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**NON-MARKOVIAN DYNAMICS
IN CONTINUOUS VARIABLE
QUANTUM SYSTEMS**

by

Ruggero Vasile

ACADEMIC DISSERTATION

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Abstract

The present manuscript represents the completion of a research path carried forward during my doctoral studies in the University of Turku. It contains information regarding my scientific contribution to the field of open quantum systems, accomplished in collaboration with other scientists.

The main subject investigated in the thesis is the non-Markovian dynamics of open quantum systems with focus on continuous variable quantum channels, e.g. quantum Brownian motion models. Non-Markovianity is here interpreted as a manifestation of the existence of a flow of information exchanged by the system and environment during the dynamical evolution. While in Markovian systems the flow is unidirectional, i.e. from the system to the environment, in non-Markovian systems there are time windows in which the flow is reversed and the quantum state of the system may regain coherence and correlations previously lost.

Signatures of a non-Markovian behavior have been studied in connection with the dynamics of quantum correlations like entanglement or quantum discord. Moreover, in the attempt to recognise non-Markovianity as a resource for quantum technologies, it is proposed, for the first time, to consider its effects in practical quantum key distribution protocols. It has been proven that security of coherent state protocols can be enhanced using non-Markovian properties of the transmission channels.

The thesis is divided in two parts: in the first part I introduce the reader to the world of continuous variable open quantum systems and non-Markovian dynamics. The second part instead consists of a collection of five publications inherent to the topic.

List of publications

This Thesis consists of an introductory part, followed by five research publications.

- I *Continuous variable entanglement dynamics in structured reservoirs*
R. Vasile, S. Olivares, M. G. A. Paris, and S. Maniscalco
Physical Review A **80** (2009) 062324 (12 pages).
- II *Nonclassical correlations in non-Markovian continuous-variable systems*
R. Vasile, P. Giorda, S. Olivares, M. G. A. Paris, and S. Maniscalco
Physical Review A **82** (2010) 012313 (10 pages).
- III *Entanglement dynamics for two harmonic oscillators coupled to independent environments*
R. Vasile
Physica Scripta T **140** (2010) 014063 (4 pages).
- IV *Quantifying non-Markovianity of continuous variable Gaussian dynamical maps*
R. Vasile, S. Maniscalco, M. G. A. Paris, H.-P. Breuer, and J. Piilo
Physical Review A **84**, (2011) 052118 (9 pages).
- V *Continuous-variable quantum key distribution in non-Markovian channels*
R. Vasile, S. Olivares, M. G. A. Paris, and S. Maniscalco
Physical Review A **83** (2011) 042321 (6 pages).

Other published material

During the doctoral studies I also had the possibility to work on other research topics. This is a list of the publications produced which have not been chosen as a part of the doctoral thesis.

- *Casimir-Polder forces, boundary conditions and fluctuations*
R. Messina, R. Passante, L. Rizzuto, S. Spagnolo and R. Vasile
J. Phys. A: Math. Theor. **41**, 164031 (2008).
- *Dynamical atom-wall Casimir-Polder forces*
R. Vasile and R. Passante
Phys. Rev. A **78**, 032108 (2008)
- *Time-dependent Maxwell fields and field energy density for an atom near a conducting wall*
R. Vasile, R. Messina, and R. Passante
Phys. Rev. A **79**, 062106 (2009).
- *Dynamical Casimir-Polder force on a partially dressed atom near a conducting wall*
R. Messina, R. Vasile, and R. Passante
Phys. Rev. A **82**, 062501 (2010).
- *Interferometry using spinor Bose-Einstein condensates*
R. Vasile, H. Mäkelä, and K.-A. Suominen
e-print on ArXiv:0812.0499.

Chapter 1

Introduction

The aim of quantum mechanics is to provide a physical description and interpretation of natural phenomena regarding the very fundamental constituents of matter such as atoms and molecules. Quantum theory has been originally formulated in the first half of 20th century to describe the dynamics of closed systems, i.e. systems whose interaction with the rest of the universe can be neglected [1]. It turns out, however, that the very nature of quantum mechanical systems makes them sensible to any external influence. Even measurement processes, whose effects on the dynamics of classical systems can be made arbitrarily small, produce non-negligible consequences in the quantum case.

Most quantum mechanical systems are made of electrically charged particles, e.g. electrons and protons, and therefore they constantly interact with the surrounding quantum electromagnetic field. Observable effects of this interaction, e.g. the existence of forces between neutral polarizable objects [2], or the phenomenon of spontaneous emission of radiation [3], witness how quantum systems need to be considered, fundamentally, open.

Once we accept the idea of the intrinsic openness of quantum systems, we may formulate a further question: is the interaction with the external environment, always, crucial? Let us consider, for instance, an atom initially in its ground state with an applied laser field able to couple the ground state to a second, excited, state. With an appropriate tuning of the atom-laser interaction, it is possible to induce a coherent dynamics between these two atomic levels, such that the atom can be effectively described as a two level closed system. However, the transition from the excited to the ground state may also be induced by the interaction with the external electromagnetic field, and, as a consequence, the coherent unitary dynamics between the atomic levels

may be compromised. If the environmental interaction is much slower than the coherent laser driven dynamics, then, at the early stages of the dynamics, the system can be effectively considered as closed. For later times, instead, the environmental interaction produces its effects and the situation becomes a true example of an open quantum system, so that appropriate, novel, mathematical tools need to be implemented to study its dynamics [4].

The theory of open quantum systems has become a very active research field in the last three decades. As a fundamental reason, we can mention the enormous improvements in the experimental techniques of preparation, control and measurement of quantum systems [5]. These techniques allow now to perform precise and fragile experiments in which it is possible to test basic quantum mechanical principles. At this level of precision quantum systems have to be considered effectively open systems and the interaction with the surroundings cannot be neglected anymore.

Another driving example can be found by looking at the advances obtained in the field of quantum information and computation [6], where properties like state coherence and quantum correlations become fundamental resources. The open system dynamics causes irreversible deterioration of these resources, threatening the accomplishment of tasks and protocols in quantum information. Hence the quantum information community has shown an increasing interest in studying ways to circumvent this problem and to develop techniques to protect the system from the detrimental effects of the environmental interactions, e.g. quantum error correction [7].

The dynamics of open quantum systems can be studied by deriving and solving an equation of motion for the state of the open system only, i.e. a master equation [8–10]. The form of the master equation depends on the physical system we are interested in, e.g. a qubit or a system of harmonic oscillators, on the properties of the environmental degrees of freedom and on the form of the system-environment interaction. One of the most commonly used and simple form of master equations is the so called Lindblad master equation [11,12]. Quantum optical systems are examples of open dynamics well described by these kind of equations [13]. On the other hand solid state systems or atom dynamics in optical cavities are more precisely described by more complicated master equations, due to the structured form of the environmental noise [14,15].

Lindblad-form equations can be derived, e.g. under the Born and Markov approximations. The Born approximation is essentially an assumption on the strength of the interaction between system and environment, considered to

be weak. The Markov approximation, instead, is a condition on the relative value of certain relevant time scales of the open system dynamics. Systems for which the Markov approximation well describes the dynamics are usually called Markovian, in contrast with non-Markovian systems. The concept of non-Markovianity has been for long time associated to a property of the master equation describing the open system dynamics. From a physical point of view non-Markovianity is connected to the existence of memory effects in the system dynamics.

Recently, however, a definition of non-Markovianity inspired to the idea of quantifying the memory effects, has been introduced and a new interpretation independent on the form of the master equation has been provided [16]. In this new view, non-Markovianity is seen as a global property of the system and environment, characterized by a back flow of information from the environment to the system which modifies qualitatively the trajectory of the state of the system. Hence non-Markovianity is defined as a property of the open system dynamics and is not strictly linked to the mathematical structure of the master equation.

Signatures of a non-Markovian behavior in open quantum systems can be observed, for instance, in the evolution of quantum correlations in multipartite systems. A Markovian evolution usually leads to a uniform degradation of these correlations as time passes, e.g. sudden death of entanglement [17]. On the other hand in a non-Markovian evolution, the back and forth communication between system and environment, makes possible revivals and increase of correlations in certain interval of times [18, 19].

The subject of this thesis is the investigation of the properties of continuous variable non-Markovian open systems. We essentially concentrate on three big topics: dynamics of Gaussian quantum correlations, definition of non-Markovianity for Gaussian states and coherent states quantum key distribution protocols in non-Markovian channels.

The first three papers of the thesis are dedicated to the study of the evolution of entanglement, quantum discord and intensity correlations in quantum Brownian motion (QBM) models [20, 21]. Particular attention is devoted to study the conditions for robustness of correlations under the dissipative open system dynamics, for various environmental structures. Quantum correlations are, indeed, important resources in many quantum information and communication protocols, e.g. quantum teleportation or quantum key distribution. Their dynamics is fundamental in the achievement or failure of these protocols and, therefore needs to be studied carefully in all situations of interests. In

non-Markovian systems it can be observed a longer survival time of correlations and the environment acts as a source capable of rebuild the already lost correlations by giving back the information about the initial state. This property is valid for correlations like entanglement [22], quantum discord [23, 24] and also for intensity correlations for light beams [25, 26].

The definition of non-Markovianity for continuous variable systems and the measure of its degree is the subject of paper IV. In this work we extend the definition given in [16] to continuous variable Gaussian states and analyze the non-Markovian behavior of coherent and squeezed states for a pair of quantum Brownian motion models.

Finally in paper V we explore, for the first time, a possible practical application of non-Markovian dynamics in the context of continuous variable quantum key distribution. We show how it is possible to exploit the non-Markovian properties of the transmission channel to enhance the security of a coherent state protocol [27] and check the presence of an eavesdropper.

The thesis is organized as follows. In Chapter 2 we introduce some fundamentals of dynamics of closed and open quantum systems. Particular attention will be devoted to the concept of master equation and its microscopic derivation, with specific emphasis to the Born-Markov and secular approximations. In the same chapter we also introduce the concepts of Markovian and non-Markovian dynamics and discuss their fundamental physical assumptions in terms of reservoir memory effects on the system dynamics. In chapter 3 we provide the basic notions and definitions for continuous variable systems. We concentrate in particular on Gaussian states, i.e. states generated by bilinear Hamiltonian dynamics on the total system. We also familiarize with the single and bimodal cases which are the main systems analyzed in the thesis. Chapter 4 is dedicated to continuous variable open systems and in particular to QBM master equations: the Hu-Paz-Zhang equation, the secular approximated equation and the Lindblad form equation. These are all the basic models employed in the contributing papers. In Chapter 5 we discuss correlations in bipartite systems introducing the concept of separability, entanglement, quantum discord and intensity correlations, with the relative expressions for bimodal continuous variable systems. At the end of the chapter we also provide the bimodal extension of the QBM master equations introduced previously. In Chapter 6 we discuss in more detail the concept of non-Markovianity and introduce its measure for a given open quantum system. Our main contribution here is the generalization of this definition to continuous variable systems, an issue presented in paper IV. In Chapter 7 we review basic concepts of quan-

tum key distribution with particular attention to continuous variable coherent state protocols. Here we also provide the basics of the novel detection method based on non-Markovian transmission channels which is the subject of paper V. Finally, Chapter 8 is dedicated to the discussion and summary of the main results of the five papers belonging to this thesis. These summaries are not however intended to substitute the papers at the end of the thesis, but they are meant to be a guide and an introduction to the more detailed discussion contained into them. We close this last chapter and the thesis with a general conclusive section about the whole research work.

Chapter 2

Open quantum systems

In this chapter we review the basic principles and tools used to investigate closed and open system dynamics in quantum mechanics. We introduce the concept of density operator, dynamical map and master equation to describe the reduced system dynamics of open systems. We also recall the basic ideas beyond the derivation of the master equation with focus on Markovian systems and dynamical semigroups. Finally we introduce the concept of non-Markovianity and discuss its physical interpretation in terms of reservoir memory effects.

2.1 Density operator and closed system dynamics

The most complete description of the state of a quantum system at a given time $t = t_0$ is provided by a vector $|\psi(t_0)\rangle$ belonging to a Hilbert space \mathcal{H} associated to the physical system. The time evolution of the state vector is determined by the solution of the Schrödinger equation

$$i\hbar \frac{d}{dt} |\psi(t)\rangle = \mathbf{H} |\psi(t)\rangle, \quad (2.1)$$

where \mathbf{H} is the Hamiltonian of the system. The dynamics generated by Eq. (2.1) is unitary

$$|\psi(t)\rangle = \mathbf{U}(t, t_0) |\psi(t_0)\rangle, \quad \mathbf{U}^\dagger(t, t_0) = \mathbf{U}^{-1}(t, t_0), \quad (2.2)$$

where, for a time-independent Hamiltonian, the evolution operator takes the form $\mathbf{U}(t, t_0) = \exp\{-i\mathbf{H}(t - t_0)/\hbar\}$.

This original formulation of state dynamics in quantum mechanics [28, 29] relies on the fundamental assumption that we can fully describe the system at the initial time in terms of a state vector. However, in some situations it is not possible to assign a state vector, for instance, when we lack a complete control on the preparation procedures of our system. In these cases only the probabilities c_i that the system is in a state $|\psi_i\rangle$ of a given ensemble $\{|\psi_i\rangle, i = 1 \dots n\}$ are known, and the state of the system must be described by a more general object, known as *density operator*. The density operator associated to the ensemble considered is given by

$$\rho = \sum_i c_i |\psi_i\rangle\langle\psi_i|, \quad \sum_i c_i = 1, \quad (2.3)$$

where the states $|\psi_i\rangle$ do not need to be orthogonal. All density operators satisfy three main properties: normalization $Tr[\rho] = 1$, hermiticity $\rho = \rho^\dagger$ and positivity $\rho \geq 0$. It follows that any density operator can be diagonalized

$$\rho = \sum_i p_i |\phi_i\rangle\langle\phi_i|, \quad \sum_i p_i = 1, \quad p_i \geq 0, \quad (2.4)$$

where in the spectral decomposition (2.4) the vectors $|\phi_i\rangle$ are orthogonal. From now on we denote with $\mathcal{D}(\mathcal{H})$ the set of density operators with support in the Hilbert space \mathcal{H} .

