming that PCT filing fee is to be paid at the $x_1{}^{\text{th}}$ month, other major costs which are associated with internationalizing a patent application and include, but are not limited to, translation and official expenses and patent agents' and attorneys' fees (see Chapter 1) are expected at the $x_2{}^{\text{th}}$ month, and all the expenses required in the application process could be discounted at the patent-filing company's corporate bond rate [58] or at risk-free rate for an individual patent applicant, namely r.

- The opportunity to invest in the follow-on patent commercialization is the second option, which has a time to maturity of t years, starting from the priority date. The corresponding underlying asset and exercise price are the present value of the commercial project's expected future profit and the one-off investment of K, respectively. It is worth emphasizing that the quantity of K varies with the nature of the underlying intellectual asset. In automotive industry, for example, a considerable amount of patents are filed to protect the working parts of cars and the processes to be used to make their various components. The follow-on commercialization, which usually requires the implementation of such inventions in the form of, for example, installing new production lines, would inevitably cause large initial investment. In chemical and pharmaceutical industries, however, patent protection is usually sought for Markush structures and thus direct investment required for their implementation is relatively marginal.

In this study, the underlying intellectual asset is a new process for the preparation of a gypsum-fibre composite held by an anonymous water chemistry company, and it remains unclear if it would be economically beneficial to file for a patent on it. Nevertheless, it should be sufficiently clear by now that the application is worth pursuing only if the value of the subsequent patent commercialization option surpasses the overall application outlays. According to the generic mechanism for compound option valuation, the value of the second option, namely the patent commercialization project, needs to be determined first. It will be demonstrated next how the pay-off method could be used to calculate the profitability of the commercialization project, which will then serve as the determinant for the initial patenting decision.

Input data

In order to account for the uncertainty concerning the income in the post-grant phase of the patent, the cash inflows from patent commercialization are depicted with four cash-flow scenarios which were collected in the same manner as demonstrated at the beginning of this section, while the cash outflows in both phases are given as crisp numbers. In particular:

- Initial local application fee, PCT filing fee and the costs associated with internationalizing a patent application, which are required in the patent application process, are fixed (there is no uncertainty involved) and thus will be represented as crisp numbers.

- The one-off investment K at the beginning of the commercialization phase is assumed to be negligible in this case study.

- The four scenarios for possible cash inflows from the patent exploitation result in four cash-flow estimates for each year into the post-grant phase, which will be represented as a series of trapezoidal fuzzy numbers in the form of $\widetilde{CI}_j = (a_j, b_j, \alpha_j, \beta_j)$ with j indicating the year.

- The representation of the outlays over the post-grant phase depends on the nature of the patent. For the purpose of simplicity and expediency, crisp numbers will be used in this study, but for a more general description they may also be described in terms of fuzzy numbers when the uncertainty concerning the costs is significant. It is important to mention that a crisp number, c, can be seen as the trapezoidal fuzzy number $\tilde{c} = (c, c, 0, 0)$. So even if we know the value of the outlays precisely, they can still be included in the model as possibility distributions.

The result

For simplicity, we assume that the the patent is granted at year $t = 3$ and is commercialized immediately. The values of the cash inflows and outflows (and their present values as of year zero) during the 8-year commercialization project are listed in Table 5.1. The resulting pay-off distribution (the cumulative present value) is then represented as a trapezoidal fuzzy number, which is $\widetilde{A} = (a, b, \alpha, \beta) = (123.46, 1983.39, 2412.04, 2231.56)$. Since $a - \alpha < 0 < a$, according to Equation 5.3, the value of the patent commercialization project is 757.76.

Table 5.1: Cash flows of a patent in the commercialization phase with four scenarios

Years (starting from year t as in Figure 4.1)	**t**	**t+1**	**t+2**	**t+3**	**t+4**	**t+5**	**t+6**	**t+7**	**t+8**
Cash inflow									
$a-\alpha$ (minimum possible)	1000	1300	1000	1800	1300	1600	1700	1400	2000
a (minimum best guess)	2000	2300	2100	2300	1500	2000	1900	1600	2500
b (maximum best guess)	2500	2900	2600	2800	1700	2600	2700	1900	3200
$b+\beta$ (maximum possible)	3500	3700	3200	3300	2000	3000	3300	2400	3500
Present value of cash inflow	opportunity cost of capital $k = 15\%$								
$a-\alpha$ (minimum possible)	657.52	743.28	497.18	778.19	488.72	523.04	483.25	346.06	429.89
a (minimum best guess)	1315.03	1315.03	1044.07	994.35	563.91	653.80	540.10	395.50	537.36
b (maximum best guess)	1643.79	1658.08	1292.66	1210.52	639.09	849.94	767.51	469.65	687.82
$b+\beta$ (maximum possible)	2301.31	2115.49	1590.97	1426.68	751.87	980.71	938.07	593.24	752.30
Cash outflows	465	580	810	1040	1155	1265	1380	1560	1560
Present value of cash outflows	risk-free rate $r = 4\%$								
	413.38	495.79	665.76	821.93	877.71	924.32	969.57	1053.88	1013.35
Profit									
$a-\alpha$	244.13	247.49	-168.58	-43.74	-388.99	-401.28	-486.32	-707.82	-583.46
a	901.65	819.25	378.31	172.43	-313.80	-270.52	-429.47	-658.38	-475.99
b	1230.41	1162.30	626.90	388.59	-238.61	-74.38	-202.06	-584.23	-325.53
$b+\beta$	1887.92	1619.70	925.20	604.75	-125.83	56.38	-31.50	-460.64	-261.04

Having proved the profitability of the commercialization project, it comes to the first question: is the patent application worth pursuing? As was noted above, the initial patenting decision resembles a call option with the underlying value being the commercialization option and the exercise price being the overall application expenses I_0. Assume the parameters are as follows: initial local application fee is 100, the PCT filing fee is 200 which is due at the 12$^{\text{th}}$ month (*i.e.*, $x_1 = 12$), and the cost required in the national phase is 150 which is paid at the 30$^{\text{th}}$ month (*i.e.*, $x_2 = 30$). It is worth emphasizing again that if there is risk perceived in the first stage, namely the patent application process, the same scenario approach could also be used for calculation. Substituting the numbers above into Equation 5.4 gives $I_0 = 428.3$. When we compare this to the value of the commercialization project, the result is

$$\text{Max}\,(757.76 - I_0, 0) = 329.46.$$

It is therefore profitable for the water chemistry company to seek patent protection on its revolutionary chemical processing technique and position itself to take advantage of the future monopoly benefits.

5.2 Justifying patent commercialization decision with the extended fuzzy Black-Scholes model

The Black-Scholes model, which was first coined by Black and Scholes [11] and subsequently extended by Merton [59], has been widely accepted as the standard for (financial) option pricing. Briefly speaking, it is obtained by solving the partial differential equation which results from the continuous application of the replicating portfolio strategy under certain assumptions. Due to its great computational simplicity and relative accuracy, the Black-Scholes model has also become prevalent in valuing real options [14]. For example, Hartmann and Hassan [35] concluded in their survey article that, in comparison to other techniques including the decision tree analysis, binomial model and Geske's compound option model, the application of the Black-Scholes model had received higher popularity in the pharmaceutical section for both R&D project and company valuations.

5.2.1 The extended fuzzy Black-Scholes model

However, despite the rationality of using both real option analysis and fuzzy set theory in making real investment decisions (see Chapters 3 and 4), the study combining both techniques, namely fuzzy real option analysis, still remains as a less exploited topic, let alone the attempt to adjust the Black-Scholes model under the fuzzy framework. Among the relatively limited literature are the works of Yoshida [96, 97] and the contribution from Carlsson and Fullér [15]. The former extends the renowned model by simultaneously accounting for two sources of uncertainties, namely randomness arising from stock price changes and fuzziness as a result of the lack of knowledge in regard to the present stock market. To do so, Yoshida introduces a fuzzy stochastic process assumed to underlie the stock price, which is defined as a family of fuzzy random variables taking on fuzzy numbers as values. The extended model then determines the prices of European-style options (see e.g., [96]) or American put option (see e.g., [97]) which are also represented as fuzzy numbers. However, the feasibility of Yoshida's extended Black-Scholes model in real option valuation, in particular in patent valuation, remains to be verified. The major concern is raised by the fact that, as was elaborated in Chapter 3, the uncertainties encountered in patent valuation are mostly human-related as they arise either from incomplete knowledge or from decision makers' limited ability to perceive, process and develop on the available information, while Yoshida's model does not seem capable of taking into consideration all these subjective interpretations. In particular, in his proposed models, all the subjective judgements from decision makers have to be condensed into one or two weighting indices, which could never be said to be easy under the context of patent.

Carlsson and Fullér [15] propose a different approach to the Black-Scholes model extension. That is, they allow the extended model to directly take on fuzzy numbers (*e.g.*, fuzzy underlying asset price and fuzzy exercise price) as input. Such modification has great practical merit of being useful in valuing real options as "from a computational point of view it is easier to use linear membership functions and, more importantly, our experience shows that senior managers prefer trapezoidal fuzzy numbers to Gaussian ones when they estimate the uncertainties associated with future cash inflows and outflows" [15]. Indeed, the extended model further improve the Black-Scholes formula by taking into consideration the practical difficulties in characterizing the present value of expected cash flows with one single number, which is especially the case when the underlying is a real asset such as real estate, agricultural land, special purpose machinery or when the underlying asset has lower liquidity or no established market such as intellectual properties. The proposed pricing model for a European-style call option is given as follows:

$$\widetilde{C}(\widetilde{S}, 0) = \widetilde{S}_0 e^{-\delta T} N(d_1) - \widetilde{X} e^{-rT} N(d_2), \tag{5.5}$$

where

$$\begin{aligned} d_1 &= \frac{\ln\left(\frac{\boldsymbol{E}(\widetilde{S}_0)}{\boldsymbol{E}(\widetilde{X})}\right) + \left(r - \delta + \frac{\sigma^2}{2}\right) T}{\sigma\sqrt{T}}, \\ d_2 &= d_1 - \sigma\sqrt{T}. \end{aligned} \tag{5.6}$$

Note that:

- Fuzzy numbers are capped by a tilde ($\tilde{}$), including the resulting option price.

- δ in Equation 5.5 stands for the constant and continuous dividend yield on the underlying stock of the (financial) option; in the case of a real option, δ denotes the value lost over the duration of the option, such as the foregone income resulting from postponing production or distribution (*e.g.*, option to defer) and the market revenues carved up by competitors as a result of "weak isolating mechanisms" [90] which are commonly seen in fast-cycle markets.

- $\boldsymbol{E}(\widetilde{S}_0)$ and $\boldsymbol{E}(\widetilde{X})$ in Equation 5.6 denote the fuzzy mean value of the underlying asset (or the present value of expected cash flows) and fuzzy mean value of the exercise price, respectively.

However, the above extension of the Black-Scholes model could be further improved with two adjustments. First, as was noted in Equation 5.6, the fuzzy mean values of the underlying asset and the exercise price are calculated respectively before their ratio is taken (*i.e.*, $\boldsymbol{E}(\widetilde{S}_0)/\boldsymbol{E}(\widetilde{X})$), which prematurely eliminates the uncertainties manifested in each (fuzzy) number. Second, it assumes a constant and continuous dividend yield which could be troublesome to determine in real option analysis. In real world scenarios, for example an information system investment opportunity available to a company for a certain period of time, this company would routinely review the opportunity at discrete time intervals, such as every month, quarter or half year since the investment concerns one of the fast-cycle markets as noted above, and justify its decision to invest or to defer. Once the decision to defer is made at one of these periodic reviews, it would cause the company to forfeit any possible revenues up to the next review, which resembles the effect dividend has on a financial option. However, in contrast to a financial option, the management could control the size of forgone cash flows by altering the review frequency [14]. That is to say, the higher the review frequency, the less foregone revenue between decision instances. Therefore, it could be argued that the forfeit could be estimated, but only as discrete projections.

Having realized the obstacles to its implementation, we propose the following extended model for fuzzy real option valuation which is based on the work of Carlsson and Fullér (*i.e.*, Equation 5.5 and 5.6):

$$\widetilde{C}(\widetilde{S},0) = \left(\widetilde{S}_0 - \sum_{i=1}^{n} D_i e^{-r\tau_i}\right) N(d_1) - \widetilde{X}e^{-rT} N(d_2), \qquad (5.7)$$

where

$$\begin{aligned} d_1 &= \frac{\ln\left(\boldsymbol{E}\left(\frac{\widetilde{S}_0-\sum_{i=1}^{n} D_i e^{-r\tau_i}}{\widetilde{X}}\right)\right)+\left(r+\frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}, \\ d_2 &= d_1-\sigma\sqrt{T}. \end{aligned} \tag{5.8}$$

The proposed extended model relies on the following assumptions in addition to the standard assumptions for option pricing:

- A finite number of discrete, dividend-like losses in value are anticipated before maturity.
- The quantities and timing of these value deductions are known.
- The option is not exercised prematurely.

Note that in Equation 5.7 and 5.8 D_i stands for the i^{th} value deduction which is anticipated at τ_i and $1 \le i \le n$. Also note that the ratio of the adjusted underlying asset value by subtracting the present value of all the known "dividends" and the exercise price is calculated prior to the mean value of the ratio (*i.e.*, $\boldsymbol{E}((\widetilde{S}_0 - \sum_{i=1}^{n} D_i e^{-r\tau_i})/\widetilde{X})$). Unfortunately, such an improvement would inevitably incur much greater computational complexities as the calculation of the ratio of two fuzzy numbers has proved to be a difficult task. It will be demonstrated in the next subsection how the membership functions of $\widetilde{S}_0$ and $\widetilde{X}$, given as two trapezoidal fuzzy numbers, would affect the value of the ratio.

The ratio of $\widetilde{S}_0 - \sum_{i=1}^{n} D_i e^{-r\tau_i}$ to $\widetilde{X}$

Recall that a γ-cut (or a γ-level set) of a fuzzy number A defined in $\mathbb{R}$ is $[A]^\gamma = \{x \in \mathbb{R} | A(x) \ge \gamma\}$ if $0 < \gamma \le 1$ and $[A]^\gamma = \text{cl}\{x \in \mathbb{R} | A(x) > \gamma\}$ (the closure of the support of A) if $\gamma = 0$. If both $\widetilde{S}_0$ and $\widetilde{X}$ are trapezoidal fuzzy numbers, they could be represented in terms of γ-cuts as

$$\begin{aligned} [\widetilde{S}_0]^\gamma &= \left[a_{\widetilde{S}_0} - \alpha_{\widetilde{S}_0} + \gamma\alpha_{\widetilde{S}_0},\ b_{\widetilde{S}_0} + \beta_{\widetilde{S}_0} - \gamma\beta_{\widetilde{S}_0}\right], \\ [\widetilde{X}]^\gamma &= \left[a_{\widetilde{X}} - \alpha_{\widetilde{X}} + \gamma\alpha_{\widetilde{X}},\ b_{\widetilde{X}} + \beta_{\widetilde{X}} - \gamma\beta_{\widetilde{X}}\right]. \end{aligned}$$

Note that $\widetilde{S}_0 - \sum_{i=1}^{n} D_i e^{-r\tau_i}$ is also a trapezoidal fuzzy number, which is labelled as $\widetilde{S}_0^*$ for simplicity. It could also be represented in terms of γ-cuts, which is

$$[\widetilde{S}_0^*]^\gamma = \left[a_{\widetilde{S}_0^*} - \alpha_{\widetilde{S}_0^*} + \gamma\alpha_{\widetilde{S}_0^*},\ b_{\widetilde{S}_0^*} + \beta_{\widetilde{S}_0^*} - \gamma\beta_{\widetilde{S}_0^*}\right].$$

The ratio of $\widetilde{S}_0^*$ to $\widetilde{X}$ is defined in terms of arithmetic operations on their γ-cuts (*i.e.*, arithmetic operation on closed intervals). That is

$$\begin{aligned}\left[\frac{\widetilde{S}_0^*}{\widetilde{X}}\right]^\gamma &= \frac{\left[a_{\widetilde{S}_0^*} - \alpha_{\widetilde{S}_0^*} + \gamma\alpha_{\widetilde{S}_0^*},\ b_{\widetilde{S}_0^*} + \beta_{\widetilde{S}_0^*} - \gamma\beta_{\widetilde{S}_0^*}\right]}{\left[a_{\widetilde{X}} - \alpha_{\widetilde{X}} + \gamma\alpha_{\widetilde{X}},\ b_{\widetilde{X}} + \beta_{\widetilde{X}} - \gamma\beta_{\widetilde{X}}\right]} \\ &= \left[a_{\widetilde{S}_0^*} - \alpha_{\widetilde{S}_0^*} + \gamma\alpha_{\widetilde{S}_0^*},\ b_{\widetilde{S}_0^*} + \beta_{\widetilde{S}_0^*} - \gamma\beta_{\widetilde{S}_0^*}\right] \\ &\quad \times \left[\frac{1}{b_{\widetilde{X}} + \beta_{\widetilde{X}} - \gamma\beta_{\widetilde{X}}},\ \frac{1}{a_{\widetilde{X}} - \alpha_{\widetilde{X}} + \gamma\alpha_{\widetilde{X}}}\right] \\ &= \left[\min\left(\frac{a_{\widetilde{S}_0^*} - \alpha_{\widetilde{S}_0^*} + \gamma\alpha_{\widetilde{S}_0^*}}{a_{\widetilde{X}} - \alpha_{\widetilde{X}} + \gamma\alpha_{\widetilde{X}}}, \frac{a_{\widetilde{S}_0^*} - \alpha_{\widetilde{S}_0^*} + \gamma\alpha_{\widetilde{S}_0^*}}{b_{\widetilde{X}} + \beta_{\widetilde{X}} - \gamma\beta_{\widetilde{X}}},\right.\right. \\ &\qquad \left.\frac{b_{\widetilde{S}_0^*} + \beta_{\widetilde{S}_0^*} - \gamma\beta_{\widetilde{S}_0^*}}{a_{\widetilde{X}} - \alpha_{\widetilde{X}} + \gamma\alpha_{\widetilde{X}}}, \frac{b_{\widetilde{S}_0^*} + \beta_{\widetilde{S}_0^*} - \gamma\beta_{\widetilde{S}_0^*}}{b_{\widetilde{X}} + \beta_{\widetilde{X}} - \gamma\beta_{\widetilde{X}}}\right), \\ &\quad \max\left(\frac{a_{\widetilde{S}_0^*} - \alpha_{\widetilde{S}_0^*} + \gamma\alpha_{\widetilde{S}_0^*}}{a_{\widetilde{X}} - \alpha_{\widetilde{X}} + \gamma\alpha_{\widetilde{X}}}, \frac{a_{\widetilde{S}_0^*} - \alpha_{\widetilde{S}_0^*} + \gamma\alpha_{\widetilde{S}_0^*}}{b_{\widetilde{X}} + \beta_{\widetilde{X}} - \gamma\beta_{\widetilde{X}}},\right. \\ &\qquad \left.\left.\frac{b_{\widetilde{S}_0^*} + \beta_{\widetilde{S}_0^*} - \gamma\beta_{\widetilde{S}_0^*}}{a_{\widetilde{X}} - \alpha_{\widetilde{X}} + \gamma\alpha_{\widetilde{X}}}, \frac{b_{\widetilde{S}_0^*} + \beta_{\widetilde{S}_0^*} - \gamma\beta_{\widetilde{S}_0^*}}{b_{\widetilde{X}} + \beta_{\widetilde{X}} - \gamma\beta_{\widetilde{X}}}\right)\right],\end{aligned}$$

provided that $0 \notin [a_{\widetilde{X}} - \alpha_{\widetilde{X}} + \gamma\alpha_{\widetilde{X}},\ b_{\widetilde{X}} + \beta_{\widetilde{X}} - \gamma\beta_{\widetilde{X}}]$.

Because $\widetilde{X}$ stands for the exercise price in the present context which is always positive, the ratio of $\widetilde{S}_0^*$ to $\widetilde{X}$ is properly defined. It is obvious that $[\widetilde{S}_0^*/\widetilde{X}]^\gamma$ varies along the left and right endpoints of $[a_{\widetilde{S}_0^*} - \alpha_{\widetilde{S}_0^*} + \gamma\alpha_{\widetilde{S}_0^*},\ b_{\widetilde{S}_0^*} + \beta_{\widetilde{S}_0^*} - \gamma\beta_{\widetilde{S}_0^*}]$. In other words, its value depends on where the zero line crosses the membership function of $\widetilde{S}_0^*$. Consequently, it could take one of the following five values:

- $a_{\widetilde{S}_0^*} - \alpha_{\widetilde{S}_0^*} \geq 0$:

$$\left[\frac{\widetilde{S}_0^*}{\widetilde{X}}\right]^\gamma = \left[\frac{a_{\widetilde{S}_0^*} - \alpha_{\widetilde{S}_0^*} + \gamma\alpha_{\widetilde{S}_0^*}}{b_{\widetilde{X}} + \beta_{\widetilde{X}} - \gamma\beta_{\widetilde{X}}},\ \frac{b_{\widetilde{S}_0^*} + \beta_{\widetilde{S}_0^*} - \gamma\beta_{\widetilde{S}_0^*}}{a_{\widetilde{X}} - \alpha_{\widetilde{X}} + \gamma\alpha_{\widetilde{X}}}\right].$$

- $a_{\widetilde{S}_0^*} - \alpha_{\widetilde{S}_0^*} < 0 < a_{\widetilde{S}_0^*}$:

$$\left[\frac{\widetilde{S}_0^*}{\widetilde{X}}\right]^\gamma = \begin{cases}\left[\frac{a_{\widetilde{S}_0^*} - \alpha_{\widetilde{S}_0^*} + \gamma\alpha_{\widetilde{S}_0^*}}{a_{\widetilde{X}} - \alpha_{\widetilde{X}} + \gamma\alpha_{\widetilde{X}}},\ \frac{b_{\widetilde{S}_0^*} + \beta_{\widetilde{S}_0^*} - \gamma\beta_{\widetilde{S}_0^*}}{a_{\widetilde{X}} - \alpha_{\widetilde{X}} + \gamma\alpha_{\widetilde{X}}}\right] & \text{if } 0 \leq \gamma \leq \frac{\alpha_{\widetilde{S}_0^*} - a_{\widetilde{S}_0^*}}{\alpha_{\widetilde{S}_0^*}}, \\ \left[\frac{a_{\widetilde{S}_0^*} - \alpha_{\widetilde{S}_0^*} + \gamma\alpha_{\widetilde{S}_0^*}}{b_{\widetilde{X}} + \beta_{\widetilde{X}} - \gamma\beta_{\widetilde{X}}},\ \frac{b_{\widetilde{S}_0^*} + \beta_{\widetilde{S}_0^*} - \gamma\beta_{\widetilde{S}_0^*}}{a_{\widetilde{X}} - \alpha_{\widetilde{X}} + \gamma\alpha_{\widetilde{X}}}\right] & \text{if } \frac{\alpha_{\widetilde{S}_0^*} - a_{\widetilde{S}_0^*}}{\alpha_{\widetilde{S}_0^*}} < \gamma \leq 1.\end{cases}$$

- $a_{\widetilde{S}_0^*} \leq 0 \leq b_{\widetilde{S}_0^*}$:

$$\left[\frac{\widetilde{S}_0^*}{\widetilde{X}}\right]^\gamma = \left[\frac{a_{\widetilde{S}_0^*} - \alpha_{\widetilde{S}_0^*} + \gamma\alpha_{\widetilde{S}_0^*}}{a_{\widetilde{X}} - \alpha_{\widetilde{X}} + \gamma\alpha_{\widetilde{X}}},\ \frac{b_{\widetilde{S}_0^*} + \beta_{\widetilde{S}_0^*} - \gamma\beta_{\widetilde{S}_0^*}}{a_{\widetilde{X}} - \alpha_{\widetilde{X}} + \gamma\alpha_{\widetilde{X}}}\right].$$

- $b_{\widetilde{S}_0^*} < 0 < b_{\widetilde{S}_0^*} + \beta_{\widetilde{S}_0^*}$:

$$\left[\frac{\widetilde{S}_0^*}{\widetilde{X}}\right]^\gamma = \begin{cases} \left[\frac{a_{\widetilde{S}_0^*} - \alpha_{\widetilde{S}_0^*} + \gamma\alpha_{\widetilde{S}_0^*}}{a_{\widetilde{X}} - \alpha_{\widetilde{X}} + \gamma\alpha_{\widetilde{X}}}, \frac{b_{\widetilde{S}_0^*} + \beta_{\widetilde{S}_0^*} - \gamma\beta_{\widetilde{S}_0^*}}{a_{\widetilde{X}} - \alpha_{\widetilde{X}} + \gamma\alpha_{\widetilde{X}}}\right] & \text{if } 0 \leq \gamma \leq \frac{b_{\widetilde{S}_0^*} + \beta_{\widetilde{S}_0^*}}{\beta_{\widetilde{S}_0^*}}, \\ \left[\frac{a_{\widetilde{S}_0^*} - \alpha_{\widetilde{S}_0^*} + \gamma\alpha_{\widetilde{S}_0^*}}{a_{\widetilde{X}} - \alpha_{\widetilde{X}} + \gamma\alpha_{\widetilde{X}}}, \frac{b_{\widetilde{S}_0^*} + \beta_{\widetilde{S}_0^*} - \gamma\beta_{\widetilde{S}_0^*}}{b_{\widetilde{X}} + \beta_{\widetilde{X}} - \gamma\beta_{\widetilde{X}}}\right] & \text{if } \frac{b_{\widetilde{S}_0^*} + \beta_{\widetilde{S}_0^*}}{\beta_{\widetilde{S}_0^*}} < \gamma \leq 1. \end{cases}$$

- $b_{\widetilde{S}_0^*} + \beta_{\widetilde{S}_0^*} \leq 0$:

$$\left[\frac{\widetilde{S}_0^*}{\widetilde{X}}\right]^\gamma = \left[\frac{a_{\widetilde{S}_0^*} - \alpha_{\widetilde{S}_0^*} + \gamma\alpha_{\widetilde{S}_0^*}}{a_{\widetilde{X}} - \alpha_{\widetilde{X}} + \gamma\alpha_{\widetilde{X}}}, \frac{b_{\widetilde{S}_0^*} + \beta_{\widetilde{S}_0^*} - \gamma\beta_{\widetilde{S}_0^*}}{b_{\widetilde{X}} + \beta_{\widetilde{X}} - \gamma\beta_{\widetilde{X}}}\right].$$

Having determined $[\widetilde{S}_0^*/\widetilde{X}]^\gamma$, the mean value of the ratio of $\widetilde{S}_0^*$ to $\widetilde{X}$ (*i.e.*, $\boldsymbol{E}((\widetilde{S}_0 - \sum_{i=1}^n D_i e^{-r\tau_i})/\widetilde{X})$ as in Equation 5.8) could be calculated by the definition of weighted fuzzy mean value in Chapter 3. That is to say, $\boldsymbol{E}(\widetilde{S}_0^*/\widetilde{X})$ is the level-weighted average of the arithmetic mean of all the γ-cuts (*i.e.*, $[\widetilde{S}_0^*/\widetilde{X}]^\gamma$), with 2γ being the weight of the arithmetic mean of the two endpoints of each $[\widetilde{S}_0^*/\widetilde{X}]^\gamma$ and $\gamma \in [0, 1]$. Without loss of generality, $[\widetilde{S}_0^*/\widetilde{X}]^\gamma$ could be simplified as

$$\left[\frac{\widetilde{S}_0^*}{\widetilde{X}}\right]^\gamma = \left[\frac{a + \gamma b}{c + \gamma d}, \frac{a^* + \gamma b^*}{c^* + \gamma d^*}\right].$$