When one of the probabilities is unit, e.g. $p_j = 1$, the system is unambiguously described by the *pure* state $|\psi_j\rangle$, and the density operator approach is equivalent to the original formulation based on state vectors. Otherwise we say that the state is *mixed*. To verify if a state is pure or mixed there is no need to find its spectral decomposition but it suffices to evaluate the so-called *purity*

$$\mu = Tr[\rho^2]. \quad (2.5)$$

It is easy to verify that $\mu \leq 1$, being equal to one only for pure states.

The density operator formulation is useful for many reasons, for instance when we want to describe the state of a multipartite system. Imagine a physical system made of two subsystems A and B described by a composite state $\rho_{AB} \in \mathcal{D}(\mathcal{H})$ with $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$. We can define the state of each subsystem by the partial trace operation, e.g.

$$\rho_A = Tr_B[\rho_{AB}] \equiv \sum_j \langle\phi_j^A| \rho_{AB} |\phi_j^A\rangle, \quad (2.6)$$

where $\{|\phi_j^A\rangle, j = 1, \dots\}$ is an orthogonal basis in \mathcal{H}_A . Even if the state of the total system ρ_{AB} is pure, the partial trace operation usually leaves us with a mixed state ρ_A of the system A , also called reduced state.

To conclude this section we provide the dynamical equations for the density operator of a closed quantum system

$$\frac{d}{dt}\rho(t) = -\frac{i}{\hbar}[\mathbf{H}, \rho], \quad (2.7)$$

known as von Neumann equation [29], which can be straightforwardly obtained from Eq. (2.1). Because the system is closed, the dynamics generated by Eq. (2.7) is unitary

$$\rho(t) = \mathbf{U}^\dagger(t, t_0)\rho(t_0)\mathbf{U}(t, t_0). \quad (2.8)$$

Under a unitary evolution the purity does not change, therefore an initially pure (mixed) states remains pure (mixed). In the next section we will see that this is not true when the system is subjected to a non-unitary evolution due to the interaction with other systems.

2.2 Open systems and dynamical maps

As we saw in the previous section, the dynamics of a closed system is described by a unitary operator, a result valid for both initial pure and mixed states. In the case of an open system the situation changes drastically and new tools to study its dynamics are needed [8–10]. We now consider a bipartite closed system, composed of the subsystems S and E , and we focus on the dynamics of the subsystem S only, from now on referred to as the system, while the other subsystem constitutes its environment or reservoir. Our goal is to investigate how the interaction with the environment E influences the dynamics of the state of the system of interest S .

We know that the dynamics of the total closed system is unitary and is determined by an evolution operator $\mathbf{U}_{SE}(t, t_0)$. Therefore if $\rho_{SE}(t_0) \equiv \rho(t_0)$ is the initial state of the full system plus environment, we have

$$\rho(t) = \mathbf{U}_{SE}^\dagger(t, t_0)\rho(t_0)\mathbf{U}_{SE}(t, t_0). \quad (2.9)$$

Applying the partial trace rule of Eq. (2.6), we get

$$\rho_S(t) = \text{Tr}_E[\mathbf{U}_{SE}^\dagger(t, t_0)\rho(t_0)\mathbf{U}_{SE}(t, t_0)]. \quad (2.10)$$

If we indicate with $\rho_S(t_0)$ the initial state of the system S , we may ask which is the connection between this state and the evolved state at time t . Can we formally write the evolution as

$$\rho_S(t) = \text{Tr}_E[\mathbf{U}_{SE}^\dagger(t, t_0)\rho(t_0)\mathbf{U}_{SE}(t, t_0)] = \Phi(t, t_0)\rho_S(t_0), \quad (2.11)$$

with $\Phi(t, t_0)$ being a superoperator acting on the initial state? In many situations Eq. (2.11) can actually be defined, and the operator $\Phi(t, t_0)$ takes the name of open system *dynamical map* [30–35]. The dynamical map satisfies two main properties: it preserves superpositions, i.e. must be linear, and it maps any density operator into another density operator

$$\begin{aligned} \Phi(t, t_0)[\lambda\rho_1 + (1 - \lambda)\rho_2] &= \lambda\Phi(t, t_0)\rho_1 + (1 - \lambda)\Phi(t, t_0)\rho_2, \\ \Phi(t, t_0)\rho &= \tilde{\rho} \in \mathcal{D}(\mathcal{H}). \end{aligned} \quad (2.12)$$

Eqs. (2.12) must be valid for any $t \geq t_0$, for any $0 \leq \lambda \leq 1$ and any $\rho, \rho_1, \rho_2 \in \mathcal{D}(\mathcal{H})$.

The map $\Phi(t, t_0)$ is not unitary, therefore in contrast with the behavior of closed systems, pure states can become mixed. This phenomenon, milestone of the theory of open quantum systems, is known under the name of *decoherence* or environment induced decoherence [36–38]. It describes the transformation from coherent superpositions (pure states) to incoherent superpositions, i.e. statistical mixtures (mixed states).

Finding the analytic expression and/or the properties of the map $\Phi(t, t_0)$ from the knowledge of the unitary operator $\mathbf{U}_{SE}^\dagger(t, t_0)$ and the initial state $\rho(t_0)$ of the system plus environment, is not always possible. A fundamental simplification consists in using *factorized initial conditions*, i.e. assuming that the initial state of the total system can be written as a product of the state of the system and of the environment: $\rho(t_0) = \rho_S(t_0) \otimes \rho_E(t_0)$. In this case it has been shown that the map $\Phi(t, t_0)$ exists and it is also *completely positive* [39, 40], i.e. any extension of the map $\Phi(t, t_0) \otimes \mathbf{I}_n$, with \mathbf{I}_n the identity map for an Hilbert space of dimension n , is a positive map. The functional form of any completely positive map Φ can be characterized by the operator sum representation, or Kraus representation [6, 41]

$$\Phi \rho = \sum_k \mathbf{A}_k \rho \mathbf{A}_k^\dagger, \quad \sum_k \mathbf{A}_k^\dagger \mathbf{A}_k = \mathbf{I}, \quad (2.13)$$

where the \mathbf{A}_k *Kraus operators*, provide a complete characterization of the dynamical map Φ .

The factorized initial condition is a very delicate assumption, because the interaction between system and environment may build correlations. However dynamical maps derived under this assumption well describe realistic experimental conditions in most of physical scenarios under investigation.

When the initial state is not factorized the situation is more complicated. Complete positivity is satisfied only for classically correlated initial states [42–44], while in general nothing can be said on both the properties and functional form of the dynamical map. In some cases it is not possible to define a dynamical map, as its expression depends on the structure and amount of correlations in the initial state of the system.

In this thesis we only deal with systems initially prepared in a factorized state with the environment.

2.3 The master equation

In the previous section we introduced the dynamical map approach to the description of the reduced system dynamics. We have seen that the dynamical map, in a sense, plays the role of the unitary operator for closed systems, providing the evolved state at a later time. One equivalent approach to look at the open system dynamics is to derive an equation of motion for the reduced state dynamics, known as *master equation* [8–10].

The starting point to the derivation of master equations is the expression of the von Neumann equation for the total closed system (2.7). There exist several approaches to the derivation of the master equation, starting from the Hamiltonian of the closed system, e.g. influence functional [45], projection operators [46, 47] or time convolutionless techniques [48–50]. Different methods may lead to different forms of master equations. We can essentially classify them in two big categories: *memory kernel* and *time-local* equations. An example of memory kernel master equation is the Nakajima-Zwanzig equation [46, 47], whose form for the only relevant assumption of factorizing initial conditions is

$$\frac{\partial}{\partial t} \mathcal{P}\rho(t) = \int_{t_0}^t ds \mathcal{K}(t, s) \mathcal{P}\rho(s), \quad (2.14)$$

where $\mathcal{K}(t, s)$ is called convolution kernel of the equation and depends essentially on the Hamiltonian of the total system and the state of the environment. $\mathcal{P}\rho(t) = Tr_E[\rho(t)] \otimes \rho_E = \rho_S(t) \otimes \rho_E$, where $\rho(t)$ is the total system plus environment state and ρ_E is a fixed state of the environment, usually taken to be a thermal state. Eq. (2.14) is an integro-differential equation for the reduced

system state, thus the solution at time t depends on the history of the state evolution. In this sense we call it a memory kernel equation, as it describes a state evolution which depends on the memory of the past evolution. On the contrary local in time equations do not contain convolution kernels and are thought to describe memoryless dynamical evolutions. As we will see later on, however, this distinction is not satisfactory, also because memory kernel master equations can be recast in a time local form [51].

We now provide an example of derivation of local in time master equation, valid for weakly coupled systems. The most general open system model can be described by the following Hamiltonian structure

$$\mathbf{H} = \mathbf{H}_S + \mathbf{H}_E + \mathbf{H}_I, \quad (2.15)$$

where \mathbf{H}_S is the system free Hamiltonian, \mathbf{H}_E is the reservoir free Hamiltonian, and \mathbf{H}_I is the interaction term, hitherto assumed to be time-independent. Situations with external laser driving fields, characterized by time-dependent Hamiltonians, have also been subjects of investigations [52–54].

In the interaction picture the von Neumann equation for the state of system plus environment reads

$$\frac{d}{dt}\tilde{\rho}(t) = -\frac{i}{\hbar}[\mathbf{H}_I(t), \tilde{\rho}(t)], \quad (2.16)$$

with $\mathbf{H}_I(t)$ and $\tilde{\rho}(t)$ the interaction Hamiltonian and the state of the total system in the interaction picture, respectively. A formal integration of the equation of motion (2.16) leads to

$$\tilde{\rho}(t) = \tilde{\rho}(0) - \frac{i}{\hbar} \int_0^t ds [\mathbf{H}_I(s), \tilde{\rho}(s)]. \quad (2.17)$$

Inserting Eq. (2.17) into (2.16), we get

$$\frac{d}{dt}\tilde{\rho}(t) = -\frac{i}{\hbar}[\mathbf{H}_I(t), \tilde{\rho}(0)] - \frac{1}{\hbar^2} \int_0^t [\mathbf{H}_I(t), [\mathbf{H}_I(s), \tilde{\rho}(s)]] ds, \quad (2.18)$$

where we assumed the initial time $t_0 = 0$, without any lack of generality. To obtain the master equation we can apply the partial trace rule (2.6) to both sides of Eq. (2.18) obtaining

$$\frac{d}{dt}\tilde{\rho}_S(t) = -\frac{1}{\hbar^2} \int_0^t Tr_E \{ [\mathbf{H}_I(t), [\mathbf{H}_I(s), \tilde{\rho}(s)]] \} ds, \quad (2.19)$$

where $\tilde{\rho}_S(t)$ is the state of system still in the interaction picture and we assumed the condition $Tr_E\{[\mathbf{H}_I(t), \tilde{\rho}(0)]\} = 0$. The following steps require a certain number of approximations compatible with the physical situation of interest. It is not our purpose to provide a full description of such approximations, but we limit ourselves to a brief discussion. The reader interested in a more complete analysis of the microscopic derivation of the master equations can consult the following references [8–10].

The first fundamental assumption is the factorized state initial condition discussed in the previous section, followed by a second condition on the strength of the interaction, i.e. the weak coupling or Born approximation. Under both these approximations, when the environment is made of an infinite number of degrees of freedom, we can assume that the system dynamics does not affect the dynamics of the environment state. It follows that system and environment remain uncorrelated at any successive time, so that $\tilde{\rho}(t) = \tilde{\rho}_S(t) \otimes \rho_E$, where ρ_E is a stationary state for the environment usually taken to be a thermal state.

Another assumption usually considered in open systems is the Markov approximation, i.e. a condition imposed on the time scales of the dynamics of the total system. Generally one can identify three time scales of interest. The first one is the free time scale of the system τ_S , which, in the case of an harmonic oscillator of frequency ω_0 is of the order of $\tau_S \simeq 1/\omega_0$. Another time scale is τ_E , known as correlation time scale of the environment, which essentially depends on the state of the environment ρ_E , on its spectral properties and on the interaction strength between system and environment. Finally we have the relaxation time scale τ_R , which quantifies the rate of system state change due to the interaction with the environment. The Markov approximation holds when $\tau_E \ll \tau_R$.

Under the factorized initial condition and Born-Markov approximation equation (2.19) reads

$$\frac{d}{dt}\tilde{\rho}_S(t) = -\frac{1}{\hbar^2} \int_0^\infty Tr_E\{[H_I(t), [H_I(t-s), \tilde{\rho}_S(t) \otimes \rho_E]]\} ds, \quad (2.20)$$

which is a master equation at second order in the interaction and in a time-local form, i.e. the solution at time t depends only on the value of the system state at time t .

2.3.1 Markovian dynamics and dynamical semigroups

Equation (2.20) can be further simplified when the interaction Hamiltonian takes the form

$$\mathbf{H}_I = \sum_i \mathbf{S}_i \otimes \mathbf{E}_i, \quad (2.21)$$

which is a sum of products of system operators \mathbf{S}_i and environmental operators \mathbf{E}_i . In this context it is possible to perform the secular approximation to eliminate fastly oscillating terms appearing into the integral of (2.20). This approximation can be performed when $\tau_S \ll \tau_R$, a condition usually satisfied, for instance, in quantum optical and nuclear magnetic resonance (NMR) systems [13, 55].