According to the definition of weighted fuzzy mean value in Chapter 3 and the membership function of $\widetilde{S}_0^*$, $\boldsymbol{E}(\widetilde{S}_0^*/\widetilde{X})$ could be defined as

$$\begin{aligned} \boldsymbol{E}\left(\frac{\widetilde{S}_0^*}{\widetilde{X}}\right) &= \int_0^1 2\gamma \frac{\frac{a+\gamma b}{c+\gamma d} + \frac{a^*+\gamma b^*}{c^*+\gamma d^*}}{2} \,\mathrm{d}\gamma \\ &= \int_0^1 \gamma \left(\frac{a + \gamma b}{c + \gamma d} + \frac{a^* + \gamma b^*}{c^* + \gamma d^*}\right) \mathrm{d}\gamma \\ &= \int_0^1 \gamma \frac{a + \gamma b}{c + \gamma d} \,\mathrm{d}\gamma + \int_0^1 \gamma \frac{a^* + \gamma b^*}{c^* + \gamma d^*} \,\mathrm{d}\gamma, \end{aligned}$$

or

$$\boldsymbol{E}\left(\frac{\widetilde{S}_0^*}{\widetilde{X}}\right) = \int_0^{\frac{\alpha_{\widetilde{S}_0^*} - a_{\widetilde{S}_0^*}}{\alpha_{\widetilde{S}_0^*}}} 2\gamma \, \frac{\frac{a+\gamma b}{c+\gamma d} + \frac{a^*+\gamma b^*}{c^*+\gamma d^*}}{2} \, \mathrm{d}\gamma + \int_{\frac{\alpha_{\widetilde{S}_0^*} - a_{\widetilde{S}_0^*}}{\alpha_{\widetilde{S}_0^*}}}^{1} 2\gamma \, \frac{\frac{e+\gamma f}{g+\gamma h} + \frac{e^*+\gamma f^*}{g^*+\gamma h^*}}{2} \, \mathrm{d}\gamma$$

$$= \int_0^{\frac{\alpha_{\widetilde{S}_0^*} - a_{\widetilde{S}_0^*}}{\alpha_{\widetilde{S}_0^*}}} \gamma \left(\frac{a+\gamma b}{c+\gamma d} + \frac{a^*+\gamma b^*}{c^*+\gamma d^*}\right) \mathrm{d}\gamma + \int_{\frac{\alpha_{\widetilde{S}_0^*} - a_{\widetilde{S}_0^*}}{\alpha_{\widetilde{S}_0^*}}}^{1} \gamma \left(\frac{e+\gamma f}{g+\gamma h} + \frac{e^*+\gamma f^*}{g^*+\gamma h^*}\right) \mathrm{d}\gamma$$

$$= \int_0^{\frac{\alpha_{\widetilde{S}_0^*} - a_{\widetilde{S}_0^*}}{\alpha_{\widetilde{S}_0^*}}} \gamma \, \frac{a+\gamma b}{c+\gamma d} \, \mathrm{d}\gamma + \int_0^{\frac{\alpha_{\widetilde{S}_0^*} - a_{\widetilde{S}_0^*}}{\alpha_{\widetilde{S}_0^*}}} \gamma \, \frac{a^*+\gamma b^*}{c^*+\gamma d^*} \, \mathrm{d}\gamma + \int_{\frac{\alpha_{\widetilde{S}_0^*} - a_{\widetilde{S}_0^*}}{\alpha_{\widetilde{S}_0^*}}}^{1} \gamma \, \frac{e+\gamma f}{g+\gamma h} \, \mathrm{d}\gamma + \int_{\frac{\alpha_{\widetilde{S}_0^*} - a_{\widetilde{S}_0^*}}{\alpha_{\widetilde{S}_0^*}}}^{1} \gamma \, \frac{e^*+\gamma f^*}{g^*+\gamma h^*} \, \mathrm{d}\gamma.$$

Clearly it suffices to show the expansion of $\int \gamma \, \frac{a+\gamma b}{c+\gamma d} \, \mathrm{d}\gamma$, which is

$$\int \gamma \, \frac{a+\gamma b}{c+\gamma d} \, \mathrm{d}\gamma = \frac{d\gamma \left(2ad - 2bc + bd\gamma\right) + 2c\left(bc - ad\right) \ln\left(c + d\gamma\right)}{2d^3} + C.$$

The rest of the calculation is rather straightforward. However, it is worth reminding that the resulting option price given by Equation 5.7 would still be a fuzzy number if any of the underlying asset price $\widetilde{S}_0$ and exercise price $\widetilde{X}$ is represented by a fuzzy number. Please refer to Chapter 3 for the rules of arithmetic operations on fuzzy numbers. It will be demonstrated in the next subsection how the proposed extended fuzzy Black-Scholes model could be applied to a patenting decision making problem.

5.2.2 Justifying commercialization decision with the extended fuzzy Black-Scholes model

Recall the aforementioned industrial case study, in which the IPR manager of an anonymous water chemistry company is faced with the decision on whether to file for a patent application on a new process for the preparation of a gypsum-fibre composite. Assume that, in addition to the patent application fees and patent maintenance fees, an outlay K at the beginning of the commercialization phase (see Figure 1 in Chapter 4) is prerequisite for the implementation of such a process and the quantity of the investment could not be determined precisely, while all other conditions remain unchanged.

Now the IPR manager is faced with another decision: is it profitable to invest in the commercial project?

As was claimed in Chapter 4, during the patent application process, the applicant is not required to make an advanced commitment to invest in patent commercialization once it is granted, but has the right to wait until more information arrives and make the investment only if the market condition turns out to be favourable for him at that time. Hence, the decision making problem regarding patent commercialization at hand is essentially the same as finding the expanded NPV of the project which, from a real option analysis standpoint, is equal to the price of a European call option with the underlying asset and exercise price being the present value of a claim on the expected commercial profits and the commercialization costs (*i.e.*, $X = K$), respectively. Since the patent application fees are paid over the life of this option which is the length of the entire application process, they could arguably be considered as a form of value loss or rather the cost of holding the option which resembles the effect dividend has on a financial option. Intuitively, the extended fuzzy Black-Scholes model is adequate to the valuation task.

Input data and result

Recall that the application period is assumed to last for three years and the patent would be kept in force for eight years once granted. Three major outlays are required in the patent application process, which are the initial local application fee (100), the PCT filing fee (200) at the 12$^{\text{th}}$ month and the cost (150) accrued in the national phase which is due at the 30$^{\text{th}}$ month. The one-off investment K is given as a trapezoidal fuzzy number, which is

$$\widetilde{K} = (a_{\widetilde{K}}, b_{\widetilde{K}}, \alpha_{\widetilde{K}}, \beta_{\widetilde{K}}) = (1500, 1700, 200, 100).$$

The revenue and maintenance cost of each year into the post-grant phase, together with their present values as of time zero, are presented in Table 5.1. The resulting NPV is represented as a trapezoidal fuzzy number, which is

$$\widetilde{S}_0 = (a_{\widetilde{S}_0}, b_{\widetilde{S}_0}, \alpha_{\widetilde{S}_0}, \beta_{\widetilde{S}_0}) = (123.46, 1983.39, 2412.04, 2231.56).$$

As was noted, the adjusted current asset value in Equation 5.7, namely $\widetilde{S}_0^* = \widetilde{S}_0 - \sum_{i=1}^{n} D_i e^{-r\tau_i}$, is also a trapezoidal fuzzy number:

$$\widetilde{S}_0^* = (a_{\widetilde{S}_0^*}, b_{\widetilde{S}_0^*}, \alpha_{\widetilde{S}_0^*}, \beta_{\widetilde{S}_0^*}) = (-304.41, 1555.50, 2412.04, 2231.56).$$

Since $a_{\widetilde{S}_0^*} < 0 < b_{\widetilde{S}_0^*}$, the ratio of $\widetilde{S}_0^*$ to $\widetilde{K}$ takes the following form (see Section 5.2.1)

$$\left[\frac{\widetilde{S}_0^*}{\widetilde{K}}\right]^{\gamma} = \left[\frac{a_{\widetilde{S}_0^*} - \alpha_{\widetilde{S}_0^*} + \gamma\alpha_{\widetilde{S}_0^*}}{a_{\widetilde{K}} - \alpha_{\widetilde{K}} + \gamma\alpha_{\widetilde{K}}}, \frac{b_{\widetilde{S}_0^*} + \beta_{\widetilde{S}_0^*} - \gamma\beta_{\widetilde{S}_0^*}}{a_{\widetilde{K}} - \alpha_{\widetilde{K}} + \gamma\alpha_{\widetilde{K}}}\right] \text{ for all } \gamma \in [0,1],$$

and hence the fuzzy mean value of this ratio is

$$\begin{aligned}\boldsymbol{E}\left(\frac{\widetilde{S}_0^*}{\widetilde{K}}\right) &= \int_0^1 2\gamma\,\frac{\frac{a_{\widetilde{S}_0^*}-\alpha_{\widetilde{S}_0^*}+\gamma\alpha_{\widetilde{S}_0^*}}{a_{\widetilde{K}}-\alpha_{\widetilde{K}}+\gamma\alpha_{\widetilde{K}}}+\frac{b_{\widetilde{S}_0^*}+\beta_{\widetilde{S}_0^*}-\gamma\beta_{\widetilde{S}_0^*}}{a_{\widetilde{K}}-\alpha_{\widetilde{K}}+\gamma\alpha_{\widetilde{K}}}}{2}\,\mathrm{d}\gamma \\ &= \int_0^1 \gamma\,\frac{a_{\widetilde{S}_0^*}-\alpha_{\widetilde{S}_0^*}+\gamma\alpha_{\widetilde{S}_0^*}}{a_{\widetilde{K}}-\alpha_{\widetilde{K}}+\gamma\alpha_{\widetilde{K}}}\,\mathrm{d}\gamma + \int_0^1 \gamma\,\frac{b_{\widetilde{S}_0^*}+\beta_{\widetilde{S}_0^*}-\gamma\beta_{\widetilde{S}_0^*}}{a_{\widetilde{K}}-\alpha_{\widetilde{K}}+\gamma\alpha_{\widetilde{K}}}\,\mathrm{d}\gamma \\ &= 0.4154.\end{aligned}$$

The volatility parameter σ in Equation 5.8 is determined by the coefficient of variation, which is defined as the ratio of the (fuzzy) standard deviation to the (fuzzy) mean value of $\widetilde{S}_0$. Hence, the coefficient of variation is found to be 1.0097. Having determined all the parameters of Equations 5.7 and 5.8, the value of the patent commercialization project (*i.e.*, the fuzzy extended NPV) could be calculated, which is

$$\begin{aligned}\widetilde{C}(\widetilde{S},0) &= \widetilde{S}_0^* N(d_1) - \widetilde{K}e^{-rT}N(d_2) \\ &= (-304.41, 1555.50, 2412.04, 2231.56)N(d_1) \\ &\quad -(1500, 1700, 200, 100)N(d_2) \\ &= (-347.90, 915.67, 1625.18, 1512.67).\end{aligned}$$

We can see from the above result that the interval [-347.90,915.67] in which the most possible value of the fuzzy expanded (strategic) NPV lies is rather large. This seemingly lackluster result, however, exactly follows from and corresponds to the high level of uncertainty associated with the cash flows estimation. It therefore dutifully unveils the possible loss as well as the potential profit based purely on the suboptimal inputs without any embellishment.

Based on our experience, a single (crisp) number value is sometimes desired by IPR managers. This can be achieved by determining the fuzzy mean value of the fuzzy expanded NPV, which is $\boldsymbol{E}(\widetilde{C}(\widetilde{S},0)) = 265.13$.

Discussion

As was discussed in Chapter 4, the expanded NPV consists of two mayor components, which are

$$\text{Expanded NPV} = \text{Static NPV} + \text{Option Premium},$$

where "Static NPV" stands for the conventional NPV of measurable expected cash flows and "Option Premium" denotes the real option value stemming from managerial flexibility.

In our case study, the "Static NPV" is

$$\begin{aligned}\text{Static NPV} &= \widetilde{S}_0^* - \frac{\widetilde{K}}{(1+r)^3} \\ &= (-304.41, 1555.50, 2412.04, 2231.56) - \frac{(1500, 1700, 200, 100)}{(1+r)^3} \\ &= (-1815.71, 222.01, 2500.94, 2409.36),\end{aligned}$$

and its fuzzy mean value is $\boldsymbol{E}(\text{Static NPV}) = -812.11$. The "Option Premium" or, in other words, the value of management's discretion to commercialize the patent is therefore

$$\text{Option Premium} = \boldsymbol{E}(\widetilde{C}(\widetilde{S}, 0)) - \boldsymbol{E}(\text{Static NPV}) = 1077.24.$$

Clearly, the conventional NPV analysis would have dismissed the consideration of a commercial project as the result of its negative outcome. The real option thinking, however, proves it worthwhile to undertake an additional investment if the future market developments are favourable.

Chapter 6

Patent valuation with a fuzzy binomial model

As was discussed in Chapter 4, a patent application, whatever patent office it is filed with, must survive several rounds of sophisticated scrutiny and revisions before receives its approval. During such a cumbersome and sometimes lengthy application process, the patent applicant will be required to decide whether to continue or forego the ongoing application at several stages. Once entering the post-grant phase, various opportunities regarding how to exploit the granted patent become available to its proprietor. For example, in addition to the opportunity to renew at the end of each year, a patent could be licensed or abandoned in exchange for royalties or its salvage value. From a real option standpoint, the opportunity to keep the patent in force resembles an European option with the underlying value and exercise price being the monopoly benefit of the coming year plus the next option to renew and the patent maintenance fee, respectively. Intuitively, such compoundness, which is also witnessed in analysing a patent application process (see Section 4.1: Real option analysis as a means to account for managerial flexibility), could be properly accounted for with a compound option. It is equally straightforward to notice that the opportunity to abandon could be likened to an American put option exercisable with a default [70]. The licensing opportunity, however, is relatively more complex to analyse as the "rent dissipation effect" [64] would unavoidably undermine the patent monopoly. Therefore, the option to license translates into the patent holder's flexibility to choose the maximum of self-generated profit without licensing, and lessened monopoly benefits plus the compensation generated through licensing agreement. However, the Black-Scholes model is arguably less efficient in evaluating such real option scenarios as the American put option does not have a closed-from solution for a finite maturity and the Black-Scholes model is not capable of dealing with multi-option situations

[55]. Therefore, the binomial model coined by Cox et al. [19] will be adopted as a basis to appraise the aforementioned flexibility.

In all the previous examples, the imprecision embedded in expert judgements on the future profitability of a given patent is directly reflected in fuzzy cash flows. In this chapter, an alternative approach to account for the imprecision will be proposed. That is, a fuzzy volatility of the underlying asset, which is the NPV of the future cash flows, will be sought from IPR experts, while the future cash flows and thus its NPV are allowed to be crisp numbers. It is important to emphasize that the use of fuzzy volatility is essentially different from the application of a stochastic volatility. As was elaborated in Chapter 3: Fuzzy set theory in patent valuation, a fuzzy volatility results from the practical difficulties in giving a precise and reliable estimate on the volatility parameter in an environment of imperfect information and it mathematically represents the accompanying imprecision of the estimation; in a stochastic volatility model, the volatility is assumed to follow a stochastic process and thus varies randomly. In other words, a fuzzy volatility could be considered as a different approach to the modelling of "volatility of volatility" (*i.e.*, heteroscedasticity) [69]. The application of fuzzy volatility serves as a valuable alternative to the methods taking fuzzy cash flows as input. For example, it would be preferable when the investment project under consideration has a long life cycle, such as a patent exploitation project.

The objective of this chapter is to propose a fuzzy binomial approach to evaluate a post-grant patent embedded with the licensing opportunity. It is necessary to point out that the proposed binomial model is also capable of accounting for coexistent options, such as the option to license and the option to abandon.

6.1 The valuation approach

6.1.1 The binomial option pricing model

The Cox-Ross-Rubinstein (CRR) model introduced in Cox et al. [19] is a discrete-time, binomial tree-based option pricing model. In particular, it provides a powerful approach to the numerical solution for the valuation of American-style options which are exercisable at any time up to maturity and do not have close-form solutions. Briefly speaking, in addition to the same underlying assumptions underpinning the Black-Scholes model, the following assumptions are made in the CRR model:

1. The life cycle of the option T is equally divided into n time intervals (or steps), that is $T = nt$.

2. At each step, the value of the underlying asset S is assumed to move

up or down at a rate propotional to its volatility σ and the length of the time interval t.

3. The upward and downward rates (also known as the jump factors), namely u and d, must satisfy $0 < d < 1 + r < u$, where r is the risk-free interest rate.

4. The binomial tree is recombinant, which means $u \cdot d = 1$.

5. At each step, the up movement and down movement are associated with (risk-neutral) probabilities of p_u and p_d, respectively, and $p_u + p_d = 1$.

To begin with, the assumed price process of the underlying asset is repeated recursively until it reaches the expiry time T. The value of the option V in each possible state (or, figuratively speaking, each node) at the maturity is determined according to the nature of the option. Working back through the binomial tree by assessing the value of the option at each previous node, which could be obtained from its expected future values (using probabilities p_u and p_d) discounted at the risk-free rate r, leads to the (present) value of the option. Without loss of generality, the values of an American put option with strike price K at maturity and intermediate stages could be determined as follows:

$$\text{At maturity:} \begin{cases} S_n^i = S_0 u^i d^{n-i} \\ V_n^i = \max\left(K - S_n^i,\, 0\right) \end{cases} \text{for all } i = 0, \ldots, n.$$

$$\text{Elsewhere:} \begin{cases} S_m^j = S_0 u^j d^{m-i} \\ V_m^j = \max\left(K - S_m^j,\, \dfrac{1}{1+r}\left(p_u V_{m+1}^{j+1} + p_d V_{m+1}^j\right)\right) \\ \qquad \text{for all } \begin{matrix} m = 0, \ldots, n-1, \\ j = 0, \ldots, m. \end{matrix} \end{cases}$$

Intuitively, V_0 signifies the current value of the put option.

The parameters u, d, p_u and p_d are defined as (see e.g., [25, 85] for a detailed derivation of the following results)

$$u = e^{\sigma\sqrt{t}}, \tag{6.1}$$

$$d = e^{-\sigma\sqrt{t}}, \tag{6.2}$$

$$p_u = \frac{(1+r) - d}{u - d}, \tag{6.3}$$

$$p_d = \frac{u - (1+r)}{u - d}. \tag{6.4}$$

6.1.2 The fuzzy binomial option pricing model

To begin with, the volatility parameter will be represented with a trapezoidal fuzzy number to account for the imprecision in its estimation. It is worth emphasizing again the fact that triangular fuzzy numbers are nothing but a special case of trapezoidal fuzzy numbers makes the proposed model applicable to both instances.

To keep good consistency in the manifestation of fuzzy numbers throughout this thesis, the fuzzy volatility is also defined in terms of its γ-cuts, which are

$$[\widetilde{\sigma}]^\gamma = [a_{\widetilde{\sigma}} - (1-\gamma)\alpha_{\widetilde{\sigma}},\, b_{\widetilde{\sigma}} + (1-\gamma)\beta_{\widetilde{\sigma}}] \text{ for all } \gamma \in [0,1].$$

As a matter of fact, the use of γ-cuts not only extends the applicability of the model to all types of fuzzy numbers, but facilitates subsequent calculations among different fuzzy quantities.

According to Equation 6.1 and 6.2, the jump factors could be defined in their γ-cuts as

$$\begin{aligned}
[\widetilde{u}]^\gamma &= \left[\mathrm{e}^{(a_{\widetilde{\sigma}} - (1-\gamma)\alpha_{\widetilde{\sigma}})\sqrt{t}},\, \mathrm{e}^{(b_{\widetilde{\sigma}} + (1-\gamma)\beta_{\widetilde{\sigma}})\sqrt{t}}\right], \\
[\widetilde{d}]^\gamma &= \left[\mathrm{e}^{-(b_{\widetilde{\sigma}} + (1-\gamma)\beta_{\widetilde{\sigma}})\sqrt{t}},\, \mathrm{e}^{-(a_{\widetilde{\sigma}} - (1-\gamma)\alpha_{\widetilde{\sigma}})\sqrt{t}}\right].
\end{aligned}$$

It is important to point out that Assumption 3 listed above must be adjusted accordingly, which now becomes

$$\mathrm{e}^{-(a_{\widetilde{\sigma}} - (1-\gamma)\alpha_{\widetilde{\sigma}})\sqrt{t}} < 1 + r < \mathrm{e}^{(a_{\widetilde{\sigma}} - (1-\gamma)\alpha_{\widetilde{\sigma}})\sqrt{t}} \text{ for all } \gamma \in [0,1]. \tag{6.5}$$

Such an extension plays a crucial role in the proposed fuzzy binomial model. I will revisit this issue in the following work.

Assume the present value of the underlying asset is S_0, the value of the asset over the next period (*i.e.*, at time t) will move either up to $S_0\widetilde{u}$ or down to $S_0\widetilde{d}$, which are (in terms of γ-cuts)

$$\begin{aligned}
[S_0\widetilde{u}]^\gamma &= \left[S_0\mathrm{e}^{(a_{\widetilde{\sigma}} - (1-\gamma)\alpha_{\widetilde{\sigma}})\sqrt{t}},\, S_0\mathrm{e}^{(b_{\widetilde{\sigma}} + (1-\gamma)\beta_{\widetilde{\sigma}})\sqrt{t}}\right], \\
[S_0\widetilde{d}]^\gamma &= \left[S_0\mathrm{e}^{-(b_{\widetilde{\sigma}} + (1-\gamma)\beta_{\widetilde{\sigma}})\sqrt{t}},\, S_0\mathrm{e}^{-(a_{\widetilde{\sigma}} - (1-\gamma)\alpha_{\widetilde{\sigma}})\sqrt{t}}\right].
\end{aligned}$$

Similarly, the values of the underlying asset at time $2t$ will be

$$
\begin{aligned}
[S_0\widetilde{u}^2]^\gamma &= \left[S_0 \mathrm{e}^{2(a_{\widetilde{\sigma}}-(1-\gamma)\alpha_{\widetilde{\sigma}})\sqrt{t}},\ S_0 \mathrm{e}^{2(b_{\widetilde{\sigma}}+(1-\gamma)\beta_{\widetilde{\sigma}})\sqrt{t}}\right], \\
[S_0\widetilde{u}\widetilde{d}]^\gamma &= \left[S_0 \mathrm{e}^{(a_{\widetilde{\sigma}}-(1-\gamma)\alpha_{\widetilde{\sigma}})\sqrt{t}-(b_{\widetilde{\sigma}}+(1-\gamma)\beta_{\widetilde{\sigma}})\sqrt{t}},\right. \\
&\qquad \left. S_0 \mathrm{e}^{(b_{\widetilde{\sigma}}+(1-\gamma)\beta_{\widetilde{\sigma}})\sqrt{t}-(a_{\widetilde{\sigma}}-(1-\gamma)\alpha_{\widetilde{\sigma}})\sqrt{t}}\right] \\
&= \left[S_0 \mathrm{e}^{((a_{\widetilde{\sigma}}-b_{\widetilde{\sigma}})-(1-\gamma)(\alpha_{\widetilde{\sigma}}+\beta_{\widetilde{\sigma}}))\sqrt{t}},\right. \\
&\qquad \left. S_0 \mathrm{e}^{((b_{\widetilde{\sigma}}-a_{\widetilde{\sigma}})+(1-\gamma)(\beta_{\widetilde{\sigma}}+\alpha_{\widetilde{\sigma}}))\sqrt{t}}\right], \\
[S_0\widetilde{d}^2]^\gamma &= \left[S_0 \mathrm{e}^{-2(b_{\widetilde{\sigma}}+(1-\gamma)\beta_{\widetilde{\sigma}})\sqrt{t}},\ S_0 \mathrm{e}^{-2(a_{\widetilde{\sigma}}-(1-\gamma)\alpha_{\widetilde{\sigma}})\sqrt{t}}\right].
\end{aligned}
$$

Without loss of generality, the value of the underlying asset at an arbitrary node could be generalized as

$$
\begin{aligned}
[S_0\widetilde{u}^i\widetilde{d}^j]^\gamma &= \left[S_0 \mathrm{e}^{i(a_{\widetilde{\sigma}}-(1-\gamma)\alpha_{\widetilde{\sigma}})\sqrt{t}-j(b_{\widetilde{\sigma}}+(1-\gamma)\beta_{\widetilde{\sigma}})\sqrt{t}},\right. \\
&\qquad \left. S_0 \mathrm{e}^{i(b_{\widetilde{\sigma}}+(1-\gamma)\beta_{\widetilde{\sigma}})\sqrt{t}-j(a_{\widetilde{\sigma}}-(1-\gamma)\alpha_{\widetilde{\sigma}})\sqrt{t}}\right] \\
&= \left[S_0 \mathrm{e}^{((ia_{\widetilde{\sigma}}-jb_{\widetilde{\sigma}})-(1-\gamma)(i\alpha_{\widetilde{\sigma}}+j\beta_{\widetilde{\sigma}}))\sqrt{t}},\right. \\
&\qquad \left. S_0 \mathrm{e}^{((ib_{\widetilde{\sigma}}-ja_{\widetilde{\sigma}})+(1-\gamma)(i\beta_{\widetilde{\sigma}}+j\alpha_{\widetilde{\sigma}}))\sqrt{t}}\right]. \qquad (6.6)
\end{aligned}
$$

One major advantage of the binomial model over the Black-Scholes model is it allows for the staged evolution and specification of the asset price through time. That is to say, the asset price is explicitly known at each node, or point in time. Such a superiority is dutifully retained in the proposed model, although the fuzzification of asset prices in terms of their γ-cuts might make them seem less definite. In other words, the membership function of each $S_0\widetilde{u}^i\widetilde{d}^j$ is indeed determinable. To clarify the matter, let's take another look at Equation 6.6.

Since the product of trapezoidal fuzzy numbers is not a trapezoidal fuzzy number, $S_0\widetilde{u}^i\widetilde{d}^j$ could not be manifested in terms of (a, b, α, β). Their core and support are nevertheless determinable by simply setting γ to 1 and 0, respectively, which are

$$
\begin{aligned}
[S_0\widetilde{u}^i\widetilde{d}^j]^1 &= \left[S_0 \mathrm{e}^{(ia_{\widetilde{\sigma}}-jb_{\widetilde{\sigma}})\sqrt{t}},\ S_0 \mathrm{e}^{(ib_{\widetilde{\sigma}}-ja_{\widetilde{\sigma}})\sqrt{t}}\right], \\
[S_0\widetilde{u}^i\widetilde{d}^j]^0 &= \left[S_0 \mathrm{e}^{((ia_{\widetilde{\sigma}}-jb_{\widetilde{\sigma}})-(i\alpha_{\widetilde{\sigma}}+j\beta_{\widetilde{\sigma}}))\sqrt{t}},\ S_0 \mathrm{e}^{((ib_{\widetilde{\sigma}}-ja_{\widetilde{\sigma}})+(i\beta_{\widetilde{\sigma}}+j\alpha_{\widetilde{\sigma}}))\sqrt{t}}\right].
\end{aligned}
$$

The membership function of $S_0\widetilde{u}^i\widetilde{d}^j$ could be determined by equalizing the endpoints of the interval in Equation 6.6 to x, respectively, which gives

$$
\begin{aligned}
x &= S_0 \mathrm{e}^{((ia_{\widetilde{\sigma}}-jb_{\widetilde{\sigma}})-(1-\gamma)(i\alpha_{\widetilde{\sigma}}+j\beta_{\widetilde{\sigma}}))\sqrt{t}}, \\
x &= S_0 \mathrm{e}^{((ib_{\widetilde{\sigma}}-ja_{\widetilde{\sigma}})+(1-\gamma)(i\beta_{\widetilde{\sigma}}+j\alpha_{\widetilde{\sigma}}))\sqrt{t}}.
\end{aligned}
$$