The dynamical maps associated to weakly coupled open systems under the Markov and secular approximation belong to a specific class of maps studied for the first time, independently, in [11, 12]. These maps possess the property of a quantum dynamical semigroup. If we indicate the map with $\Phi(t, 0) \equiv \Phi(t)$, this is a dynamical semigroup if

$$\Phi(t_1) \cdot \Phi(t_2) = \Phi(t_1 + t_2), \quad \forall t_1, t_2 \geq 0. \quad (2.22)$$

Any dynamical semigroup can be written as $\Phi(t) = \exp\{\mathcal{L}t\}$, where \mathcal{L} is called *generator* of the semigroup. The master equation associated to this type of dynamical map attains the so-called Lindblad form

$$\frac{d}{dt}\rho_S = \mathcal{L}\rho_S, \quad (2.23)$$

with

$$\mathcal{L}\rho_S = -\frac{i}{\hbar}[\mathbf{H}_S, \rho_S] + \frac{1}{2} \sum_{k=1}^{N^2-1} \gamma_k (2C_k \rho_S C_k^\dagger - C_k^\dagger C_k \rho_S - \rho_S C_k^\dagger C_k), \quad (2.24)$$

where \mathbf{H}_S is the system free Hamiltonian, C_k are the *jump* or Lindblad operators and the positive constant coefficients γ_k are called decay rates. The summation is limited by the dimension $\dim\{\mathcal{H}_S\} = N$ of the Hilbert space of the system.

Systems described by master equations in the Lindblad form are usually known in the literature as *Markovian*. This denomination, however, is not fully recognized among the scientific community, and different definitions of non-Markovian/Markovian dynamics have been recently proposed. In the next section we investigate this issue in more detail.

2.3.2 Non-Markovian vs Markovian dynamics

It is widely accepted that Lindblad theory fails in describing the open system dynamics in various physical situations, e.g. in solid state systems at low temperature [14], or in superconducting circuits [15], as well as with photonic band gap materials [56,57]. The reason can be found in the failure of the Born approximation, i.e. strong system environment coupling, or in the failure of the Markov approximation. This last assumption may be inappropriate in the presence of non-trivial structured environments, as it happens in the case of photonic band gap materials, or in solid state systems subjected to $1/f$ noise.

In order to describe more precisely these *non-Markovian* systems, we need to abandon some of these approximations. As a consequence, the equations describing the dynamics of the reduced system will be mathematically more complicated, e.g. memory kernel or time dependent decay rates, and thus become more difficult to handle analytically, often requiring the implementation of numerical algorithms.

In section (2.3.1) we have already shown an example of exact form of master equation (2.14) which have an integro-differential form with a memory kernel. Because this equation is derived without the Markov assumption, people tend to associate memory-kernel master equations with non-Markovian systems, hence interpreting non-Markovianity as the presence of system-reservoir memory effects. According to this view, for instance, equations of the following Lindblad-type form

$$\frac{d}{dt}\rho_S = -\frac{i}{\hbar}[\mathbf{H}_S, \rho_S] + \frac{1}{2} \sum_{k=1}^{N^2-1} \gamma_k(t) (2C_k \rho_S C_k^\dagger - C_k^\dagger C_k \rho_S - \rho_S C_k^\dagger C_k), \quad (2.25)$$

should describe Markovian dynamics. However these kind of equations can be found, for instance, in the exact dynamics of an atom interacting with a lossy structured cavity [8, 58], hence in a derivation which does not include any Born-Markov approximation. Another example of exact local in time master equation is provided by the Hu-Paz-Zhang equation describing the exact dynamics of an harmonic particle interacting with a thermal bosonic bath under the only assumption of factorized initial conditions [59].

These examples show how the definition of Markovianity/non-Markovianity cannot be given in relation with the only form of the associated master equation, but a different point of view should be seriously considered. To support this new view, we can mention a recent reference [51], in which the authors claim that it is possible for a generic open system to be described both by memory kernel and local in time equations.

Non-Markovianity of physical processes can be defined looking at observable physical effects on the state of the system, due to the existence of reservoir memory affecting its dynamics. For instance in [60,61] memory effects are connected to the appearance of reverse quantum jumps able to restore coherence in the reduced state trajectory. In the particular case of Eq. (2.25) a Markovian dynamics, obtained for decay rates $\gamma_k(t) \geq 0$, does not present reverse jumps in the state trajectory, therefore no restoration of state coherence is observed. On the contrary a non-Markovian dynamics is characterized by some of the coefficients $\gamma_k(t) < 0$ for some k and some $t > 0$. This phenomenon can be interpreted as if part of the information on the state lost to the environment is stored (memory) and given back to the system at later time, i.e. during the negativity periods of the decay rates. The back flow of information is exactly the characteristic of a non-Markovian dynamics [16, 62]. According to this novel definition, Mazzola et al. demonstrate that memory kernel master equations do not necessarily describe reservoir memory effects [63].

In most of the examples provided in this thesis (Papers I-III,V), the non-Markovian behavior is shown to have beneficial effects with respect to a Markovian dynamics. Non-Markovianity in fact has the power of restoring coherence and correlations like entanglement or quantum discord. The pure detrimental effect on these properties due to a Markovian dynamics is, partially, washed away when the information travel backwards, i.e. from the environment to the system. For this reason we believe in a possible future role of non-Markovianity as a resource to partially prevent the detrimental effect of the open system dynamics. In this thesis we provide a practical example of this idea in the field of quantum key distribution (see paper V), where we prove how to use the properties of non-Markovian channels to increase the security of coherent state protocols.

Chapter 3

Continuous variable systems

In this chapter we introduce the central system of the thesis: an ensemble of harmonic oscillators. This is the prototype of what it is usually called a *continuous variable* (CV) system [64, 65]. Here we provide basic definitions and concepts about CV systems, together with the notation used throughout the manuscript.

3.1 Definitions and notation

The Hamiltonian for a system of N non interacting quantum harmonic oscillators, or bosonic modes, can be written as follows

$$\mathbf{H}_0 = \sum_{k=1}^N \frac{\mathbf{p}_k^2}{2m_k} + \frac{1}{2}m_k\omega_k^2\mathbf{q}_k^2, \quad (3.1)$$

where ω_k and m_k are respectively the free frequencies and the masses of the oscillators, while \mathbf{q}_k and \mathbf{p}_k denote the position and momentum operators, satisfying the canonical commutation relations $[\mathbf{q}_j, \mathbf{p}_k] = i\hbar\delta_{jk}$. The total Hilbert space of the ensemble is given by a tensor product of the Hilbert spaces for each mode, i.e. $\mathcal{H} = \bigotimes_{k=1}^N \mathcal{H}_k$ and it is spanned by vectors given by tensor products of single mode Hamiltonian eigenstates, or Fock states

$$|m_1, m_2, \dots, m_N\rangle = |m_1\rangle \otimes |m_2\rangle \dots \otimes |m_N\rangle, \quad (3.2)$$

with $m_k \in \mathbb{N}$, $\forall k = 1, \dots, N$, being $\{|m_k\rangle, m_k \in \mathbb{N}\}$ a basis for the space \mathcal{H}_k . The vector $|0\rangle = |0\rangle_1 \otimes |0\rangle_2 \dots \otimes |0\rangle_N$ is the ground state of the system, also called *vacuum state*.

For the sake of simplicity we adopt here a common convention in quantum optics and define the scaled position and momentum operators

$$\mathbf{Q}_k = \sqrt{\frac{m_k \omega_k}{\hbar}} \mathbf{q}_k, \quad \mathbf{P}_k = \frac{\mathbf{p}_k}{\sqrt{m_k \omega_k \hbar}}, \quad (3.3)$$

and the vector $\mathbf{R} = (\mathbf{Q}_1, \mathbf{P}_1, \mathbf{Q}_2, \dots, \mathbf{P}_{N-1}, \mathbf{Q}_N, \mathbf{P}_N)^T$, i.e. an ordered string of all the position and momentum quadrature operators of the system. These operators are particular examples of *quadrature operators* [13, 66], defined in general as

$$\mathbf{x}_\phi = \mathbf{Q}_k \cos \phi + \mathbf{P}_k \sin \phi, \quad (3.4)$$

where $0 \leq \phi \leq 2\pi$. The components of the quadrature vector satisfy the following commutation relation

$$[\mathbf{R}_j, \mathbf{R}_k] = i\Omega_{jk}, \quad \Omega = \bigoplus_{k=1}^n \omega, \quad \omega = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \quad (3.5)$$

In some occasions, throughout the text, we also make use of the creation and annihilation operators, defined as

$$a_k = \frac{\mathbf{Q}_k + i\mathbf{P}_k}{\sqrt{2}}, \quad a_k^\dagger = \frac{\mathbf{Q}_k - i\mathbf{P}_k}{\sqrt{2}}. \quad (3.6)$$

The quadrature operators in terms of a_k and a_k^\dagger take the form

$$\mathbf{x}_\phi = \frac{a_k e^{-i\phi} + a_k^\dagger e^{i\phi}}{\sqrt{2}}, \quad (3.7)$$

while the free Hamiltonian (3.1) becomes

$$\mathbf{H}_0 = \sum_{k=1}^N \hbar \omega_k \left(a_k^\dagger a_k + \frac{1}{2} \right). \quad (3.8)$$

3.2 Covariance matrix and symplectic transformations

Given a quantum state ρ of the system of N modes, we define two fundamental quantities: the first one is a vector collecting the mean values of the quadrature

vector components $\bar{\mathbf{X}}$. The second is a $2N \times 2N$ matrix σ , called *covariance matrix* [66]. Their form is the following

$$\begin{aligned}\bar{\mathbf{X}}_j &= \langle \mathbf{R}_j \rangle, \\ \sigma_{jk} &= \frac{1}{2} \langle \{ \mathbf{R}_j, \mathbf{R}_k \} \rangle - \langle \mathbf{R}_j \rangle \langle \mathbf{R}_k \rangle,\end{aligned}\tag{3.9}$$

where $\langle \mathbf{A} \rangle$ indicates the mean value of the operator \mathbf{A} over the chosen state ρ . These quantities are called first moments and second moments of the state ρ , analogously to the correspondent definitions for probability distributions in statistical sciences [67]. Higher order functions can be also defined, however, in the following chapters we concentrate on the dynamics of a specific class of states, the Gaussian states, which are completely characterized by quadrature first and second moments only.

In quantum mechanics a transformation of dynamical variables is canonical if it preserves the commutation relations between them. For a system of N modes a transformation $\mathbf{R}'_k = f_k(\mathbf{R}_1 \dots \mathbf{R}_{2N})$ is canonical if and only if

$$[\mathbf{R}'_j, \mathbf{R}'_k] = i\Omega_{jk}.\tag{3.10}$$

Eq. (3.10) states that any unitary operation, e.g. the unitary evolution of any closed system, defines a canonical transformation. We now consider only linear transformations, which can be characterized by a $2N \times 2N$ matrix \mathbf{F} and a displacement vector \mathbf{d}_R such that

$$\mathbf{R}' = \mathbf{F}\mathbf{R} + \mathbf{d}_R.\tag{3.11}$$

The vector \mathbf{d}_R describes a phase space translation of the operators, while \mathbf{F} induces a rotation of the variables in phase space. It can be proven that the commutation relations (3.10) are preserved if and only if the canonical matrix satisfies $\mathbf{F}\Omega\mathbf{F}^T = \Omega$. In this case the canonical transformation is called *symplectic* and the associated matrix \mathbf{F} is called *symplectic matrix* [66].

Under a given symplectic transformation, the covariance matrix and mean quadrature vector transform as

$$\sigma \rightarrow \mathbf{F}\sigma\mathbf{F}^T, \quad \bar{\mathbf{X}} \rightarrow \mathbf{F}\bar{\mathbf{X}} + \mathbf{d}_R.\tag{3.12}$$

We note that phase space translations do not influence the form of the covariance matrix.

3.3 Bilinear operations

Since now we have considered only a system of non-interacting harmonic oscillators, for which the Hamiltonian is simply given by Eq. (3.1). Here instead we focus on more general Hamiltonian models containing also interaction terms. In particular we are interested in studying interaction models at most quadratic in the annihilation and creation operators, since they can be easily implemented in realistic experimental situations [68–70].

The most general form of a bilinear interaction Hamiltonian is the following

$$\begin{aligned} \mathbf{H}_I = & \sum_{k=1}^N g_k^{(1)} a_k^\dagger + \sum_{k=1}^N \delta\omega_k a_k^\dagger a_k + \sum_{k \neq l=1}^N g_{kl}^{(2)} a_k^\dagger a_l \\ & + \sum_{k=1}^N g_k^{(3)} (a_k^\dagger)^2 + \sum_{k \neq l=1}^N g_k^{(4)} a_k^\dagger a_l^\dagger + h.c. \end{aligned} \quad (3.13)$$

We refer to the first term of (3.13), as linear term, or *displacement* [71]. The evolution operator, denoted by $\mathbf{D}(\lambda)$, is given by a product of N independent displacement operators $\mathbf{D}_k(\lambda_k)$

$$\mathbf{D}(\lambda) = \prod_{k=1}^N \mathbf{D}_k(\lambda_k) = \prod_{k=1}^N e^{\lambda_k a_k^\dagger - \lambda_k^* a_k}, \quad (3.14)$$

where $\lambda_k = ig_k^{(1)} t/\hbar$, while the symbol $\lambda = (\lambda_1, \lambda_1^*, \dots, \lambda_N, \lambda_N^*)^T$ collects all the parameters involved in the displacement operation. The evolution of the quadrature operators is the following

$$\mathbf{D}_k^\dagger(\lambda_k) \mathbf{R}_k \mathbf{D}_k(\lambda_k) = \mathbf{R}_k + \Lambda_k, \quad (3.15)$$

where $\Lambda \equiv (b_1, c_1, \dots, b_N, c_N)^T$, with $\lambda_k = (b_k + ic_k)/\sqrt{2}$, b_k and c_k being real parameters. The canonical transformation associated to the displacement operation is thus symplectic and characterized by $\mathbf{F} = \mathbf{I}$ and $\mathbf{d}_R = \Lambda$.

The second term of (3.13) is of the same form of the free Hamiltonian (3.8), therefore it corresponds to a shift $\delta\omega_k$ of the free frequency of each oscillator independently. It is usually called, improperly, *renormalization term*, and may appear when considering certain models of open system dynamics (see Chapter 4). Moreover it can be implemented in experiments with light modes using an optical element known as *phase shifter* [68].