Expressing γ in terms of x leads to

$$\gamma = \frac{1}{(i\alpha_{\widetilde{\sigma}} + j\beta_{\widetilde{\sigma}})\sqrt{t}} \ln \frac{x}{S_0} - \frac{ia_{\widetilde{\sigma}} - jb_{\widetilde{\sigma}}}{i\alpha_{\widetilde{\sigma}} + j\beta_{\widetilde{\sigma}}} + 1,$$

$$\gamma = -\frac{1}{(i\beta_{\widetilde{\sigma}} + j\alpha_{\widetilde{\sigma}})\sqrt{t}} \ln \frac{x}{S_0} + \frac{ib_{\widetilde{\sigma}} - ja_{\widetilde{\sigma}}}{i\beta_{\widetilde{\sigma}} + j\alpha_{\widetilde{\sigma}}} + 1,$$

and thus the membership function of the asset price $S_0\widetilde{u}^i\widetilde{d}^j$ could be given as follows:

$$S_0\widetilde{u}^i\widetilde{d}^j = \begin{cases} \frac{1}{(i\alpha_{\widetilde{\sigma}} + j\beta_{\widetilde{\sigma}})\sqrt{t}} \ln \frac{x}{S_0} - \frac{ia_{\widetilde{\sigma}} - jb_{\widetilde{\sigma}}}{i\alpha_{\widetilde{\sigma}} + j\beta_{\widetilde{\sigma}}} + 1 \\ (\text{if } S_0 \mathrm{e}^{((ia_{\widetilde{\sigma}} - jb_{\widetilde{\sigma}}) - (i\alpha_{\widetilde{\sigma}} + j\beta_{\widetilde{\sigma}}))\sqrt{t}} \leq x < S_0 \mathrm{e}^{(ia_{\widetilde{\sigma}} - jb_{\widetilde{\sigma}})\sqrt{t}}) \\ 1 \\ (\text{if } S_0 \mathrm{e}^{(ia_{\widetilde{\sigma}} - jb_{\widetilde{\sigma}})\sqrt{t}} \leq x \leq S_0 \mathrm{e}^{(ib_{\widetilde{\sigma}} - ja_{\widetilde{\sigma}})\sqrt{t}}) \\ -\frac{1}{(i\beta_{\widetilde{\sigma}} + j\alpha_{\widetilde{\sigma}})\sqrt{t}} \ln \frac{x}{S_0} + \frac{ib_{\widetilde{\sigma}} - ja_{\widetilde{\sigma}}}{i\beta_{\widetilde{\sigma}} + j\alpha_{\widetilde{\sigma}}} + 1 \\ (\text{if } S_0 \mathrm{e}^{(ib_{\widetilde{\sigma}} - ja_{\widetilde{\sigma}})\sqrt{t}} < x \leq S_0 \mathrm{e}^{((ib_{\widetilde{\sigma}} - ja_{\widetilde{\sigma}}) + (i\beta_{\widetilde{\sigma}} + j\alpha_{\widetilde{\sigma}}))\sqrt{t}}) \\ 0 \\ (\text{otherwise}). \end{cases}$$

Using the fuzzy version of the volatility parameter requires that Equations 6.3 and 6.4 be adjusted accordingly. In Paper 4, the approach suggested by Muzzioli and Reynaerts (see e.g., [67, 68, 69]) is adopted, which models the probabilities p_u and p_d as trapezoidal fuzzy numbers which are approximated by

$$\left[\frac{(1+r) - \bar{d}}{\bar{u} - \bar{d}}, \frac{(1+r) - \underline{d}}{\underline{u} - \underline{d}}\right], \tag{6.7}$$

and

$$\left[\frac{\underline{u} - (1+r)}{\underline{u} - \underline{d}}, \frac{\bar{u} - (1+r)}{\bar{u} - \bar{d}}\right], \tag{6.8}$$

respectively, where $\underline{u}$ (respectively $\underline{d}$) and $\bar{u}$ (respectively $\bar{d}$) represent the endpoints of the γ-cuts of $\widetilde{u}$ (respectively $\widetilde{d}$). The above fuzzy numbers, namely those manifested by 6.7 and 6.8, are determinable by solving the following non-linear programming problems:

$$\max_{u,d} \frac{(1+r) - d}{u - d}, \quad \min_{u,d} \frac{(1+r) - d}{u - d};$$

and

$$\max_{u,d} \frac{u - (1+r)}{u - d}, \quad \min_{u,d} \frac{u - (1+r)}{u - d}.$$

In Paper 5, an alternative approach to the derivation of probabilities p_u and p_d is proposed, which is based on fuzzy arithmetic as defined in Chapter 3. Recall that, given two fuzzy numbers $\widetilde{A}$ and $\widetilde{B}$, their subtraction and ratio could be defined in terms of their γ-cuts as

$$\begin{aligned}
[\widetilde{A}-\widetilde{B}]^{\gamma} &= \left[\underline{A}-\bar{B},\, \bar{A}-\underline{B}\right], \\
[\widetilde{A}/\widetilde{B}]^{\gamma} &= \left[\min\left(\underline{A}/\underline{B}, \underline{A}/\bar{B}, \bar{A}/\underline{B}, \bar{A}/\bar{B}\right), \max\left(\underline{A}/\underline{B}, \underline{A}/\bar{B}, \bar{A}/\underline{B}, \bar{A}/\bar{B}\right)\right].
\end{aligned}$$

Accordingly, the numerator and denominator of $\frac{(1+r)-\widetilde{d}}{\widetilde{u}-\widetilde{d}}$ take the form

$$[(1+r)-\widetilde{d}]^{\gamma} = \left[(1+r)-\bar{d},\, (1+r)-\underline{d}\right],$$

and

$$[\widetilde{u}-\widetilde{d}]^{\gamma} = \left[\underline{u}-\bar{d},\, \bar{u}-\underline{d}\right],$$

respectively. The fuzzy probability $\widetilde{p_u}$ is

$$\begin{aligned}
[\widetilde{p_u}]^{\gamma} &= \left[\frac{(1+r)-\bar{d}}{\bar{u}-\underline{d}}, \frac{(1+r)-\underline{d}}{\underline{u}-\bar{d}}\right] \\
&= \Bigg[\frac{(1+r)-\mathrm{e}^{-(a_{\widetilde{\sigma}}-(1-\gamma)\alpha_{\widetilde{\sigma}})\sqrt{t}}}{\mathrm{e}^{(b_{\widetilde{\sigma}}+(1-\gamma)\beta_{\widetilde{\sigma}})\sqrt{t}}-\mathrm{e}^{-(b_{\widetilde{\sigma}}+(1-\gamma)\beta_{\widetilde{\sigma}})\sqrt{t}}}, \\
&\qquad \frac{(1+r)-\mathrm{e}^{-(b_{\widetilde{\sigma}}+(1-\gamma)\beta_{\widetilde{\sigma}})\sqrt{t}}}{\mathrm{e}^{(a_{\widetilde{\sigma}}-(1-\gamma)\alpha_{\widetilde{\sigma}})\sqrt{t}}-\mathrm{e}^{-(a_{\widetilde{\sigma}}-(1-\gamma)\alpha_{\widetilde{\sigma}})\sqrt{t}}}\Bigg].
\end{aligned} \tag{6.9}$$

Similarly, the fuzzy probability $\widetilde{p_d}$ could be written as

$$\begin{aligned}
[\widetilde{p_d}]^{\gamma} &= \left[\frac{\underline{u}-(1+r)}{\bar{u}-\underline{d}}, \frac{\bar{u}-(1+r)}{\underline{u}-\bar{d}}\right] \\
&= \Bigg[\frac{\mathrm{e}^{(a_{\widetilde{\sigma}}-(1-\gamma)\alpha_{\widetilde{\sigma}})\sqrt{t}}-(1+r)}{\mathrm{e}^{(b_{\widetilde{\sigma}}+(1-\gamma)\beta_{\widetilde{\sigma}})\sqrt{t}}-\mathrm{e}^{-(b_{\widetilde{\sigma}}+(1-\gamma)\beta_{\widetilde{\sigma}})\sqrt{t}}}, \\
&\qquad \frac{\mathrm{e}^{(b_{\widetilde{\sigma}}+(1-\gamma)\beta_{\widetilde{\sigma}})\sqrt{t}}-(1+r)}{\mathrm{e}^{(a_{\widetilde{\sigma}}-(1-\gamma)\alpha_{\widetilde{\sigma}})\sqrt{t}}-\mathrm{e}^{-(a_{\widetilde{\sigma}}-(1-\gamma)\alpha_{\widetilde{\sigma}})\sqrt{t}}}\Bigg].
\end{aligned} \tag{6.10}$$

Recall the extended assumption manifested in inequality 6.5, which is

$$\mathrm{e}^{-(a_{\widetilde{\sigma}}-(1-\gamma)\alpha_{\widetilde{\sigma}})\sqrt{t}} < 1+r < \mathrm{e}^{(a_{\widetilde{\sigma}}-(1-\gamma)\alpha_{\widetilde{\sigma}})\sqrt{t}} \text{ for all } \gamma \in [0,1].$$

Evidently, this assumption guarantees that both $\widetilde{p_u}$ and $\widetilde{p_d}$ are positive fuzzy numbers.

It is hoped that the previous discussion has paved the way for subsequent arithmetic operations on fuzzy numbers. If so, the only technical difficulty remaining unresolved is the binary operation max on fuzzy numbers. In consistence with the previous work, the operation max will also be defined in terms of γ-cuts. Formally speaking, for any two fuzzy numbers $\widetilde{A}$ and $\widetilde{B}$, their maximum is defined as

$$[\max(\widetilde{A}, \widetilde{B})]^\gamma = \left[\max([\underline{\widetilde{A}}]^\gamma, [\underline{\widetilde{B}}]^\gamma),\ \max([\bar{\widetilde{A}}]^\gamma, [\bar{\widetilde{B}}]^\gamma)\right] \text{ for all } \gamma \in [0,1]. \tag{6.11}$$

However, implementing the above method imposes considerable computational load, and the amount of additional computation could become enormous when valuing an American-style option as the same operation needs to be repeated at each time step. Therefore, for the sake of fast computation and neat manifestation of option value varying through time, the operation defined in 6.11 will be restricted on the cores (*i.e.*, $\gamma = 1$) and supports (*i.e.*, $\gamma = 0$) of the fuzzy numbers under comparison. That is to say, the operation max will be simplified to

$$\begin{aligned}\max(\widetilde{A}, \widetilde{B}) = \Big\{ &\max([\underline{\widetilde{A}}]^0, [\underline{\widetilde{B}}]^0), \max([\underline{\widetilde{A}}]^1, [\underline{\widetilde{B}}]^1), \\ &\max([\bar{\widetilde{A}}]^1, [\bar{\widetilde{B}}]^1), \max([\bar{\widetilde{A}}]^0, [\bar{\widetilde{B}}]^0) \Big\}. \end{aligned} \tag{6.12}$$

It is worth pointing out that

- The resulting quadruplet listed in 6.12 also forms a trapezoidal-shaped fuzzy number. However, as a result of the simplification, its membership function is not determinable from the quadruplet.
- The above quadruplet (*i.e.*, the fuzzy number) is grouped using curly braces $\{\ldots\}$ as contrasted to trapezoidal fuzzy numbers which are conventionally given in parenthesis, such as $\widetilde{A} = (a_{\widetilde{A}}, b_{\widetilde{A}}, \alpha_{\widetilde{A}}, \beta_{\widetilde{A}})$.
- The quadruplet in 6.12 is arranged in ascending order, which makes the arithmetic operations defined earlier also legitimate on it.

To sum up, the values of an American put option with fuzzy volatility, $\widetilde{\sigma} = (a_{\widetilde{\sigma}}, b_{\widetilde{\sigma}}, \alpha_{\widetilde{\sigma}}, \beta_{\widetilde{\sigma}})$, at maturity and intermediate stages could be determined as follows:

$$\text{At maturity: } \begin{cases} \widetilde{S}_n^i = S_0 \widetilde{u}^i \widetilde{d}^{n-i} \\ \widetilde{V}_n^i = \max\left(K - \widetilde{S}_n^i,\ 0\right) \end{cases} \text{ for all } i = 0, \ldots, n,$$

$$\text{Elsewhere: } \begin{cases} \widetilde{S}_m^j = S_0 \widetilde{u}^j \widetilde{d}^{m-i} \\ \widetilde{V}_m^j = \max\left(K - \widetilde{S}_m^j,\ \dfrac{1}{1+r}\left(\widetilde{p_u}\widetilde{V}_{m+1}^{j+1} + \widetilde{p_d}\widetilde{V}_{m+1}^{j}\right)\right) \\ \qquad \text{for all } \begin{array}{l} m = 0, \ldots, n-1, \\ j = 0, \ldots, m, \end{array} \end{cases}$$

where $\widetilde{V}_0$ represents the price of the option, and the other parameters could be given in terms of γ-cuts as