The third term contains the product of a creation and an annihilation operator for different modes and it is responsible for the exchange of excitations

among the modes. For this reason it is usually referred to as *mixing term*. In the case of light beams it can be implemented using a passive optical element, e.g. a *beam splitter* [68]. When the j -th and k -th modes are involved, the unitary operator associated to the mixing term reads

$$\mathbf{U}_{jk}(\zeta_{jk}) = \exp\{\zeta_{jk}a_j^\dagger a_k - \zeta_{jk}^* a_k^\dagger a_j\}, \quad (3.16)$$

with $\zeta_{jk} = ig_{jk}^{(2)}t/\hbar$.

Finally the fourth and fifth terms of (3.13) are called *single mode squeezing* and *two-mode squeezing* terms, respectively, and their implementation requires non-linear parametric processes [72]. The unitary operators associated are

$$\mathbf{S}_k^{(1)}(\xi_k) = \exp\left\{\frac{\xi_k(a_k^\dagger)^2 - \xi_k^* a_k^2}{2}\right\}, \quad (3.17)$$

and

$$\mathbf{S}_{jk}^{(2)}(\xi_{jk}) = \exp\{\xi_{jk}a_j^\dagger a_k^\dagger - \xi_{jk}^* a_j a_k\}, \quad (3.18)$$

where $\xi_k = 2ig_k^{(3)}t/\hbar$ and $\xi_{jk} = ig_{jk}^{(4)}t/\hbar$ are the complex squeezing amplitudes.

Except for the linear term, all the other terms in the Hamiltonian (3.13) produce a non-trivial ($\mathbf{F} \neq \mathbb{I}$) symplectic evolution. Moreover a generic symplectic evolution can always be obtained by a suitable combination of some of the interaction terms contained in (3.13). We can then conclude that the Hamiltonian of Eq. (3.13) is the generator of symplectic evolutions for N modes [66]. From the expression of the symplectic matrices it is then possible to calculate the evolution of the vector $\bar{\mathbf{X}}$ and the covariance matrix $\boldsymbol{\sigma}$ using Eq. (3.12).

To summarize, the dynamics of mean quadratures values and covariance matrices for a system of oscillators under bilinear Hamiltonians can be evaluated using finite dimensional matrix analysis. In the next section we will introduce a class of states, the Gaussian states, characterized only by quadrature first and the second moments only. Therefore their evolution under bilinear interactions can be easily evaluated using the results presented in this section.

3.4 Gaussian states

The state of a quantum system is uniquely determined given the expression of the density operator ρ . However for continuous variable systems it is possible to provide an alternative but equivalent description of quantum states in terms

of quasi-probability distributions in phase space. For an exhaustive discussion on phase space distributions we refer to Refs. [73,74]. Here we introduce only the symmetrically ordered characteristic function $\chi(\lambda)$, defined as

$$\chi(\lambda) = \text{Tr}[\mathbf{D}(\lambda)\rho]. \quad (3.19)$$

The formulation based on phase space distributions is equivalent to that of density operators for two reasons: there is a one to one correspondence between characteristic functions and density operators, and, moreover, any matrix element of any operator of the system can be evaluated from the expression of the characteristic function.

Furthermore, dynamical equations for the density operators, e.g. the von Neumann equation or master equations, can be converted into partial differential equations for the phase space distributions [73].

This equivalent approach is particularly useful when we are interested in dynamical problems involving a specific class of continuous variable states: the Gaussian states. Among the different definitions provided in the literature, Gaussian states can be introduced by defining the form of their associated characteristic function. A quantum state ρ of a continuous variable system is Gaussian if its characteristic function is Gaussian shaped. The most general form of the characteristic function of a Gaussian state is

$$\chi(\Lambda) = \exp\left\{-\frac{1}{2}\Lambda^T \boldsymbol{\sigma} \Lambda + i\Lambda^T \overline{\mathbf{X}}\right\}. \quad (3.20)$$

We notice that the characteristic function of a Gaussian state depends only on the expression of the mean value of the quadratures $\overline{\mathbf{X}}$ and the covariance matrix $\boldsymbol{\sigma}$. This means that any Gaussian state can be obtained from another Gaussian state through only symplectic evolutions.

Though a general theory for N mode Gaussian states could be formulated, in this thesis we concentrate only on the cases $N = 1$ and $N = 2$.

3.4.1 Single mode Gaussian states

For a single mode system, the Hamiltonian (3.13) simplifies considerably, containing only the linear term and the single mode squeezing. From now on we neglect the renormalization term, assuming that it is included into the free Hamiltonian.

Any single mode Gaussian state can be generated applying squeezing and displacement operations to a thermal state [66,75]

$$\rho_G(n, \alpha, \xi) = \mathbf{D}(\alpha)\mathbf{S}_1(\xi)\nu(n)\mathbf{S}_1^\dagger(\xi)\mathbf{D}^\dagger(\alpha), \quad (3.21)$$

where $\nu(n)$ is a thermal state of temperature parameter n , $\alpha = |\alpha|e^{i\phi}$ is the complex displacement amplitude and $\xi = re^{i\psi}$ is the squeezing complex parameter. For a fixed value of the temperature parameter n , Eq. (3.21) describes a subclass of Gaussian states all characterized by the same symplectic spectrum [66], with symplectic eigenvalue $d_1 = n + 1/2$. A simple calculation leads to the expression of the covariance matrix, which, as expected, depends only on the temperature and squeezing parameters

$$\boldsymbol{\sigma} = \frac{2n+1}{2} \begin{pmatrix} \cosh(2r) + \sinh(2r) \cos \psi & -\sinh(2r) \sin \psi \\ -\sinh(2r) \sin \psi & \cosh(2r) - \sinh(2r) \cos \psi \end{pmatrix}. \quad (3.22)$$

3.4.2 Two-mode Gaussian states

Two-mode Gaussian states are generated by applying a general two-mode bilinear evolution \mathbf{U}_S to a product of thermal states

$$\rho = \mathbf{U}_S \nu_1(n_1) \otimes \nu_2(n_2) \mathbf{U}_S^\dagger, \quad (3.23)$$

with, the symplectic eigenvalues being $d_+ = n_1 + 1/2$ and $d_- = n_2 + 1/2$, assuming $n_1 \geq n_2$. The general form of the covariance matrix in terms of the squeezing and mixing parameters ξ_1, ξ_2, ξ_{12} and ζ_{12} is quite complicated, as it depends on the order in which each operation is performed. In many situations, it suffices to know the expression of the covariance matrix in its *normal form*

$$\boldsymbol{\sigma} = \begin{pmatrix} \mathbf{A} & \mathbf{C} \\ \mathbf{C}^T & \mathbf{B} \end{pmatrix} = \begin{pmatrix} a & 0 & c_1 & 0 \\ 0 & a & 0 & c_2 \\ c_1 & 0 & b & 0 \\ 0 & c_2 & 0 & b \end{pmatrix}, \quad (3.24)$$

obtainable starting from any bimodal gaussian state and applying suitable local symplectic operations. The following quantities, remaining unchanged under local symplectic operations, are known as *symplectic invariants*

$$I_1 = \text{Det}[\mathbf{A}], \quad I_2 = \text{Det}[\mathbf{B}], \quad I_3 = \text{Det}[\mathbf{C}], \quad I_4 = \text{Det}[\boldsymbol{\sigma}]. \quad (3.25)$$

The quantity I_4 is the only one invariant also under global unitaries.

We want to stress the fact that, Gaussian states characterized by a pair of covariance matrices $\boldsymbol{\sigma}_1$ and $\boldsymbol{\sigma}_2$ connected only by local symplectic operations are different states. However some properties of the total state of the two-mode system are unaffected by local operations, e.g. quantum correlations. This is

one of the reasons why the normal-form of the two-mode covariance matrix is useful.

To conclude we provide also the expression of the symplectic eigenvalues of the covariance matrix in terms of the symplectic invariants

$$d_{\pm} = \sqrt{\frac{I_1 + I_2 + 2I_3 \pm \sqrt{(I_1 + I_2 + 2I_3)^2 - 4I_4}}{2}} \quad (3.26)$$

3.5 Detection of mode quadratures: balanced homodyne detection

In the previous sections we discussed the properties of continuous variable systems with particular attention to the class of bilinear operations and Gaussian states. Gaussian states constitute a privileged class within CV systems, for their simple mathematical description and physical interpretation in terms of generating Hamiltonian. In quantum information and quantum optics many theoretical and experimental achievements rely on the use of this class of states. The successful implementation of quantum communication and information processing protocols in CV systems, e.g. quantum teleportation [76, 77] and quantum key distribution [27, 78], has been possible thanks to the development of measurements techniques such as photodetection [79] or homodyne detection [80, 81]. Hence we wish to conclude this chapter with a brief review of the basic notions about balanced homodyne detection, a method we refer to in paper V when exploring coherent states quantum key distribution.

Any measurement process in quantum theory can be thought as a controlled induced interaction between the system of interest and a measurement apparatus, consisting of a certain number of probe modes. The measured quantity is described by an observable \mathbf{Z} defined on the total Hilbert space $\mathcal{H} \otimes \mathcal{K}$ of the system and the probe modes. If we denote by \mathcal{Z} the spectrum of the operator \mathbf{Z} , by $z \in \mathcal{Z}$ the eigenvalues and by $E(z) = |z\rangle\langle z|$ the corresponding orthogonal projectors, the probability distribution of obtaining the value z in the measurement process is given by

$$p(z) = \text{Tr}[\rho \otimes \rho_M E(z)], \quad (3.27)$$

where ρ is the state of the system and ρ_M is a default state of the apparatus. If we want to define the measurement process only in the space of the system we need to trace out the degrees of freedom of the apparatus, getting

$$p(z) = \text{Tr}_{\mathcal{H}}[\rho \Pi(z)], \quad (3.28)$$

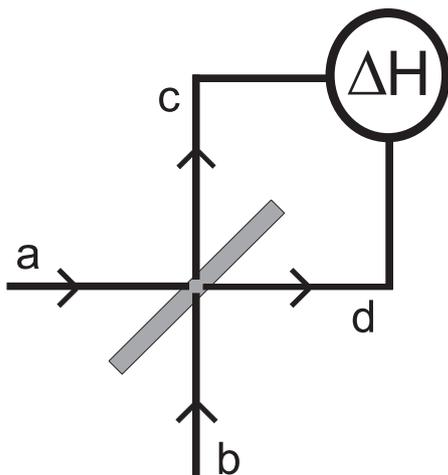


Figure 3.1: Basic balanced homodyne detection scheme.

where the $\Pi(z) = \text{Tr}_{\mathcal{K}}[\rho_M E(z)]$ are positive selfadjoint operators. The set of $\Pi(z)$ defines a POVM (positive operator valued measure) which completely characterizes the measurement process [6].

Balanced homodyne detection is a measurement scheme exploited to measure one of the quadrature operators \mathbf{x}_ϕ of a single field mode. The scheme is depicted in Fig. 3.1 and shows how the system mode a and a probe mode b are mixed in a fifty-fifty beam splitter. The outer modes c and d then are subjected to a photodetection measurement and their current difference $c^\dagger c - d^\dagger d$ is evaluated. The probe mode is usually prepared in a coherent state characterized by a given mean value and fluctuations around it. However if the probe is prepared in a high amplitude coherent state $|\alpha|e^{i\phi}$ with $|\alpha|^2 \gg 1$ (high amplitude limit), the fluctuations can be neglected. The measured operator is then defined only on the space of the system and, in the case of ideal photodetection, reads

$$\Delta H = \frac{c^\dagger c - d^\dagger d}{\sqrt{2}|\alpha|} = \frac{a^\dagger b + ab^\dagger}{\sqrt{2}|\alpha|} \simeq \frac{a^\dagger e^{i\phi} + a e^{-i\phi}}{\sqrt{2}} = \mathbf{x}_\phi. \quad (3.29)$$

The angle ϕ measures the phase difference between the system and probe modes and can be tuned to modify the choice of quadrature to measure. In the high amplitude limit and for ideal photodetection processes the POVM associated to the measurement consists of a set of projectors $\Pi^{(\phi)}(x) = |x\rangle_\phi \langle x|$. Slightly

different detection scheme have been also used to measure the quadratures of a single beam and two-mode systems [82].

Chapter 4

Quantum Brownian motion

In this chapter we introduce a paradigmatic class of continuous variable open quantum systems: the quantum Brownian motion (QBM) models [20,21]. This class contains systems consisting of a finite number N of free, or interacting, quantum particles subjected to an external potential $V(x)$ and coupled to M independent bosonic environments. In this thesis we concentrate the attention to the case of at most $N = 2$ non-interacting particles in harmonic potential, or modes, coupled to one common environment or to two independent environments.

The class of QBM open system models is capable of describing the open system dynamics in many different physical situations. For instance we may mention the dissipative dynamics of light modes in optical cavities [83], or the propagation of light modes in optical fibers [66] or even damped oscillations in mesoscopic optomechanical systems [84–86]. In the following sections we review the basic concepts of the QBM models and their open system dynamics.

4.1 Classical Brownian motion

The classical Brownian motion is a phenomenon appearing in many situations in nature. The first detailed observation comes from the English botanist R. Brown, who studied the mechanical behavior of pollen grains suspended in water solutions [87]. The stochastic behavior observed in the grains motion can be explained microscopically as a result of the many collisions suffered by the relatively big pollen particle with the water molecules.

The first exhaustive physical explanation of the Brownian motion comes from A. Einstein [88], and independently from M. Smoluchovsly [89], who

provided also a mathematical model in terms of a diffusion equation for the density $\rho(\mathbf{r}, t)$ of the Brownian particle in \mathbf{r} at time t . In the one dimensional case the equation reads

$$\frac{\partial \rho(x, t)}{\partial t} = D \frac{\partial^2 \rho(x, t)}{\partial x^2}, \quad (4.1)$$

where D is called diffusion coefficient and its value depends on the solution density and temperature. Assuming that the particle density is initially peaked in x_0 and the initial speed is zero, the solution of this equation can be expressed as

$$\rho(x, t) = \frac{1}{\sqrt{4\pi Dt}} e^{-(x-x_0)^2/4Dt}, \quad (4.2)$$

Einstein's solution reveals the stochastic nature of the Brownian motion, showing that the average position of the particle does not change with time, while, its mean quadratic value is different from zero and increasing. This means that in each realization of the stochastic process the particle actually travels across the solution.

The classical Brownian motion is considered a good model when applied to the description of the motion of macroscopic objects subjected to random forces and friction, like the pollen grains case. However as we consider smaller particles, or when we want to apply the same ideas to purely quantum systems, e.g. few photon states in optical fibers or cavities, we need to extend the theory within quantum mechanics. In the next section we briefly introduce the most simple and convenient way of describing Brownian like phenomena in the quantum regime.

4.2 The quantum Brownian motion model

The open system model discussed in this section consists of one mode interacting with one environment made of an ensemble of independent harmonic oscillators, i.e. a bosonic reservoir or bath. Situations like this can be found, for instance, when considering an optical monomodal cavity dynamics [83]. Because the cavity walls are not perfect mirrors, the internal mode is unavoidably coupled with the external degrees of freedom. This coupling phenomenon can be modeled as if the cavity mode is actually exchanging energy with other modes constituting the environment. Same idea applies for instance in the dynamics of a light mode propagating into a monomodal optical fiber, or in photonic band-gap materials [56, 57].

The Hamiltonian describing the total system dynamics may be written as

$$\begin{aligned}
\mathbf{H} &= \mathbf{H}_S + \mathbf{H}_E + \mathbf{H}_I + \mathbf{H}_R, \\
\mathbf{H}_S &= \frac{\mathbf{p}^2}{2M} + \frac{1}{2}M\omega_0^2\mathbf{q}^2, \\
\mathbf{H}_E &= \sum_n \frac{p_n^2}{2m_n} + \frac{1}{2}m_n\omega_n^2q_n^2,
\end{aligned} \tag{4.3}$$

where \mathbf{q} and \mathbf{p} are the position and momentum operators of the system mode, M is the mass of the harmonic particle and ω_0 is the bare frequency. The operators q_n and p_n are the dynamical variables associated to the bath oscillators. The functional form of the interaction term \mathbf{H}_I depends on the particular physical situation we are interested in. The simplest forms of system-environment coupling are given by

$$\begin{aligned}
\mathbf{H}_I^{RWA} &= \alpha \sum_n k_n (ab_n^\dagger + a^\dagger b_n), \\
\mathbf{H}_I &= -\alpha\mathbf{q} \sum_n k_n q_n,
\end{aligned} \tag{4.4}$$

where a and a^\dagger are the annihilation and creation operator for the system oscillator, and b_n 's and b_n^\dagger 's are the same operators for the bath modes. The constants k_n define the strength of the interaction as determined by the properties of the environment and the actual physical interaction process, while α is an overall coupling constant. The first model is called *rotating wave*, or quantum optical coupling, and contains only terms conserving the number of excitations in the total system. It is especially employed to describe quantum optical systems, i.e. photon states propagating in optical fibers. The second interaction model is known under the name of *position-position* coupling. Other models can be also considered, e.g. momentum-momentum coupling [90, 91], but in the following we are interested only in the ones provided by expression (4.4).

One of the consequences of the open system dynamics is a change of the free energies of the system mode: $\omega_0 \rightarrow \omega_0 + \delta\omega_0$. To cancel this unphysical effect the term \mathbf{H}_R is added to the Hamiltonian. The form of this *renormalization* term depends on the choice of the interaction Hamiltonian, see for instance Ref. [8].

To complete the description of the total system we need to provide a full characterization of the environmental modes, i.e. their number and how they

are distributed in frequency. In practical situations the reduced dynamics is derived under the assumption of an infinite ensemble of oscillators distributed in a continuum of modes. The properties of the environment are then specified by the spectral density $J(\omega)$ describing the coupling of the system with the reservoir oscillators of frequency ω [9, 10].

The exact reduced dynamics of the model (4.3)-(4.4) in the master equation approach can be derived in the case of factorized initial condition with the environment in a thermal stationary state. The derivation can be obtained for instance using path integrals and influence functional approaches [20, 45, 59]. Other master equations can be derived within Born and/or Markov approximation in high or low temperature regimes [20]. In the next section we provide a panoramic view of the most important master equation models used in this thesis.

4.2.1 The High-T Caldeira-Leggett model

The first master equation we present is the high-temperature Markovian Caldeira-Leggett model [20]

$$\frac{d\rho_S}{dt} = -\frac{i}{\hbar} [\mathbf{H}_S, \rho_S] - \frac{i\gamma}{\hbar} [\mathbf{q}, \{\mathbf{p}, \rho_S\}] - \frac{2M\gamma k_B T}{\hbar^2} [\mathbf{q}, [\mathbf{q}, \rho_S]], \quad (4.5)$$

where T is the temperature of the environment, supposed to be in a stationary thermal state, γ is a coupling constant usually called *damping coefficient* and M is the mass of the particle.

This equation has been derived under the Born-Markov approximation and factorized initial conditions, within a position-position interaction model [8]. Moreover there is also an assumption of high-temperature, i.e. the thermal energy $k_B T$ is assumed to be bigger than any other excitation energies. For instance if the particle is also subjected to a harmonic potential, we put $k_B T \gg \hbar\omega_0$, with ω_0 being the free oscillator frequency. The spectral density used is of the form $J(\omega) \propto \gamma\omega$, known as *Ohmic* or linear frequency distribution [9].

Although Eq. (4.5) is not employed in any of the papers of this thesis, it serves as a good and simple example to describe the main features of the reduced dynamics in open quantum systems. Moreover this model shows many common properties with the classical Brownian motion discussed previously.

For instance let us look at the dynamics of a free particle, i.e. when the free Hamiltonian $\mathbf{H}_S = \mathbf{p}^2/2M$ contains only a kinetic term. The mean values

for the position and momentum operators evolve as

$$\begin{aligned}\langle \mathbf{p}(t) \rangle &= e^{-2\gamma t} \langle \mathbf{p}(0) \rangle, \\ \langle \mathbf{q}(t) \rangle &= \langle \mathbf{q}(0) \rangle + \frac{1 - e^{-2\gamma t}}{2M\gamma} \langle \mathbf{p}(0) \rangle.\end{aligned}\tag{4.6}$$

The properties of these solutions resemble the ones observed in the classical case. The momentum mean value is damped to zero, while the position mean value reaches a constant value which depends on its initial value $\langle \mathbf{q}(0) \rangle$ and on the initial momentum mean value $\langle \mathbf{p}(0) \rangle$. This feature of the dynamics is called *damping* and it depends only on the coupling constant, or damping coefficient γ , while it is independent on the temperature. The damping phenomenon is present also in the classical Brownian motion case, therefore we cannot consider it a genuine quantum effect.

To see quantum mechanical effects we must study the dynamics of higher order moments of position and momentum. For instance if we consider the solution for the position quadratic value $\langle \mathbf{q}^2(t) \rangle$ in the short time limit ($\gamma t \ll 1$), we get

$$\langle \mathbf{q}^2(t) \rangle = \langle \mathbf{q}^2(0) \rangle + \frac{\hbar^2 t^2}{4m^2 \langle \mathbf{q}^2(0) \rangle} + \frac{4\gamma k_B T}{3m} t^3,\tag{4.7}$$

where we assumed the initial condition $\langle \mathbf{q}(0) \rangle = \langle \mathbf{p}(0) \rangle = \langle \{\mathbf{q}(0), \mathbf{p}(0)\} \rangle = 0$ and we consider as initial state a Gaussian minimum uncertainty wave packet $\langle \mathbf{q}^2(0) \rangle = \hbar/4 \langle \mathbf{p}^2(0) \rangle$, to be as near as possible to a classical situation. The spreading of the packet, also present in the classical Brownian particle density, contains three terms. The first one is the value at time zero, compatible with the Heisenberg uncertainty relation. The second term is the quantum mechanical free particle spreading term, while the third term is temperature dependent and of a quantum mechanical nature. It describes an additional spreading of the wave packet due to the interaction with a thermal environment. However it is not a classical term, being proportional to the third power of t . This additional contribution is called *quantum diffusion* term and it is responsible for the loss of quantum coherence of the system state, or *decoherence*, as one can see by checking the value of the purity of the system state.

In this section, with a very basic example of master equation, we described the main essential features of the dynamics of continuous variable open systems, *damping* and *quantum diffusion*, i.e. decoherence. These features are also common in many other open system models as we will see in the next sections.

4.2.2 The Hu-Paz-Zhang master equation

The exact reduced dynamics of an harmonic particle of mass M and free frequency ω_0 interacting with a reservoir with a position-position coupling can be derived within the only assumptions of factorized initial conditions and environment initially in a thermal state at temperature T . The associated master equation, expressed in terms of the quadrature operators \mathbf{Q} and \mathbf{P} defined in Eq. (3.3), is known as the Hu-Paz-Zhang equation and has the following local in time form [59]

$$\begin{aligned} \frac{d\rho_S}{dt} = & \frac{1}{i\hbar} [\mathbf{H}_S, \rho_S] + ir(t) [\mathbf{Q}^2, \rho_S] - i\gamma(t) [\mathbf{Q}, \{\mathbf{P}, \rho_S\}] \\ & - \Delta(t) [\mathbf{Q}, [\mathbf{Q}, \rho_S]] + \Pi(t) [\mathbf{Q}, [\mathbf{P}, \rho_S]]. \end{aligned} \quad (4.8)$$

The first term describes the unitary evolution of the free harmonic oscillator while all the other terms appear due to the interaction with the thermal bath. The second term, depending on the time-dependent coefficient $r(t)$, is called *renormalization term* and its effect is to change the free oscillator frequency. The third term describes *damping*-like phenomena, it depends on the form of the damping coefficient $\gamma(t)$ and it is temperature independent. The last two terms, containing the direct $\Delta(t)$ and anomalous $\Pi(t)$ diffusion coefficients, describe diffusive dynamics and decoherence. The analytic form of the time dependent coefficients is in general quite complicated, depending essentially on the state of the environment and on its spectral distribution. Each of them can be usually written in a power series in the coupling constant α . The first non zero contributions, are at second order in α , and read [18, 92, 93]

$$\begin{aligned} \Delta(t) &= \alpha^2 \int_0^t ds \int_0^\infty d\omega J(\omega) [2N(\omega) + 1] \cos(\omega s) \cos(\omega_0 s), \\ \Pi(t) &= \alpha^2 \int_0^t ds \int_0^\infty d\omega J(\omega) [2N(\omega) + 1] \cos(\omega s) \sin(\omega_0 s), \\ \gamma(t) &= \alpha^2 \int_0^t ds \int_0^\infty d\omega J(\omega) \sin(\omega s) \sin(\omega_0 s), \\ r(t) &= \alpha^2 \int_0^t ds \int_0^\infty d\omega J(\omega) \sin(\omega s) \cos(\omega_0 s), \end{aligned} \quad (4.9)$$

where $J(\omega)$ is the spectral density of the environmental modes and $N(\omega) = (\exp\{\hbar\omega/k_B T\} - 1)^{-1}$ is the mean number of photons at frequency ω . The solution of the Eq. (4.8) depends on the form of the coefficients (4.9) and has

a simple form in terms of the characteristic function

$$\chi(\Lambda, t) = \chi(\exp\{-\Gamma(s)/2\}\mathbf{R}^{-1}(t)\Lambda, 0) \exp\{-\Lambda^T \tilde{\mathbf{W}}(t)\Lambda\}, \quad (4.10)$$

where

$$\begin{aligned} \tilde{W}(t) &= [\mathbf{R}^{-1}(t)]^T \left[\int_0^t e^{\Gamma(s)-\Gamma(t)} \mathbf{R}^T(s) M(s) \mathbf{R}(s) \right] \mathbf{R}^{-1}(t), \\ M(s) &= \begin{pmatrix} \Delta(s) & -\Pi(s)/2 \\ -\Pi(s)/2 & 0 \end{pmatrix}, \end{aligned} \quad (4.11)$$

and we defined $\Gamma(t) = 2 \int_0^t \gamma(s) ds$. The renormalization coefficient $r(t)$ appears only in the 2×2 matrix $\mathbf{R}(t)$, whose form is in general quite complicated. At the second order expansion in α it is possible to neglect the renormalization effect and the matrix takes the free oscillator form

$$\mathbf{R}(t) = \begin{pmatrix} \cos(\omega_0 t) & \sin(\omega_0 t) \\ -\sin(\omega_0 t) & \cos(\omega_0 t) \end{pmatrix}. \quad (4.12)$$

From the solution (4.10) we may notice that Eq. (4.8) describes a Gaussian preserving dynamical map. The characteristic function at any later time t is, in fact, a product of a Gaussian term and a rotated characteristic function at the initial time, which means that if the initial state is Gaussian, also the final state is Gaussian. This expected behavior is due to the bilinear form of the Hamiltonian (4.3). A simple calculation provides the dynamical evolution of the mean quadrature value and covariance matrix as

$$\begin{aligned} \bar{\mathbf{X}}(t) &= e^{-\Gamma(t)/2} \mathbf{R}(t) \bar{\mathbf{X}}(0), \\ \boldsymbol{\sigma}(t) &= e^{-\Gamma(t)} \mathbf{R}(t) \boldsymbol{\sigma}(0) \mathbf{R}^T(t) + 2\tilde{W}(t). \end{aligned} \quad (4.13)$$

In the weak coupling limit $\bar{\mathbf{X}}(t)$ is damped towards the center of phase space and rotates with frequency ω_0 due to the free unitary dynamics. The covariance matrix instead experiences both damping dynamics, as the information on the initial value $\boldsymbol{\sigma}(0)$ is washed away, and decoherence due to the term proportional to $\tilde{W}(t)$ which in general is typical of a thermal squeezed state even at zero temperature. The asymptotic time qualitative behavior is independent of the form of the environment spectral distribution $J(\omega)$. However the form of the spectrum is important in the early stages of the dynamics, i.e. within a time scale comparable with the reservoir correlation time τ_E . The correlation time scale τ_E , which depends on the form of the spectrum, roughly corresponds to the interval of time in which the coefficients (4.9) are time-dependent. For $t \gg \tau_E$, and within the weak coupling limit, the coefficients reach a constant value, i.e. their *Markovian value*.