$$
\begin{aligned}
[\widetilde{u}]^\gamma &= \left[\mathrm{e}^{(a_{\widetilde{\sigma}}-(1-\gamma)\alpha_{\widetilde{\sigma}})\sqrt{t}}, \mathrm{e}^{(b_{\widetilde{\sigma}}+(1-\gamma)\beta_{\widetilde{\sigma}})\sqrt{t}}\right], \\
[\widetilde{d}]^\gamma &= \left[\mathrm{e}^{-(b_{\widetilde{\sigma}}+(1-\gamma)\beta_{\widetilde{\sigma}})\sqrt{t}}, \mathrm{e}^{-(a_{\widetilde{\sigma}}-(1-\gamma)\alpha_{\widetilde{\sigma}})\sqrt{t}}\right], \\
[\widetilde{p_u}]^\gamma &= \left[\frac{(1+r)-\bar{d}}{\bar{u}-\underline{d}}, \frac{(1+r)-\underline{d}}{\underline{u}-\bar{d}}\right] \\
&= \left[\frac{(1+r)-\mathrm{e}^{-(a_{\widetilde{\sigma}}-(1-\gamma)\alpha_{\widetilde{\sigma}})\sqrt{t}}}{\mathrm{e}^{(b_{\widetilde{\sigma}}+(1-\gamma)\beta_{\widetilde{\sigma}})\sqrt{t}}-\mathrm{e}^{-(b_{\widetilde{\sigma}}+(1-\gamma)\beta_{\widetilde{\sigma}})\sqrt{t}}}, \right. \\
&\qquad \left. \frac{(1+r)-\mathrm{e}^{-(b_{\widetilde{\sigma}}+(1-\gamma)\beta_{\widetilde{\sigma}})\sqrt{t}}}{\mathrm{e}^{(a_{\widetilde{\sigma}}-(1-\gamma)\alpha_{\widetilde{\sigma}})\sqrt{t}}-\mathrm{e}^{-(a_{\widetilde{\sigma}}-(1-\gamma)\alpha_{\widetilde{\sigma}})\sqrt{t}}}\right], \\
[\widetilde{p_d}]^\gamma &= \left[\frac{\underline{u}-(1+r)}{\bar{u}-\underline{d}}, \frac{\bar{u}-(1+r)}{\underline{u}-\bar{d}}\right] \\
&= \left[\frac{\mathrm{e}^{(a_{\widetilde{\sigma}}-(1-\gamma)\alpha_{\widetilde{\sigma}})\sqrt{t}}-(1+r)}{\mathrm{e}^{(b_{\widetilde{\sigma}}+(1-\gamma)\beta_{\widetilde{\sigma}})\sqrt{t}}-\mathrm{e}^{-(b_{\widetilde{\sigma}}+(1-\gamma)\beta_{\widetilde{\sigma}})\sqrt{t}}}, \right. \\
&\qquad \left. \frac{\mathrm{e}^{(b_{\widetilde{\sigma}}+(1-\gamma)\beta_{\widetilde{\sigma}})\sqrt{t}}-(1+r)}{\mathrm{e}^{(a_{\widetilde{\sigma}}-(1-\gamma)\alpha_{\widetilde{\sigma}})\sqrt{t}}-\mathrm{e}^{-(a_{\widetilde{\sigma}}-(1-\gamma)\alpha_{\widetilde{\sigma}})\sqrt{t}}}\right].
\end{aligned}
$$

6.2 Valuing the option to out license with the fuzzy binomial model

As was discussed in Chapter 4, patent out-licensing has become one major approach to capitalize on intangible assets for those operating in technology-driven industries, such as electronic data processing/telecommunication companies and technical services providers. Just to give the readers a glimpse into the astonishing financial stakes in the smartphone-related patent license deals, in recently disclosed data, Samsung Electronics Ltd. was reported to have paid at least a staggering one billion US dollars in 2013 to Microsoft Corp. as royalty fees for using the latter's technology in Samsung's Android-powered devices. In addition to the increased earnings, granting licenses could also prevent litigation, increase visibility on the market and facilitate standardisation of a company's products. However, out-licensing might potentially jeopardize the licensor's monopoly benefits, which is known as the "rent dissipation effect". There is, therefore, a trade-off between the two patent exploitation strategies. In other words, the decisive factor in licensing is "the licensor's hope for profits from the exploitation of technology that is not going to greatly disadvantage the licensor" [78], which, from a real option analysis standpoint, could translate into patent proprietor's flexibility to choose the maximum of self-generated profit without licensing and licensing revenues plus the lessened monopoly benefits, which could be theoretically approximated by an American-style option.

To capture the side effect on the monopoly benefits due to patent licensing, a dissipation factor, namely δ, will be introduced. For the sake of generality, we represent it with a trapezoidal fuzzy number $\widetilde{\delta} = (a_\delta, b_\delta, \alpha_\delta, \beta_\delta)$, which satisfies $0 < a_\delta - \alpha_\delta < b_\delta + \beta_\delta < 1$. The aforementioned licensing decision can then be mathematically expressed as

$$\max(\widetilde{S}, \widetilde{\delta}\widetilde{S} + \widetilde{L}), \tag{6.13}$$

where $\widetilde{S}$ stands for the self-generated profit without licensing and $\widetilde{L}$ denotes the rolling royalties. It is important to point out the royalty rate determination is an equally complex task to patent valuation, and the conventional patent valuation techniques, namely the income approach, cost approach and comparable market approach, are also used for this purpose in practice. In this thesis, it is assumed for simplicity that the royalty and the monopoly benefits of the patent are positively correlated, and hence the royalty also follows a multiplicative binomial process. It should be noted the assumption that licensing royalty and patent monopoly are correlated is not new and it refers to the early findings of Trigeorgis [92] who claims the similar correlation is witnessed between a project's value in its current use and in its best alternative use (or its salvage value).

Assume that the volatility parameters of $\widetilde{S}$ and $\widetilde{L}$ (*i.e.*, the standard deviations of the rate of change of $\widetilde{S}$ and $\widetilde{L}$) are both fuzzy numbers, namely $\widetilde{\sigma}_S$ and $\widetilde{\sigma}_L$. The value of a granted patent together with the value of the option to license (*i.e.*, the extended NPV) could be determined by solving the following system:

$$\text{At maturity: } \begin{cases} \widetilde{S}_n^i = S_0 \widetilde{u}_S^i \widetilde{d}_S^{n-i} \\ \widetilde{L}_n^i = L_0 \widetilde{u}_L^i \widetilde{d}_L^{n-i} \\ \widetilde{V}_n^i = \max\left(\widetilde{S}_n^i, \widetilde{\delta}\widetilde{S}_n^i + \widetilde{L}_n^i\right) \end{cases} \quad \text{for all } i = 0, \ldots, n.$$

$$\text{Elsewhere: } \begin{cases} \widetilde{S}_m^j = S_0 \widetilde{u}_S^j \widetilde{d}_S^{m-i} \\ \widetilde{L}_m^j = L_0 \widetilde{u}_L^j \widetilde{d}_L^{m-i} \\ \widetilde{V}_m^j = \max\left(\widetilde{\delta}\widetilde{S}_m^j + \widetilde{L}_m^j, \dfrac{1}{1+r}\left(\widetilde{p_u}\widetilde{V}_{m+1}^{j+1} + \widetilde{p_d}\widetilde{V}_{m+1}^{j}\right)\right) \\ \qquad \text{for all } \begin{array}{l} m = 0, \ldots, n-1, \\ j = 0, \ldots, m. \end{array} \end{cases}$$

6.2.1 Example

To demonstrate the implementation of the proposed model, I use a simple numerical example of a fictional patent on an ethical drug. For illustration purposes, the life of this patent would be limited to 4 years after grant; the risk-free rate is 4%; the dissipation factor, which captures the damage of out-licensing, is assumed to be a crisp number 0.65; the decision to licence or not will be reviewed on a annual basis. The (fuzzy) volatility parameter of $\widetilde{S}$ could be approximated by the implied volatilities of traded options with the same maturity on major pharmaceutical companies. According to Schwartz [84], the average implied volatility for the call options on nine pharmaceutical stocks is around 35%. In this illustrative example, the volatilities of $\widetilde{S}$ and $\widetilde{L}$ are given as $\widetilde{\sigma}_S = (0.355, 0.36, 0.035, 0.02)$ and $\widetilde{\sigma}_L = (0.31, 0.365, 0.03, 0.025)$, respectively. The corresponding upward and downward rates (the jump factors) are

$$\begin{aligned}
\widetilde{u}_S &= \{1.377, 1.426, 1.433, 1.462\}, \\
\widetilde{d}_S &= \{0.684, 0.698, 0.701, 0.726\}, \\
\widetilde{u}_L &= \{1.323, 1.363, 1.441, 1.477\}, \\
\widetilde{d}_L &= \{0.677, 0.694, 0.733, 0.756\}.
\end{aligned}$$

Note that the fuzzy jump factors are specified by the quadruplet representation as introduced in Equation 6.12 for simplicity. Given that the NPVs of S and L are 500 and 250, respectively, the binomial trees for the self-generated profit without out-licensing and the royalty will be as depicted in Figures 6.1 and 6.2, respectively.

According to 6.13, the payoff at the maturity (*i.e.*, at the end of year 4) is

$$\begin{aligned}
\widetilde{V}_4^4 &= \{1935.122, 2208.468, 2448.216, 2675.679\}, \\
\widetilde{V}_4^3 &= \{972.541, 1097.612, 1219.139, 1346.695\}, \\
\widetilde{V}_4^2 &= \{488.879, 545.725, 607.336, 677.956\}, \\
\widetilde{V}_4^1 &= \{245.804, 271.435, 302.675, 341.374\}, \\
\widetilde{V}_4^0 &= \{123.615, 135.061, 150.903, 171.932\}.
\end{aligned}$$

The up and down probabilities could be determined by Equations 6.9 and 6.10, which are

$$\begin{aligned}
\widetilde{p_u} &= \{0.403, 0.461, 0.472, 0.547\}, \\
\widetilde{p_d} &= \{0.433, 0.525, 0.543, 0.649\}.
\end{aligned}$$

The option prices at each earlier nodes are given in Figure 6.3. The price of this American-style option at time zero is $\widetilde{V}_0 = \{575, 575, 635.663, 1332.238\}$, and its fuzzy mean value (*i.e.*, the extended NPV) could be approximated which is 721.43. The embedded flexibility to out licence the underlying patent is therefore $721.43 - 500 = 221.43$. That is, the value of the option to license at any time before the patent lapses is worth more than 40% of the NPV of the self-generated profit without licensing.

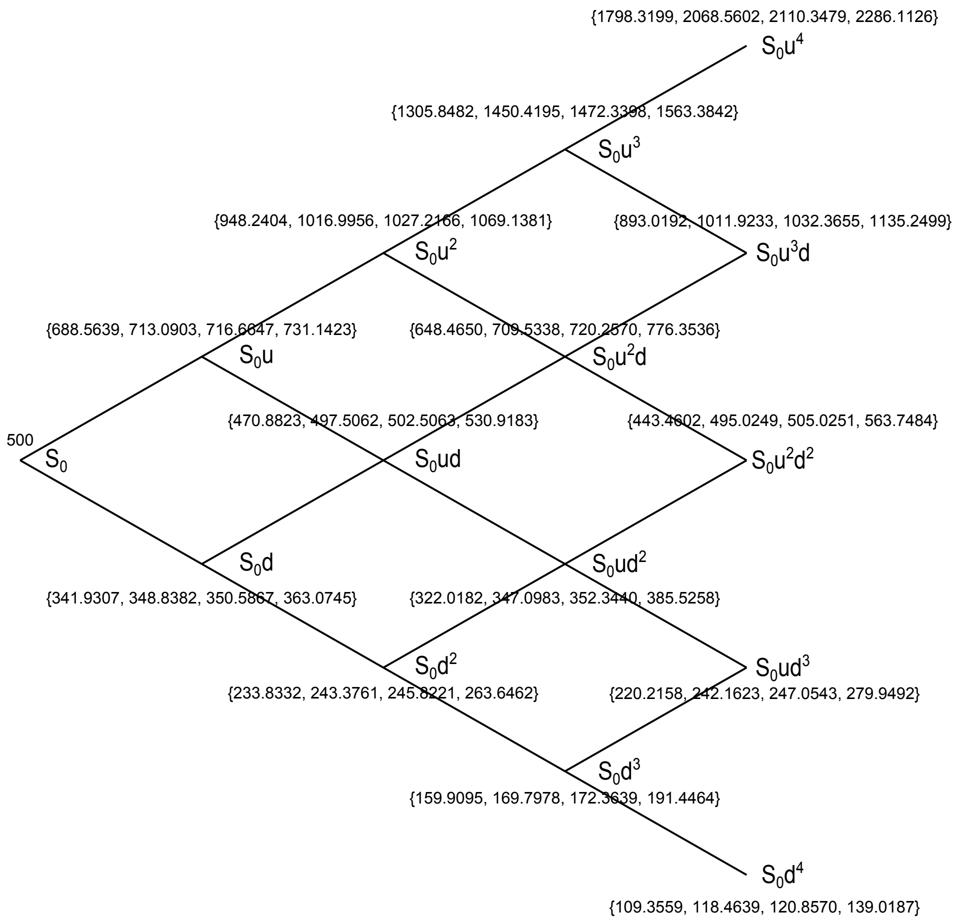


Figure 6.1: Fuzzy binomial tree of the self-generated profit without patent licensing