4.2.3 The secular approximated master equation

The Hu-Paz-Zhang equation is an exact equation of motion for the reduced dynamics of the system oscillator. However in some situations it is possible to use a simplified form. We present here a QBM master equation derived under weak coupling limit and secular approximation. In the interaction picture the equation reads

$$\begin{aligned} \frac{d\rho_S}{dt} = & \frac{\Delta(t) + \gamma(t)}{2} (2a\rho_S a^\dagger - a^\dagger a \rho_S - \rho_S a^\dagger a) \\ & + \frac{\Delta(t) - \gamma(t)}{2} (2a^\dagger \rho_S a - a a^\dagger \rho_S - \rho_S a a^\dagger), \end{aligned} \quad (4.14)$$

where the damping and diffusion coefficients $\Delta(t)$ and $\gamma(t)$ are of the same form of (4.9). The secular approximation is an assumption on the relative value of the free system dynamics timescale $\tau_S \simeq \omega_0^{-1}$ and the relaxation dynamics time scale τ_R . When $\tau_S \ll \tau_R$ it is possible to neglect some terms in the master equation, i.e. the non-secular terms, which oscillate in time very fast giving a negligible contribution to the dynamics. The validity of this approximation, and therefore of Eq. (4.14), depends on the system-reservoir parameters and in particular on the form of the spectral distribution [94].

The solution of Eq. (4.14) for an initial Gaussian state reads

$$\begin{aligned} \overline{\mathbf{X}}(t) &= e^{-\Gamma(t)/2} \overline{\mathbf{X}}(0), \\ \boldsymbol{\sigma}(t) &= e^{-\Gamma(t)} \boldsymbol{\sigma}(0) + 2\Delta_\Gamma(t) \mathbf{I}, \end{aligned} \quad (4.15)$$

where $\Delta_\Gamma(t) = e^{-\Gamma(t)} \int_0^t e^{\Gamma(s)} \Delta(s) ds$. In these equations the free dynamics has been eliminated by using the transformation into the interaction picture, hence the mean quadrature vector is simply damped towards the center of the phase space. The covariance matrix instead tends to a simple thermal state without any squeezing contribution.

4.2.4 Damped harmonic oscillator: Markovian treatment

The simplest model of a CV single mode open quantum system is given by a master equation in the Lindblad form [8, 66]. In the interaction picture the equation reads

$$\frac{d\rho_S}{dt} = \frac{\gamma}{2} (2a\rho_S a^\dagger - a^\dagger a \rho_S - \rho_S a^\dagger a). \quad (4.16)$$

This master equation can be derived microscopically starting from a rotating wave coupling \mathbf{H}_I^{RWA} under Born-Markov and secular approximation. Eq.

(4.16) is usually the starting point of any dissipative dynamics treatment among the quantum optics community as it well describes, for instance, the lossy dynamics of propagating light beams or intracavity light modes. The solution for an initial Gaussian state can be written as follows

$$\begin{aligned}\overline{\mathbf{X}}(t) &= e^{-\gamma t} \overline{\mathbf{X}}(0), \\ \boldsymbol{\sigma}(t) &= e^{-2\gamma t} \boldsymbol{\sigma}(0) + (1 - e^{-2\gamma t}) \frac{\mathbf{I}}{2}.\end{aligned}\quad (4.17)$$

Dissipation and decoherence processes are here interrelated and depending on the only free parameter of the model, i.e. the decay rate γ .

A phenomenological generalization of Eq. (4.16) can be also used in some contexts, where the decay rate is now time dependent, i.e. $\gamma(t)$,

$$\frac{d\rho_S}{dt} = \frac{\gamma(t)}{2} (2a\rho_S a^\dagger - a^\dagger a \rho_S - \rho_S a^\dagger a). \quad (4.18)$$

The solution is

$$\begin{aligned}\overline{\mathbf{X}}(t) &= e^{-\Gamma(t)/2} \overline{\mathbf{X}}(0), \\ \boldsymbol{\sigma}(t) &= e^{-\Gamma(t)} \boldsymbol{\sigma}(0) + (1 - e^{-\Gamma(t)}) \frac{\mathbf{I}}{2},\end{aligned}\quad (4.19)$$

with $\Gamma(t) = 2 \int_0^t \gamma(s) ds$. The generalization (4.18) is explored in papers *IV* and *V* in two different physical contexts. Whether this kind of open dynamics can be actually implemented for any form of the rate $\gamma(t)$ is still an open question. If we assume $\gamma(t) > 0$ for any $t \geq 0$ we may imagine that the equation could describe the propagation of light modes in optical fibers where the fiber coating changes through the line.

Moreover Eq. (4.18) can be also considered a special case of Eq. (4.14) when $|\Delta(t) - \gamma(t)| \ll |\Delta(t) + \gamma(t)|$. Because both coefficients depend on the same spectral distribution $J(\omega)$ we could argue that, with appropriate reservoir engineering techniques, in principle, we may be able to satisfy the previous requirement.

Chapter 5

Correlations in quantum systems

In this section we review some basic properties of bipartite systems, focusing on the correlations in bipartite quantum states. We first define the concept of correlated quantum states and introduce some of the most common measures for correlations. In the last sections we focus on the dynamical evolution of correlations in the presence of an environment.

5.1 Correlated states in quantum mechanics

A quantum mechanical state ρ_{AB} of a bipartite system A and B is said to be correlated if it cannot be written as a product state $\rho_{AB} = \rho_A \otimes \rho_B$. This definition, valid for both pure and mixed states, relies on the fact that when the state is not a product, joint measurement outcomes are, in general, statistically correlated. A paradigmatic example is provided by a Bell state for a two qubits system

$$|\Psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle_A|0\rangle_B + |1\rangle_A|1\rangle_B), \quad (5.1)$$

with $\langle 0|1\rangle = 0$. If we perform a joint projective measurement onto the canonical basis $\{|0, 0\rangle, |1, 0\rangle, |0, 1\rangle, |1, 1\rangle\}$, we find that both subsystems are in the state $|0\rangle$ or $|1\rangle$.

Measurement outcomes, however, may be correlated also in classical systems. For instance imagine to put one red ball and one blue ball into a bag. If we pick up the red ball first, then we know for certain that the second ball is

blue and viceversa. The nature of correlations expressed by Eq. (5.1) is however radically different from the classical balls example. Quantum mechanics shows different types and more general correlations than classical mechanics.

Recently [95] there have been attempts to classify correlations in terms of their quality, and consequently, in terms of properties of the bipartite states involved. Depending on the nature of correlations, we can define different classes, or sets, of bipartite states:

- *Product states:* $\rho_{AB} = \rho_A \otimes \rho_B,$
- *Classical states:* $\rho_{AB} = \sum_{kl} p_{kl} |\psi_k^{(A)}\rangle\langle\psi_k^{(A)}| \otimes |\psi_l^{(B)}\rangle\langle\psi_l^{(B)}|,$
- *Separable states:* $\rho_{AB} = \sum_k p_k \rho_k^{(A)} \otimes \rho_k^{(B)},$
- *Entangled states:* $\rho_{AB} \neq \sum_k p_k \rho_k^{(A)} \otimes \rho_k^{(B)},$

with $\{|\psi_k^{(A),(B)}\rangle\}$ being an orthonormal basis in $\mathcal{H}_{A,B}$. These sets are not disjoint, e.g. classically correlated states are also separable and product states are trivially classically correlated and separable.

In the next section we will review the properties of these classes of states, discussing their nature and their physical meaning.

5.2 Separability and entanglement

Following an historical path, the interest of the physics community about bipartite correlations may be traced back to the concept of *separability* and *entanglement*. Separability was first introduced in a seminal paper by R. Werner [96] where its physical meaning was given in terms of preparation procedures of bipartite quantum states. According to his definition, a state of a bipartite system is separable if can be prepared using only local operations and classical communication (LOCC) on each subsystem. In practice to prepare these states we need two local machines capable of preparing the states $\rho_k^{(A)}$ and $\rho_k^{(B)}$ in each subsystem location, and a classical random number generator which selects numbers $1\dots r$ with probabilities p_k 's. When the number k is drawn, this information is sent to each machine through completely classical channels. The machines then use this information to prepare the states $\rho_k^{(A)}$ and $\rho_k^{(B)}$. The composite state can be described by a statistical mixture of the form

$$\rho_{AB} = \sum_k p_k \rho_k^{(A)} \otimes \rho_k^{(B)}. \quad (5.2)$$

If a state is not separable then it is called entangled. Product states are simply separable states where the probability of obtaining a given result by the classical random machine is one. Classically correlated states are obtained by local machines which prepare locally distinguishable states (set of orthogonal states). The preparation of entangled states requires instead a more general procedure, which usually consists in the implementation of a direct, i.e. non-local, interaction between the subsystems. The problem of finding, for a generic system, sufficient and necessary conditions for separability is considered a fundamental research topic. Until now a complete answer to this question exists for the class of pure states only. In this case the Schmidt decomposition theorem [6] tells that pure separable states are given by products of pure states.

For mixed states it exists only a necessary condition for separability valid for any bipartite system. This condition, provided in 1996 by A. Peres [97], is called PPT (positivity under partial transposition) criterion. Peres introduces first the definition of *transposition map* \mathbf{T} which, applied to a generic state ρ acts as $\mathbf{T}[\rho] = \rho^{\mathbf{T}}$. The transposition map is positive but it is not completely positive. Therefore if we extend the map to a bigger system $\mathbf{T}_A \otimes \mathbf{I}_B$, the extended map, known as *partial transposition* in respect to the first subsystem, may lead to non positive states ρ .

If partial transposition is applied to a separable state we get

$$(\mathbf{T}_A \otimes \mathbf{I}_B) [\rho_{sep}] \equiv \rho^{\mathbf{T}_A} = \sum_k p_k (\rho_k^{(A)})^T \otimes \rho_k^{(B)}, \quad (5.3)$$

which is still a positive operator. We can conclude that positivity under partial transposition is a necessary condition for separability. This criteria turns out to be also sufficient for some specific systems, e.g. two qubit states [98] or bimodal continuous variable Gaussian states [99].

5.3 Entanglement measures

Entangled states are considered important tools in quantum information and computation theory. A striking example is provided by the original quantum teleportation protocol [100]. In this protocol Alice possesses an unknown one-qubit quantum state and wants to teleport it to the receiver Bob. If the two parties pre-share a two-qubit Bell state, like (5.1), it is then possible, by means of measurements performed only by Alice, to prepare the initial unknown quantum state in Bob's location, and achieve teleportation of the quantum state. This operation is possible only because the two parties share

entanglement contained in the Bell state. If the shared state is not entangled, teleportation cannot be achieved. This and other examples, e.g quantum dense coding [101], quantum key distribution [102] or search algorithms [103], show the role of entanglement as a resource in quantum information theory. The teleportation protocol works also when the shared state is not a Bell state, but it is still not separable. In this case the protocol efficiency is diminished. The reason stands from the fact that among two-qubit entangled states, Bell states seem to possess more entanglement than others.

To quantify the amount of entanglement in a generic bipartite system we need to define a measure valid for any system and any density operator. In the case of a pure state $|\psi\rangle$, entanglement $\mathbf{E}(|\psi\rangle\langle\psi|)$ can be measured, for instance, by the von Neumann entropy $\mathbf{S}(\rho) = -Tr[\rho \ln \rho]$ of the reduced state of one of the subsystems [6]

$$\mathbf{E}_V(|\psi\rangle\langle\psi|) = \mathbf{S}(\rho_A) = \mathbf{S}(\rho_B), \quad \rho_{A(B)} = Tr_{B(A)}[|\psi\rangle\langle\psi|]. \quad (5.4)$$

For mixed states, the lack of a sufficient and necessary condition for separability complicates also the problem of finding suitable measures of entanglement.

During the last decades many conceptually and operatively different entanglement measures have been presented in the literature [22]. These quantities possess a conceptual meaning independent from the specification of the system of interest, and hence they can, in principle, be applied to any bipartite system state. A quantity $\mathbf{E}(\rho_{AB})$ needs to satisfy certain requirements in order to be eligible as a good measure of bipartite entanglement. The two main requirements are

$$\begin{aligned} \mathbf{E}(\rho_{AB}) = 0 & \quad \Leftrightarrow \quad \rho_{AB} = \sum_k p_k \rho_k^{(A)} \otimes \rho_k^{(B)} \\ \mathbf{E}(\Phi_L \rho_{AB}) & \leq \mathbf{E}(\rho_{AB}). \end{aligned} \quad (5.5)$$

The first line of (5.5) states that the measure is zero only for separable states. The second requirement states that the measure is monotonically decreasing when local actions and classical communication only are performed, where Φ_L is a LOCC map [104]. Moreover $\mathbf{E}(\Phi_L \rho_{AB}) = \mathbf{E}(\rho_{AB})$ only when Φ_L is a local unitary operation. Other requirements, e.g. continuity or convexity, may be demanded. More detailed results can be found, e.g., in [22].

For most of the bipartite systems of interest, the evaluation of entanglement measures is a very difficult task, and usually numerical algorithms are required to calculate entanglement. Analytic results can be found in very few systems, for instance in two-qubit states or in continuous variable Gaussian states [98].

5.3.1 Logarithmic negativity and entanglement of formation

In this section we introduce two conceptually distinct measures for entanglement of bipartite states. In both cases there exist analytical expressions for bimodal continuous variable Gaussian states.

The first measure is known as *logarithmic negativity* and it is defined as follows [105]

$$\mathbf{E}_N(\rho_{AB}) = \ln(2\mathcal{N} + 1), \quad (5.6)$$

where the quantity $\mathcal{N}(\rho_{AB})$, called *negativity*, is given by the absolute value of the sum of the negative eigenvalues of the partially transposed density operator $\rho_{AB}^{T_A}$ with respect to one of the subsystems [106]. If the partial transpose operator is a density operator, then both negativity and logarithmic negativity are zero. This means that, when the PPT criterion is also sufficient for separability, then the state is separable. Otherwise the state may or may not be separable and logarithmic negativity ceases to behave as a good entanglement measure.