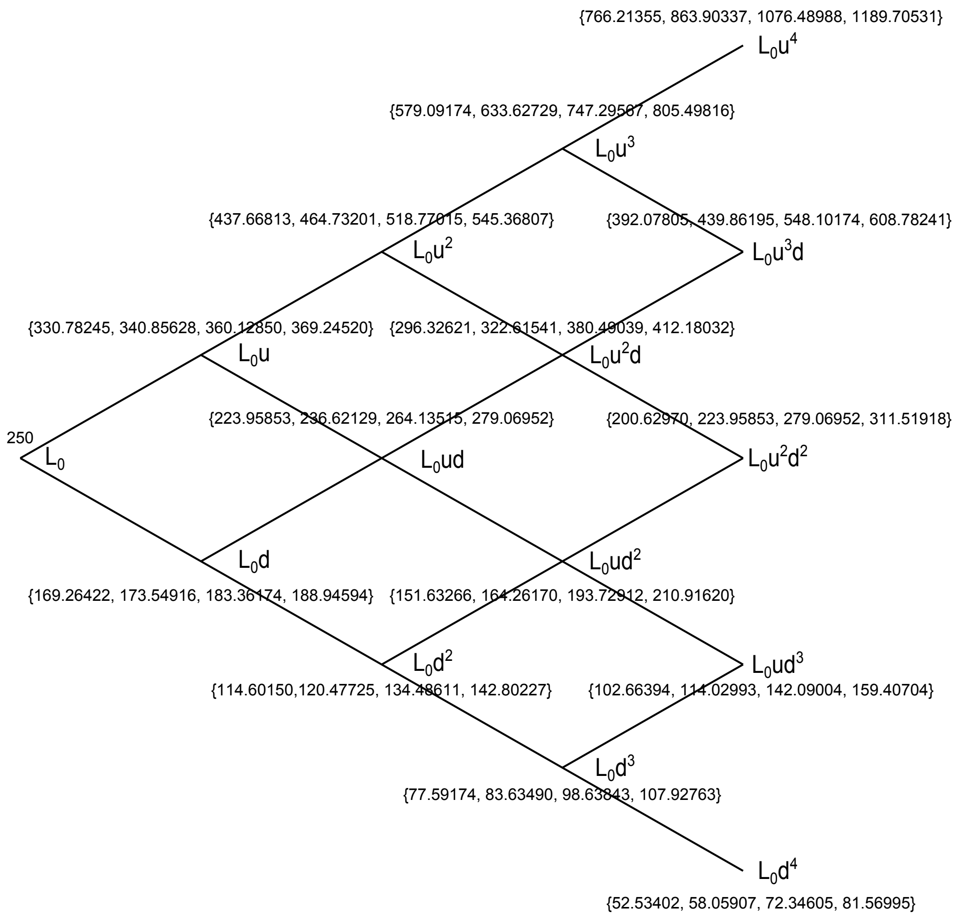


Figure 6.2: Fuzzy binomial tree of the licensing revenue

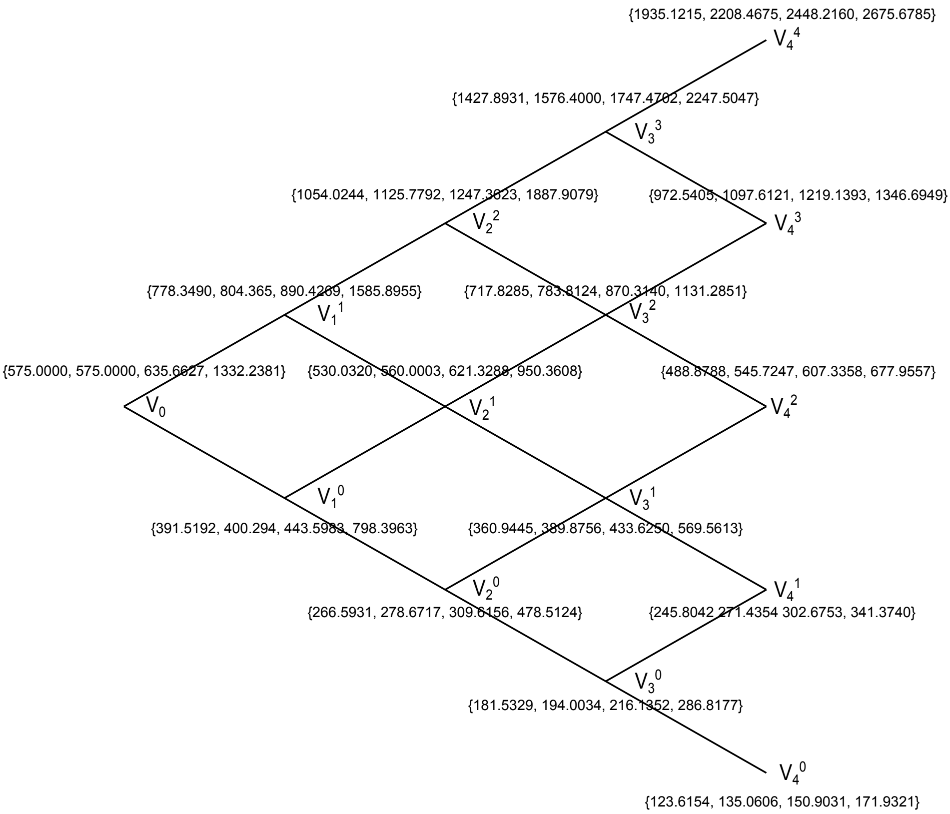


Figure 6.3: The American-style option prices

Chapter 7

Discussion and future research

In addition to following the traditional routine in writing the final chapter of doctoral theses/dissertations, that is to reaffirm the contributions of my work and summarize the answers to my research questions, I would also like to explain how my cognition has developed in my research period and discuss how future research could improve my work.

My commitment to this research topic was prompted by the fact that, in the wake of the evolution of patent from merely a defensive barrier to financial asset and a form of alternative investment, the demand for reliable and systematic patent valuation has reached an unprecedented level. However, the three conventional monetary valuation models for patents are essentially based on discounted cash flow analysis, which proves to be efficient only under such restricted conditions that the future cash flows of the project under consideration could be accurately estimated and the management has little discretion in adjusting its original operating strategy once the project has been initiated, none of which, however, could be satisfied in the case of patent. In other words, the distinctive characteristics of patent, the remarkable inefficiencies as witnessed in the patent market and hence the economic opportunities arising from these pricing inefficiencies hamper the applicability of DCF analysis in patent related decision making and in patent valuation. In Chapter 1, the major factors impeding its adoption have been put under the microscope, which include the market illiquidity, the poor data availability, discriminatory cash-flow estimations, and its incapability to account for changing risk and managerial flexibility.

These impeding factors, especially the first three as listed above, have inevitably resulted in a situation in which one has not only very limited knowledge and biased information on the possible future states of nature, but little reference about what has already happened. Apparently, the uncertainty encountered in patent valuation or rather the uncertainty underlying patent cash flows cannot be simply assumed to follow a stochastic process, and such a realization has really pushed me out of my comfort zone and prompted me to delve into its nature and explore the fundamental differences between this type of uncertainty and those subject to inherent randomness. All these thoughts naturally led to my first research question:

RQ1 What is the nature of the uncertainties inherent in patent cash flow estimations?

To answer this question, I first argued that there exist two levels of uncertainties to be considered:

- The first kind stems from the imprecision and vagueness inherent in human judgements, which, in our context, result from information deficiency and the limited ability of human being in perceiving and processing the available information.
- The second kind is derived from the discrepancies among the individual expert estimates of future cash flows.

Further investigation revealed that both levels of uncertainties fall under the categorization of subjective uncertainty, which differs from objective uncertainty originating from inherent randomness in that uncertainties labelled as subjective are highly related to the behavioural aspects of decision making and are usually witnessed whenever human judgement, evaluation or reasoning is crucial to the system under consideration and there exists a lack of complete knowledge on its variables.

Having elaborated the nature of the uncertainties in patent cash flow estimation, the next logical step would be to choose an appropriate uncertainty theory, which therefore raised my second research question:

RQ2 Is fuzzy set theory a feasible uncertainty theory to capture the patent-related uncertainties?

To rationalize the use of fuzzy set theory, I emphasized on the fact that fuzzy set theory facilitates a gradual transition from membership to non-membership which enables it to outperform classical set theory and two-value logic upon which probability theory is based in expressing the inaccuracy of human perception which prevails in patent cash flow estimations. All the aforementioned mental work, together with the justification of fuzzy set theory for this problem, have been elaborately depicted with text in Chapter 3.

As for the other two impeding issues as listed above (*i.e.*, the changing risk and managerial flexibility), one of the candidates capable of accounting for them is the real option analysis. The reason for selecting it for these problems is rather straightforward: real option analysis implicitly assumes that there is an underlying source of uncertainty (*e.g.*, the cash flows from a investment project and the price of a commodity) and explicitly captures the value of future flexibility derived from management's discretion to adjust its operating strategy in response to the unveiling of the underlying uncertainty. Naturally, the theoretical attractiveness of real option analysis motivated me to scrutinize patents through the real option lens, which in turn systematised my answer to the third research question:

RQ3 How to conceptualize and quantify the embedded managerial flexibility in a patent with real option analysis?

Accordingly, Chapter 4 has been devoted to an explicit identification of the types of managerial flexibility inherent in patent-related decision making problems and in patent valuation, and a discussion on how they could be interpreted in terms of options. Given the wide variety of courses of action available to patent applicants and patentees, I first treated patent application and patent exploitation as a unity before I limited the focus to each process individually. It is important to point out that much work still remains in exploiting the practical use of real option analysis in the field of patent valuation, and hence the examples demonstrated in this chapter can by no means be said to serve as a comprehensive and complete collection of feasible real option based valuation methodologies. For example, similar to the compound option framework I have treated the patent application and the subsequent patent exploitation with, in which a patent application is filed for a new invention in the first stage and irreversible market entry is initiated in the second, the characteristics of sequential investments could also be witnessed in patent litigation, that is, the first option is to sue the patent infringer and the subsequent option is to pursue collection of awards from the infringer. Additionally, one highly sought-after research topic is to employ a real game option approach in analysing patent-investment race, which is triggered by the fact that "the optimal patent acquisition decision represents a tradeoff between the benefit of holding the real option of developing the substitute product and the loss in profit flow rate resulting from being preempted" [54].

Having justified the feasibility of fuzzy set theory and real option analysis in the context of patents, the next task faced by me was to show how the proposed techniques could be implemented in practice, or in other words to investigate:

- the application of fuzzy real option analysis in patent-related decision making problems.

I have been fortunate enough to participate in an industrial-funded research project, the aim of which was to design patent application and commercialization decision making tools for our industrial collaborator, a leading water chemistry company in possession of a large number of patent families and a strong innovation portfolio. The mathematical models presented in Chapters 5 are fully motivated by their industrial applications. For example, two fuzzy logic based real option pricing models, namely the pay-off method and the extended fuzzy Black-Scholes model, have been employed to investigate the profitability of a patent application project for a new process for the preparation of a gypsum-fibre composite and to justify the subsequent patent commercialization decision, respectively. In particular, the use of trapezoidal fuzzy numbers which are derived from the cash-flow scenarios provided by patent experts from our industrial collaborator makes it possible to formulate the internal uncertainties (*i.e.*, the uncertainty stemming from subjective judgements) in a straightforward way; the proposed fuzzy real option models are capable of accounting for the external uncertainties, which are implied and evolved by effects drawn from the future behaviour of the real world.

Additionally, a patent licensing agreement is a highly lucrative instrument available to patent-holding firms to benefit from their intangible capitals, which therefore prompted me to also investigate:

- the application of fuzzy real option analysis in valuing a patent licensing opportunity.

In Chapter 6, a fuzzy binomial model for patent valuation was proposed under the assumptions that the licensing royalty and patent monopoly are correlated and that the volatilities of the self-generated profit without licensing and royalty incomes are both trapezoidal fuzzy numbers. A binomial tree-based model was selected for this purpose because the option to license translates into the patent proprietor's flexibility to choose the maximum of self-generated profit without licensing and lessened monopoly benefits plus the compensation generated through licensing agreement, and hence resembles an American-styled option, whose value could be easily approximated with the binomial model.

7.1 Future research

It is very important to emphasize that the usefulness of all the aforementioned mathematical models depends largely on the extent to which the embedded assumption match the characteristics of the investment proposal being evaluated. Therefore, one direction for future research is to assess the effects of any simplified assumptions. For example, it would make perfect sense to propose a patent-specific discount rate, the unattended compoundness stemming from the series of nested options (*e.g.*, the options to continue patent application after search and examination reports are published, and the options to renew the patent at the end of each year in the post-grant phase) should be reexamined, just to name a few.

Additionally, the mathematical models presented in this dissertation have paid little attention to the potential upward bias of patent value. That is to say, they actually assess the combined value of the patent under consideration plus the underlying innovation asset it protects. Therefore, some future effort should aim at disentangling the incremental value of a patent from the overall value of the innovation project under its protection.

To sum up, this dissertation is merely the first step towards a rigorous application of fuzzy real option analysis to patent valuation. It is nevertheless hoped that it will at least lay the foundation for any future exploration.

Bibliography

[1] Euro-PCT Guide: PCT procedure at the EPO. Retrieved from `http://www.epo.org/applying/international/guide-for-applicants.html`.

[2] How to get a European patent, Guide for applicants - Part 1. Retrieved from `http://www.epo.org/applying/european/Guide-for-applicants.html`.

[3] Understanding Industrial Property. WIPO Publication No. 895(E).

[4] What is Intellectual Property? WIPO Publication No. 450(E).

[5] WIPO Intellectual Property Handbook: Policy, Law and Use. WIPO Publication No. 489 (E).

[6] Bilal M Ayyub and George J Klir. *Uncertainty modeling and analysis in engineering and the sciences*. CRC Press, 2010.

[7] Louis Bachelier. Théorie de la Spéculation. In *Annales Scientifiques de l'Ecole Normale Supérieure*, volume 17, pages 21–88, 1900. [English translation in P. Cootner (ed.) The Random Character of Stock Prices. MIT Press, 1964, reprinted Risk Books, London 2000].

[8] Cédric Baudrit and Didier Dubois. Comparing methods for joint objective and subjective uncertainty propagation with an example in a risk assessment. In *ISIPTA*, volume 5, 2005.

[9] Mario Benassi and Alberto Di Minin. Playing in between: patent brokers in markets for technology. *R&D Management*, 39(1):68–86, 2009.

[10] James Bessen and Michael James Meurer. *Patent failure: How judges, bureaucrats, and lawyers put innovators at risk*. Princeton University Press, 2008.

[11] Fischer Black and Myron Scholes. The pricing of options and corporate liabilities. *The journal of political economy*, pages 637–654, 1973.

[12] George Bojadziev and Maria Bojadziev. *Fuzzy logic for business, finance, and management.* World Scientific Publishing Co., Inc., 2007.

[13] M. Brunelli. *Some advances in mathematical models for preference relations.* PhD thesis, Åbo Akad. Univ., Turku Centre for Computer Science, 2011.

[14] John A Campbell. Real options analysis of the timing of IS investment decisions. *Information & Management*, 39(5):337–344, 2002.

[15] Christer Carlsson and Robert Fullér. A fuzzy approach to real option valuation. *Fuzzy sets and systems*, 139(2):297–312, 2003.

[16] Brandon J Casey, Jake M Logan, Erik M Cressall, and Stephen H Cotterill. Gesture entry techniques, October 12 2012. US Patent App. 13/651,118.

[17] Mikael Collan, Robert Fullér, and József Mezei. A fuzzy pay-off method for real option valuation. *Advances in Decision Sciences*, 2009, 2009.

[18] Tom Copeland and Peter Tufano. A real-world way to manage real options. *Harvard business review*, 82(3):90–99, 2004.