For bimodal continuous variable Gaussian states, the PPT criterion is also sufficient for separability [99] and the logarithmic negativity assumes the form [105]

$$\mathbf{E}_N^{Gau}(\rho_{AB}) = \max\{0, -\log(2\tilde{d}_-)\}, \quad (5.7)$$

where \tilde{d}_- is the smallest symplectic eigenvalue of the partially transposed covariance matrix $\tilde{\sigma}$ associated to the state. The partially transposed covariance matrix can be obtained from the actual covariance matrix σ as follows

$$\tilde{\sigma} = \mathbf{Y}\sigma\mathbf{Y}, \quad \mathbf{Y} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \quad (5.8)$$

Then the expression of the eigenvalue \tilde{d}_- can be entirely written in terms of the local and global symplectic invariants I_i 's

$$\tilde{d}_- = \sqrt{\frac{I_1 + I_2 - 2I_3 - \sqrt{(I_1 + I_2 - 2I_3)^2 - 4I_4}}{2}}, \quad (5.9)$$

thus showing how local symplectic operations cannot change the value of the entanglement.

The second measure we consider is called *entanglement of formation* and is denoted by $\mathbf{E}_F(\rho_{AB})$ [104]. Given a good entanglement measure for pure states,

e.g. the von Neumann entropy of the reduced states (5.4), the entanglement of formation is given by

$$\mathbf{E}_F(\rho_{AB}) = \min \sum_i p_i \mathbf{E}_V(|\psi_i\rangle\langle\psi_i|), \quad (5.10)$$

where the minimum is taken over the ensemble of states and related probabilities $\{|\psi_i\rangle, p_i\}$ which realize the given state, $\rho_{AB} = \sum_i p_i |\psi_i\rangle\langle\psi_i|$. Conceptually this quantity tells how many pure maximally entangled states are needed, at most, to realize the incoherent superposition ρ_{AB} . In this sense a highly entangled state is a state given by a mixture of highly entangled pure states.

The expression of entanglement of formation for bimodal Gaussian states reads [107, 108]

$$\mathbf{E}_F^{Gau}(\rho_{AB}) = \max\{0, h(\tilde{d}_-)\}, \quad (5.11)$$

with

$$h(x) = \frac{(1+2x)^2}{8x} \log\left[\frac{(1+2x)^2}{8x}\right] - \frac{(1-2x)^2}{8x} \log\left[\frac{(1-2x)^2}{8x}\right]. \quad (5.12)$$

5.4 Quantum discord

Entangled states can be used in quantum information and computation theory to perform tasks beyond the classical efficiency level. The reasons seems to lie on the qualitatively different kind of correlations exhibited by quantum systems, which allow for more efficient computational algorithms and information transfer compared to classical systems. We may ask now if entanglement is the only type of correlations of quantum nature, or stated in another way, if among the class of separable states there are states exhibiting correlations not present in a classical framework. A partial positive answer to this question has been suggested through the concept of *quantum discord* [23, 24], which quantifies the total amount of correlations of quantum nature in a given quantum state. Within bipartite quantum systems, quantum discord is defined as the algebraical difference between the total amount of correlations in a state, quantified by the quantum mutual information [6]

$$\mathcal{I}(\rho_{AB}) = \mathbf{S}(\rho_A) + \mathbf{S}(\rho_B) - \mathbf{S}(\rho_{AB}), \quad (5.13)$$

and the correlations of classical nature. The fundamental and delicate point of this definition consists in choosing the appropriate measure for the classical correlations of a given bipartite quantum state. In [24] classical correlations

are measured by quantifying the maximum amount of knowledge gained on one subsystem when the other is measured. The result of this operation depends in general on which of the two subsystems is measured, thus two different measures of classical correlations can be introduced

$$\begin{aligned}\mathcal{C}^{\rightarrow}(\rho_{AB}) &= \max_{\Pi_{i,B}} \left\{ \mathbf{S}(\rho_A) - \sum_i p_i^B \mathbf{S}(\rho_{A|B}^{\Pi_{i,B}}) \right\}, \\ \mathcal{C}^{\leftarrow}(\rho_{AB}) &= \max_{\Pi_{i,A}} \left\{ \mathbf{S}(\rho_B) - \sum_i p_i^A \mathbf{S}(\rho_{B|A}^{\Pi_{i,A}}) \right\}.\end{aligned}\tag{5.14}$$

The measurement set is usually restricted to projective measurements, characterized by a set of projection operators $\{\Pi_{i,Y}, i = 1, \dots, \dim \mathcal{H}_Y\}$ when the system $Y = A, B$ is measured. The quantity $\rho_{X|Y}^{\Pi_{i,Y}} = \text{Tr}_Y[\rho_{AB} \mathbf{I} \otimes \Pi_{i,Y}]$ is the post measurement state of system X after the i -th result is obtained on system Y with probability $p_i^Y = \text{Tr}_{AB}[\rho_{AB} \mathbf{I} \otimes \Pi_{i,Y}]$.

Being $\mathcal{C}^{\rightarrow}(\rho_{AB}) \neq \mathcal{C}^{\leftarrow}(\rho_{AB})$, quantum discord can be defined in two versions, depending on which system is measured

$$\begin{aligned}\mathcal{D}^{\rightarrow}(\rho_{AB}) &= \mathcal{I}(\rho_{AB}) - \mathcal{C}^{\rightarrow}(\rho_{AB}), \\ \mathcal{D}^{\leftarrow}(\rho_{AB}) &= \mathcal{I}(\rho_{AB}) - \mathcal{C}^{\leftarrow}(\rho_{AB}).\end{aligned}\tag{5.15}$$

It is worth mentioning that there are also situations in which only one version of the discord is different from zero. In this case one usually talks of quantum/classical or classical/quantum correlations measured by the discord. The possibility that discord asymmetry comes from a limitation in the choice of the measurement schemes has been recently investigated, and the definitions (5.14) and (5.15) have been extended to allow for more general measurements schemes, i.e. POVM [109–113]. Other definitions of discord, independent on measurements protocols but rather based on distance measures, have also been recently provided [95, 114, 115].

As in the case of entanglement, quantum correlations as measured by quantum discord have attracted the attention of the scientific community due to their possible role in enhancing the efficiency of quantum information protocols, in particular in mixed state quantum computation, where the role of entanglement becomes unclear. Is quantum discord a resource useful for quantum technologies? This question is still under investigation and up to our knowledge the only example of quantum efficient computation where discord plays a key role is the deterministic quantum computation (DCQ1) model [116]. In this protocol a single qubit interacts with a register of N qubits subjected to a unitary operation \mathbf{U} . After the interaction, a measurement of the system

qubit allows to evaluate the trace of the operator \mathbf{U} . The computation runs without creation of entanglement between the system and the register, while instead quantum discord is present [117].

Due to the maximization procedure needed to obtain the classical correlations (5.14), the evaluation of quantum discord is not an easy task. Even for the simplest system made of two qubits there is no analytic formula except for some specific classes of states [109, 111, 112] for projective measurements, or for certain classes of POVM's [110]. For bimodal continuous variable systems an analytic expression for quantum discord exists only in the case of Gaussian states and when Gaussian measurements are performed [118, 119]. This expression depends only on the symplectic invariants, thus showing that also quantum discord does not change when we act locally with Gaussian unitary operations. For the particular case of symmetric states we are interested in, the general analytic expression [119] reduces to

$$\mathcal{D}(\rho_{AB}) = f(\sqrt{I_1}) - f(\nu_-) - f(\nu_+) + R(I_1, I_3, I_4), \quad (5.16)$$

where ν_{\pm} are the symplectic eigenvalues of the covariance matrix,

$$f(x) = \frac{x+1}{2} \log \left[\frac{x+1}{2} \right] - \frac{x-1}{2} \log \left[\frac{x-1}{2} \right], \quad (5.17)$$

and

$$R(I_1, I_3, I_4) = \frac{2I_3^2 + (I_1 - 1)(I_4 - I_1) + 2|I_3| \sqrt{I_3^2 + (I_1 - 1)(I_4 - I_1)}}{(I_1 - 1)^2}, \quad (5.18)$$

if $(I_4 - I_1^2)^2 \leq (I_1 + 1)I_3^2(I_1 + I_4)$ otherwise

$$R(I_1, I_3, I_4) = \frac{I_1^2 - I_3^2 + I_4 - \sqrt{I_3^4 + (I_4 - I_1^2)^2 - 2I_3^2(I_4 + I_1^2)}}{2I_1}. \quad (5.19)$$

5.5 Intensity correlations and non-classicality in CV systems

Entanglement and quantum discord are examples of correlations present in any kind of composed quantum system, consequences of the superposition principle and the partial trace rule (2.6). Correlations of a different nature can be defined for specific systems in relationship with specific measurement schemes. As a paradigmatic example we may consider a pair of independently propagating

light beams prepared in a given quantum state ρ_{AB} , and ask about the features of the joint probability distribution of photon detection in each beam.

If we are interested in correlations between beam intensities, the quantity of interest is a second order correlation function of the form [25, 26]

$$\mathcal{I} = 1 - \frac{\langle \Delta \mathcal{I}_-^2 \rangle}{\langle \mathcal{I}_+ \rangle} = 1 - \frac{\langle \mathcal{I}_-^2 \rangle - \langle \mathcal{I}_- \rangle^2}{\langle \mathcal{I}_+ \rangle}, \quad (5.20)$$

where $\mathcal{I}_\pm = \mathbf{n}_A \pm \mathbf{n}_B$, with $\mathbf{n}_A = a^\dagger a$, $\mathbf{n}_B = b^\dagger b$ are the number operators for the two light modes. The intensity correlation marker \mathcal{I} quantifies the quantumness of the total state of the two beams exhibited in the intensity measurement statistics. If the beams are in a product of coherent states we have $\mathcal{I} = 0$, which defines the quantum-classical detection threshold, or shot-noise limit [120–122]. When $0 \leq \mathcal{I} \leq 1$ the marker indicates the presence of genuine non-classical correlations, i.e. intensity correlations which cannot be observed in interference experiments with classical light. It is worth noticing that such result can be obtained also for product states, e.g. in the case of single mode squeezing, showing that the intensity correlation marker does not witness correlations like entanglement or discord. It is, on the other hand, a quantifier of classicality/quantumness present in joint intensity measurements. Finally, when $\mathcal{I} < 0$ the state ρ_{AB} does not show non-classical features.

The dynamics of intensity correlations is investigated in paper II, where we consider only bimodal Gaussian states with zero displacement, i.e. $\bar{\mathbf{X}}_A = \bar{\mathbf{X}}_B = 0$. In this case the expression for the marker \mathcal{I} can be entirely written as a function of the elements of the bimodal state covariance matrix as

$$\mathcal{I} = 1 - \frac{\sigma_{11}^2 + \sigma_{22}^2 + 2\sigma_{13}^2 - \sigma_{14}^2 - \sigma_{23}^2 - \sigma_{24}^2 - \frac{1}{2}}{\sigma_{11} + \sigma_{22} - 1}. \quad (5.21)$$

As expected Eq. (5.21) cannot be written as a function of the symplectic invariants because, unlike entanglement or discord, local evolutions may change the value of \mathcal{I} .

States possessing non-classical intensity correlations have been proven to be useful in quantum optical protocols like ghost-imaging/diffraction experiments [25, 123].

5.6 Evolution of correlations in open quantum systems

In the previous sections we provided a basic picture of the most important properties of correlated states in quantum mechanics and shortly summarized

their role in quantum information and quantum optics.

Because quantum systems are subjected to free dynamics as well as to interactions with other systems, their state and, consequently, their correlations change in time. One can identify at least two independent sources of state evolution. The first one is the unitary dynamics within the bipartite system, given by the free evolution of each subsystem, and a direct interaction among the subsystems. A second source, on which we have minor or no control at all, is the interaction with the surrounding environment, and/or with the measurement apparatus.

In the following we assume that there is no direct interaction between the subsystems, so that the free dynamics and the interaction with the environment become the only sources of dynamical evolutions of correlations.

Under this assumption we may consider two different scenarios. In the first scenario the subsystems interact locally with their own environment, therefore the total dynamical map $\Phi(t)$ can be written as a product of two independent maps $\Phi_A(t) \otimes \Phi_B(t)$ for any $t \geq 0$. If the initial state carries no correlations, i.e. $\rho_{AB}(0) = \rho_A(0) \otimes \rho_B(0)$ is a product state, then at any successive time the state remains uncorrelated

$$\rho_{AB}(t) = \rho_A(t) \otimes \rho_B(t) = \Phi_A(t)\rho_A(0) \otimes \Phi_B(t)\rho_B(0). \quad (5.22)$$

On the other hand if the initial state is correlated, classically or quantum mechanically, the amount of correlations changes with time. As for entanglement, since its measures are monotonically decreasing under LOCC operations, we must always have

$$\mathbf{E}\{\rho_{AB}(t)\} \leq \mathbf{E}\{\rho_{AB}(0)\}. \quad (5.23)$$

This phenomenon is usually referred to as *disentanglement*, since it describes the detrimental effect on correlations due to the local open system dynamics. As reported in paper II, the same phenomenon exists for quantum discord and intensity correlations in the continuous variable case. The decrease of intensity correlations has however two different origins: the first one is the loss of quantum correlations and the second one is the loss of non-classicality features, like squeezing or purity, driving a quantum to classical transition in the system [38].

A second scenario consists in both subsystems interacting with the same common environment. The open system dynamics in a common reservoir is described by a dynamical map $\Phi(t)$ which cannot be factorized as in the case of independent reservoir. As a consequence, the common environment gives rise to an effecting coupling between the two subsystems. It is then expected that

quantum correlations can either decrease or increase depending on the initial state of the system, the coupling strength, the temperature and the spectral density of the environment.

In the next sections we will briefly introduce the master equations describing the dynamics of a bimodal CV system in independent and common reservoir scenarios.