[19] John C Cox, Stephen A Ross, and Mark Rubinstein. Option pricing: A simplified approach. *Journal of Financial Economics*, 7(3):229–263, 1979.

[20] Rose-Anne Dana and Monique Jeanblanc. The Black-Scholes Formula. *Financial Markets in Continuous Time*, pages 81–125, 2007.

[21] Mark Davis. Mathematics of Financial Markets. In *in Engquist B., Schmid W.," Mathematics Unlimited–2001 & Beyond", Springer Berlin 2001.* Citeseer.

[22] Mark Davis et al. *The evaluation of venture capital as an instalment option: Valuing real options using real options.* Springer, 2004.

[23] Avinash K Dixit and Robert S Pindyck. *Investment under uncertainty.* Princeton university press, 1994.

[24] Didier Dubois. Possibility theory and statistical reasoning. *Computational Statistics & Data Analysis*, 51(1):47–69, 2006.

[25] Robert James Elliott and P Ekkehard Kopp. *Mathematics of financial markets*, volume 10. Springer, 2005.

[26] Paul N Finlay and John M Wilson. The paucity of model validation in operational research projects. *Journal of the Operational Research Society*, pages 303–308, 1987.

[27] Paul Flignor and David Orozco. Intangible Asset & Intellectual Property Valuation: A Multidisciplinary Perspective. Retrieved from `http://www.wipo.int/sme/en/documents/ip_valuation_fulltext.html`.

[28] Robert Fullér et al. *Fuzzy reasoning and fuzzy optimization*. Number 9. Citeseer, 1998.

[29] Robert Fullér and Péter Majlender. On weighted possibilistic mean and variance of fuzzy numbers. *Fuzzy Sets and Systems*, 136(3):363–374, 2003.

[30] Robert Fullér and Péter Majlender. On interactive fuzzy numbers. *Fuzzy Sets and Systems*, 143(3):355–369, 2004.

[31] Andrei Hagiu and David B Yoffie. *Intermediaries for the IP Market*. Harvard Business School, 2011.

[32] Andrei Hagiu and David B Yoffie. The new patent intermediaries: platforms, defensive aggregators, and super-aggregators. *The Journal of Economic Perspectives*, 27(1):45–65, 2013.

[33] Randolph W Hall. What's so scientific about MS/OR? *Interfaces*, 15(2):40–45, 1985.

[34] Mark Harris. Snapping up Kodak. *Spectrum, IEEE*, 51(2):30–62, 2014.

[35] Marcus Hartmann and Ali Hassan. Application of real options analysis for pharmaceutical R&D project valuation–Empirical results from a survey. *Research Policy*, 35(3):343–354, 2006.

[36] Jon C Helton and David E Burmaster. Guest editorial: treatment of aleatory and epistemic uncertainty in performance assessments for complex systems. *Reliability Engineering & System Safety*, 54(2):91–94, 1996.

[37] F Herrera and JL Verdegay. Fuzzy sets and operations research: perspectives. *Fuzzy Sets and Systems*, 90(2):207–218, 1997.

[38] Frederick S. Hillier and Gerald J. Lieberman. *Introduction to operations research*. McGraw-Hill Higher Education, 2010.

[39] F Owen Hoffman and Jana S Hammonds. Propagation of uncertainty in risk assessments: the need to distinguish between uncertainty due to lack of knowledge and uncertainty due to variability. *Risk Analysis*, 14(5):707–712, 1994.

[40] Adam B Jaffe and Josh Lerner. *Innovation and its discontents: How our broken patent system is endangering innovation and progress, and what to do about it.* Princeton University Press, 2011.

[41] Robert Jarrow and Philip Protter. A short history of stochastic integration and mathematical finance: the early years, 1880-1970. *Lecture Notes-Monograph Series*, pages 75–91, 2004.

[42] S. Kamiyama, J. Sheehan, and C. Martinez. Valuation and Exploitation of Intellectual Property. OECD Publishing, 2006. OECD Science, Technology and Industry Working Papers.

[43] David Kellogg and John M Charnes. Real-options valuation for a biotechnology company. *Financial Analysts Journal*, pages 76–84, 2000.

[44] Alexander Kirsch. Securitization of Intellectual Property as a funding alternative. Master's thesis, 2005.

[45] Armen Der Kiureghian and Ove Ditlevsen. Aleatory or epistemic? Does it matter? *Structural Safety*, 31(2):105–112, 2009.

[46] George Klir and Bo Yuan. *Fuzzy sets and fuzzy logic*, volume 4. Prentice Hall New Jersey, 1995.

[47] George J Klir. *Uncertainty and information: foundations of generalized information theory.* John Wiley & Sons, 2005.

[48] George J Klir and Richard M Smith. On measuring uncertainty and uncertainty-based information: recent developments. *Annals of Mathematics and Artificial Intelligence*, 32(1-4):5–33, 2001.

[49] Frank H Knight. Risk, uncertainty and profit. *New York: Hart, Schaffner and Marx*, 1921.

[50] Tim Koller, Marc Goedhart, David Wessels, et al. *Valuation: measuring and managing the value of companies*, volume 499. John Wiley and Sons, 2010.

[51] Céline Lagrost, Donald Martin, Cyrille Dubois, and Serge Quazzotti. Intellectual property valuation: how to approach the selection of an appropriate valuation method. *Journal of Intellectual Capital*, 11(4):481–503, 2010.

[52] Maurice Landry, Jean-Louis Malouin, and Muhittin Oral. Model validation in operations research. *European Journal of Operational Research*, 14(3):207–220, 1983.

[53] Cooper H Langford, Jeremy Hall, Peter Josty, Stelvia Matos, and Astrid Jacobson. Indicators and outcomes of Canadian university research: Proxies becoming goals? *Research Policy*, 35(10):1586–1598, 2006.

[54] Chi Man Leung and Yue Kuen Kwok. Real options game analysis of sleeping patents. *Decisions in Economics and Finance*, 34(1):41–65, 2011.

[55] Shu-Hsien Liao and Shiu-Hwei Ho. Investment project valuation based on a fuzzy binomial approach. *Information Sciences*, 180(11):2124–2133, 2010.

[56] Rogemar S Mamon and Robert J Elliott. *Hidden markov models in finance*, volume 4. Springer, 2007.

[57] Daryl Martin and David C Drews. Intellectual property valuation techniques. *The Licensing Journal*, page 16, 2006.

[58] Scott Mathews, Vinay Datar, and Blake Johnson. A practical method for valuing real options: the boeing approach. *Journal of Applied Corporate Finance*, 19(2):95–104, 2007.

[59] Robert C Merton. Theory of rational option pricing. *The Bell Journal of Economics and Management Science*, pages 141–183, 1973.

[60] J. Mezei. *A quantitative view on fuzzy numbers.* PhD thesis, Åbo Akad. Univ., Turku Centre for Computer Science, 2011.

[61] Ashby HB Monk. The emerging market for intellectual property: drivers, restrainers, and implications. *Journal of Economic Geography*, 9(4):469–491, 2009.

[62] Adam Moore. Intellectual Property. In Edward N. Zalta, editor, *The Stanford Encyclopedia of Philosophy.* Summer 2011 edition, 2011.

[63] Millett Granger Morgan and Mitchell Small. *Uncertainty: a guide to dealing with uncertainty in quantitative risk and policy analysis.* Cambridge University Press, 1992.

[64] Kazuyuki Motohashi. Licensing or not licensing? An empirical analysis of the strategic use of patents by Japanese firms. *Research Policy*, 37(9):1548–1555, 2008.

[65] Federico Munari and Raffaele Oriani. *The economic valuation of patents: methods and applications*. Edward Elgar Publishing, 2011.

[66] Paul H. Munter, Jr. Orzechowski, Robert J., and Irina Zavoronkova. Appraisal Foundation Issues Guidance on Contributory Asset Charges Used in Fair Value Measurements, August 2010. Publication of KPMG's Department of Professional Practice.

[67] Silvia Muzzioli and Huguette Reynaerts. Fuzzy linear systems of the form A1x+ b1= A2x+ b2. *Fuzzy Sets and Systems*, 157(7):939–951, 2006.

[68] Silvia Muzzioli and Huguette Reynaerts. The solution of fuzzy linear systems by non-linear programming: a financial application. *European Journal of Operational Research*, 177(2):1218–1231, 2007.

[69] Silvia Muzzioli and Huguette Reynaerts. American option pricing with imprecise risk-neutral probabilities. *International Journal of Approximate Reasoning*, 49(1):140–147, 2008.

[70] Stewart C Myers and Saman Majd. *Real Options and Investment under Uncertainty: Classical Readings and Recent Contributions*, chapter Abandonment value and project life, pages 295–312. MIT press, 2001.

[71] Stewart C Myers and Stuart M Turnbull. Capital budgeting and the capital asset pricing model: Good news and bad news. *The Journal of Finance*, 32(2):321–333, 1977.

[72] CV Negoita, LA Zadeh, and HJ Zimmermann. Fuzzy sets as a basis for a theory of possibility. *Fuzzy sets and systems*, 1:3–28, 1978.

[73] Andrew J Nelson. Measuring knowledge spillovers: What patents, licenses and publications reveal about innovation diffusion. *Research Policy*, 38(6):994–1005, 2009.

[74] H. Otsuyama. *Patent Strategy Handbook*, chapter Patent Valuation and Intellectual Assets Management. Chuckeizaisha, Tokyo, 2003.

[75] Heiko Paeth, Robin Girmes, Gunter Menz, and Andreas Hense. Improving seasonal forecasting in the low latitudes. *Monthly weather review*, 134(7):1859–1879, 2006.

[76] M Elisabeth Paté-Cornell. Uncertainties in risk analysis: Six levels of treatment. *Reliability Engineering & System Safety*, 54(2):95–111, 1996.

[77] Enrico Pennings and Luigi Sereno. Evaluating pharmaceutical R&D under technical and economic uncertainty. *European Journal of Operational Research*, 212(2):374–385, 2011.

[78] Robert H Pitkethly. Intellectual property strategy in Japanese and UK companies: patent licensing decisions and learning opportunities. *Research Policy*, 30(3):425–442, 2001.

[79] Robert H Pitkethly. *The Management of Intellectual Property*, chapter Patent Valuation and Real Options, pages 268–292. Edward Elgar, 2006.

[80] Erik P Rau. Combat science: the emergence of operational research in World War II. *Endeavour*, 29(4):156–161, 2005.

[81] Bertrand Russell. Vagueness. *The Australasian Journal of Psychology and Philosophy*, 1(2):84–92, 1923.

[82] Paul A Samuelson. Proof that properly anticipated prices fluctuate randomly. *Industrial management review*, 6(2):41–49, 1965.

[83] Paul A Samuelson. Rational theory of warrant pricing. *Industrial management review*, 6:13–31, 1965.

[84] Eduardo S Schwartz. Patents and R&D as real options. *Economic Notes*, 33(1):23–54, 2004.

[85] Steven Shreve. *Stochastic calculus for finance I: the binomial asset pricing model.* Springer Science & Business, 2012.

[86] Roman Slowinski et al. *Fuzzy sets in decision analysis, operations research and statistics.* Kluwer Academic Publishers USA, 1998.

[87] Han TJ Smit and Lenos Trigeorgis. *Strategic investment: Real options and games.* Princeton University Press, 2012.

[88] Gordon V Smith and Russell L Parr. *Valuation of intellectual property and intangible assets.* Wiley Chichester, 2000.

[89] Alexander Suetin. Post-crisis developments in international financial markets. *International Journal of Law and Management*, 53(1):51–61, 2011.

[90] Andrew G Sutherland and Jeffrey R Williams. Valuing Real Options: Insight from Competitive Strategy. *Tepper School of Business*, page 540, 2008.

[91] A Çagri Tolga. Fuzzy multicriteria R&D project selection with a real options valuation model. *Journal of Intelligent and Fuzzy Systems*, 19(4):359–371, 2008.

[92] Lenos Trigeorgis. *Real options: Managerial flexibility and strategy in resource allocation*. MIT press, 1996.

[93] W Weaer. Science and complexity. *American scientist*, 36(4):536, 1948.

[94] Harald Wirtz. Valuation of Intellectual Property: A Review of Approaches and Methods. *International Journal of Business and Management*, 7(9):p40, 2012.

[95] Bo K Wong and Vincent S Lai. A survey of the application of fuzzy set theory in production and operations management: 1998–2009. *International Journal of Production Economics*, 129(1):157–168, 2011.

[96] Yuji Yoshida. The valuation of European options in uncertain environment. *European Journal of Operational Research*, 145(1):221–229, 2003.

[97] Yuji Yoshida, Masami Yasuda, Jun-ichi Nakagami, and Masami Kurano. A new evaluation of mean value for fuzzy numbers and its application to American put option under uncertainty. *Fuzzy Sets and Systems*, 157(19):2614–2626, 2006.

[98] Lotfi A Zadeh. Fuzzy sets. *Information and control*, 8(3):338–353, 1965.

[99] Lotfi A Zadeh. Outline of a new approach to the analysis of complex systems and decision processes. *Systems, Man and Cybernetics, IEEE Transactions on*, (1):28–44, 1973.

[100] Lotfi A Zadeh. Is there a need for fuzzy logic? *Information Sciences*, 178(13):2751–2779, 2008.

[101] H-J Zimmermann. An application-oriented view of modeling uncertainty. *European Journal of Operational Research*, 122(2):190–198, 2000.

[102] HJ Zimmermann. *Fuzzy Set Theory and Its Applications, Second Revised Edition*. Springer, 1992.

Xiaolu Wang

Fuzzy Real Option Analysis in Patent Related Decision Making and Patent Valuation

This dissertation attempts to overcome the five impeding barriers that limit the usability and reliability of the classic approaches for patent valuation by rationalizing the use of two techniques, namely fuzzy set theory and real option analysis.

The use of the proposed techniques in practical applications is demonstrated by three fuzzy real option analysis based models, namely the pay-off method, the extended fuzzy Black-Scholes model and a fuzzy binomial model.

9 789521 232275 >

ISBN 978-952-12-3227-5