5.6.1 Independent reservoirs scenario

The Hamiltonian describing the independent reservoir scenario is given by the direct sum of terms of the form (4.3), where each oscillator interacts with its own environment. Following the lines of the derivation of the Hu-Paz-Zhang master equation (4.8) for one oscillator, the master equation for the system of two oscillators can be extended as [18]

$$\begin{aligned} \frac{d\rho_{AB}}{dt} = & \sum_{j=A,B} \frac{1}{i\hbar} [\mathbf{H}_S^{(j)}, \rho_{AB}] + ir_j(t) [\mathbf{Q}_j^2, \rho_{AB}] - i\gamma_j(t) [\mathbf{Q}_j, \{\mathbf{P}_j, \rho_{AB}\}] \\ & - \Delta_j(t) [\mathbf{Q}_j, [\mathbf{Q}_j, \rho_{AB}]] + \Pi_j(t) [\mathbf{Q}_j, [\mathbf{P}_j, \rho_{AB}]], \end{aligned} \quad (5.24)$$

where ρ_{AB} is the state of the two modes, $\mathbf{H}_S^{(j)}$ indicates the free Hamiltonian of the j -th subsystem, and where we assume that the properties and the coupling strength of the two environments with each mode may differ, leading to different values of the dissipation, renormalization and diffusion coefficients for each channel.

The master equation (5.24) is given by the sum of two parts each containing only operators and coefficients pertinent to the j -th system-environment. The absence of cross terms shows that if the modes are initially in an uncorrelated product state, then they will evolve independently. However in the presence of correlations in the initial state $\rho_{AB}(0)$, Eq. (5.24) describes a non-trivial dynamics. This physical situation is the subject of papers I-II-III, where the time evolution of quantum features like entanglement, discord or intensity correlations is investigated.

5.6.2 Common reservoir scenario

The Hamiltonian for the common reservoir scenario can be also written in the form (4.3), with a different interaction term

$$\mathbf{H}_I = -\alpha(\mathbf{Q}_A + \mathbf{Q}_B) \sum_n \kappa_n q_n, \quad (5.25)$$

In (5.25) we assumed that the two oscillators interact symmetrically, i.e. in the same way, with the common bath. The master equation can be derived using a canonical transformation on the system variables $(\mathbf{Q}_A, \mathbf{P}_A, \mathbf{Q}_B, \mathbf{P}_B) \rightarrow (\mathbf{Q}_+, \mathbf{P}_+, \mathbf{Q}_-, \mathbf{P}_-)$, leading to a new Hamiltonian in which one oscillator interacts with the bath and the other evolves freely [90, 91]. Therefore the master equation for the transformed state $\tilde{\rho}_{AB}$ reads

$$\begin{aligned} \frac{d\tilde{\rho}_{AB}}{dt} = & \sum_{j=+,-} \frac{1}{i\hbar} [\mathbf{H}_S^{(j)}, \tilde{\rho}_{AB}] + ir(t) [\mathbf{Q}_+^2, \tilde{\rho}_{AB}] - i\gamma(t) [\mathbf{Q}_+, \{\mathbf{P}_+, \tilde{\rho}_{AB}\}] \\ & - \Delta(t) [\mathbf{Q}_+, [\mathbf{Q}_+, \tilde{\rho}_{AB}]] + \Pi(t) [\mathbf{Q}_+, [\mathbf{P}_+, \tilde{\rho}_{AB}]]. \end{aligned} \quad (5.26)$$

Applying the inverse canonical transformation we get the dynamics of the original state of the oscillators and we can study the evolution of correlations. This is the main subject of paper II where we consider the dynamics of entanglement, discord and intensity correlations in independent and common environments.

Chapter 6

Degree of non-Markovianity of dynamical maps

In chapter 2 we introduced the concept of dynamical map as a one parameter family of maps $\Phi(t, t_0) \equiv \Phi(t)$ (without loss of generality we fix $t_0 = 0$ from now on), describing the evolution of the quantum state of an open system. Within factorized initial conditions, the dynamical map is also completely positive and, for each time t , can be characterized by a set of Kraus operators $\{\mathbf{A}_k(t)\}$. Among the class of completely positive dynamical maps, we introduced the dynamical semigroups and identified the generator of such processes, i.e. a Lindblad generator [11, 12]. The associated Lindblad master equations are usually derived under the Born-Markov approximation, an assumption justified for weakly coupled systems when the relaxation dynamics is much slower than the reservoir correlation functions dynamics.

However, there exist many physical situations which need to be described by more general equations, derived without the Markov or the Born approximation. These non-Markovian systems can be characterized by a dynamics influenced by information flow between the system and the environment. Their evolution presents very interesting properties regarding the dynamics, for instance, of quantum correlations. In this chapter we want to discuss in more detail an established definition of non-Markovianity of quantum processes, introducing a way to measure its degree with particular attention to continuous variable systems.

6.1 Non-Markovianity of quantum dynamical maps

Due to the reduced non unitary dynamics, an open quantum system experiences phenomena like decoherence and dissipation, as discussed in Chapter 4. A given initial state follows a "trajectory" in the space of density operators $\mathcal{D}(\mathcal{H})$, starting from the initial point $\rho(0)$ to the final configuration $\rho(t \rightarrow \infty)$. In many situations the final configuration is the same for any initial state. We then refer to the common final state as the *asymptotic state*, denoted by ρ_∞ . It follows that two initially different states ρ_1 and ρ_2 , become asymptotically indistinguishable. This feature can be interpreted as a loss of information on the initial state configuration due to the complete open dynamics, and appears as a characterizing property in many open system dynamical maps, for instance in dynamical semigroups.

Because the evolution from the initial to the final state is continuous, the information loss is a phenomenon occurring during the whole dynamics and we may then ask if it is possible to monitor its effects. A first step requires the definition and quantification of distinguishability of two quantum states. One possible way is to define a metric in the space of density operators, such as the trace distance [6]

$$\mathcal{D}(\rho_1, \rho_2) = \frac{1}{2} \text{Tr} |\rho_1 - \rho_2|, \quad (6.1)$$

Based on this definition, two quantum states ρ_1 and ρ_2 are said to be distinguishable if $\mathcal{D}(\rho_1, \rho_2) > 0$ and the degree of their distinguishability is provided by the value of their trace distance.

Trace distance also possesses a *contractivity* property under any completely positive map [6], i.e. for any completely positive map Ψ it is

$$\mathcal{D}(\rho_1, \rho_2) \geq \mathcal{D}(\Psi\rho_1, \Psi\rho_2), \quad (6.2)$$

where the equality is satisfied in the case of unitary dynamics. This means that for any given completely positive dynamical map $\Phi(t)$ and for any initial pair of states ρ_1 and ρ_2 , the trace distance decreases. Hence two initially different states become less distinguishable with respect to the initial time $t = 0$. Within the factorized initial condition any dynamical map is CPT, therefore we always have

$$\mathcal{D}(\rho_1(0), \rho_2(0)) \geq \mathcal{D}(\rho_1(t), \rho_2(t)). \quad (6.3)$$

Compared to the initial degree of distinguishability, at any later time two states become less distinguishable, and Eq. (6.3) describes a flow of information about the initial states, from the system to the environment. We may ask now which

are the properties of such flow, e.g. if it is constant or if it is reversible. Stated in another way, we may investigate if it is possible to find two instants of time t_1 and t_2 , such that, if $t_1 < t_2$, then

$$\mathcal{D}(\rho_1(t_2), \rho_2(t_2)) \geq \mathcal{D}(\rho_1(t_1), \rho_2(t_1)). \quad (6.4)$$

To answer this question we need to study the properties of the dynamical map in the interval $[t_1, t_2]$, and in particular its *divisibility*. A completely positive dynamical map (CPT) $\Phi(t)$ is divisible if, for any t and any intermediate time s , it is possible to divide the map into a composition of two CPT maps

$$\Phi(t) = \Phi(t, s) \cdot \Phi(s). \quad (6.5)$$

Among divisible processes we can mention the dynamical semigroups and time-dependent Lindblad maps described by a generator of the form (2.25) with positive decay rates $\gamma_k(t)$ for any $t \geq 0$.

We now can conclude that, if a dynamical map is divisible then the condition (6.4) is never satisfied because of the contractivity property of the trace distance under CPT maps. In this case the trace distance $\mathcal{D}(\rho_1(t), \rho_2(t))$ has a monotonic behavior as a function of time for any $\rho_1(0)$ and $\rho_2(0)$, showing that divisible processes describe phenomena with irreversible flow of information from the system to the environment.

On the other hand, if the map is not divisible we may, or may not, find pairs of initial states and/or time intervals for which Eq. (6.4) is satisfied, thus implying that, in certain cases, the environment is able to give back information (information backflow) to the open system. In [16] this feature is considered to be the key ingredient of non-Markovianity of the open system dynamics. An open quantum system is then Markovian if for any pair of initial states the trace distance is always monotonic as a function of time. On the other hand it is non-Markovian if there exists at least one pair of initial states whose trace distance is not globally monotonic.

We would like to stress here the fact that many dynamical maps derived without the Markov approximation, are indeed Markovian in the sense specified before. For instance many Lindblad type maps of the form (2.25) are usually derived without assumptions on the system-reservoir time scales, but in certain regimes are Markovian.

For completeness we also mention alternative definitions of non-Markovianity present in the literature [124, 125]. These definitions, in many physical situations, do not agree with the concept introduced in [16].

6.2 A measure of non-Markovianity of Gaussian dynamical maps

In addition to the definition of non-Markovianity of quantum processes, in [16,62], the authors introduce a measure $\mathbf{N}(\Phi)$ of the non-Markovian behavior, quantifying how much information flows back from the environment to the system. The expression for the measure reads

$$\mathbf{N}(\Phi) = \max_{\rho_1, \rho_2} \int_{\mathcal{D}' > 0} \mathcal{D}'(\rho_1(t), \rho_2(t)) dt, \quad (6.6)$$

where \mathcal{D}' indicates the time derivative of the trace distance and the maximum value is taken over all the initial pair of states. Definition (6.10) is independent of the specific open system considered, and requires the evaluation of the trace distance between the evolved states, together with a maximization procedure over all the initial states. This task is in general very difficult to implement except for simple situations like qubit systems [62]. When the dimension of the Hilbert space becomes large, or even infinite, the evaluation of the non-Markovianity measure may be computationally demanding.

In paper IV we extend the definition (6.10) to the case of continuous variable dynamical maps, concentrating on Gaussian processes described by master equations of the form (4.14) and (4.18). Because there is no available expression for the trace distance of harmonic oscillator states, we need to introduce another suitable distance measure. The choice employed in the paper is the fidelity [6], defined as

$$\mathcal{F}(\rho_1, \rho_2) = \text{Tr} \sqrt{\sqrt{\rho_1} \rho_2 \sqrt{\rho_1}}. \quad (6.7)$$

The fidelity is not a proper distance measure but it is related to the Bures distance as follows

$$\mathcal{D}_{\mathcal{F}}(\rho_1, \rho_2) = \sqrt{2 - 2\sqrt{\mathcal{F}(\rho_1, \rho_2)}}. \quad (6.8)$$

Moreover it is $\mathcal{F}(\rho_1, \rho_2) \leq \mathcal{F}(\Phi\rho_1, \Phi\rho_2)$ for any CPT map Φ , i.e., as the trace distance $\mathcal{D}_{\mathcal{F}}$ is contractive. We can then use the definition of fidelity to study the property of information flow and define a non-Markovianity measure.

An analytical expression for the fidelity of CV states is known only for the class of Gaussian states [126]. Hence the measure we define in paper IV is limited to those states only and to dynamical maps preserving their Gaussian structure. These are all the classes of maps derived under the assumption

that the system-environment interaction is at most bilinear in the dynamical variables, e.g. the quantum Brownian motion modes. The analytic expression of fidelity depends only on the covariance matrices $\boldsymbol{\sigma}_1$ and $\boldsymbol{\sigma}_2$ and mean quadrature vectors $\bar{\mathbf{X}}_1$ and $\bar{\mathbf{X}}_2$

$$\mathcal{F}(\rho_1^G, \rho_2^G) = \frac{2}{\sqrt{\Delta + \delta} - \sqrt{\delta}} e^{-\frac{1}{2}(\bar{\mathbf{X}}_1 - \bar{\mathbf{X}}_2)^T (\boldsymbol{\sigma}_1 + \boldsymbol{\sigma}_2)^{-1} (\bar{\mathbf{X}}_1 - \bar{\mathbf{X}}_2)}, \quad (6.9)$$

with $\Delta = 4\text{Det}[\boldsymbol{\sigma}_1 + \boldsymbol{\sigma}_2]$ and $\delta = (4\text{Det}[\boldsymbol{\sigma}_1] - 1)(4\text{Det}[\boldsymbol{\sigma}_2] - 1)$. The Gaussian non-Markovianity measure is then defined as

$$\mathbf{N}(\Phi) = \max_{\rho_1^G, \rho_2^G} (-1) \int_{\mathcal{F}' < 0} \mathcal{F}'(\rho_1^G(t), \rho_2^G(t)) dt, \quad (6.10)$$

where the maximum is taken over all possible pairs of Gaussian states (ρ_1^G, ρ_2^G) , parameterizable by ten parameters, i.e. two squeezing complex amplitudes, two displacement complex amplitudes and two real thermal parameters. A summary of the main results of the paper can be found in Chapter 8.

Chapter 7

Quantum key distribution

Cryptography is a branch of information science aiming at studying and developing techniques to share valuable information among selected users in a secure way [127]. A message is shared in a secure way if any illegitimate user, or eavesdropper, cannot obtain the information contained in the message without being noticed, or if the information is obtained after its value is lost.

Classical cryptography is essentially based on two main ingredients: message encryption and key distribution. The information is first encrypted using a secret key chosen by the sender and then can travel securely through a communication channel. Any eavesdropper cannot, in fact, access the message without the knowledge of the key. However legitimate users need also the key to decrypt the content of the message. Therefore the subject of cryptography reduces to the study of protocols to distribute the secret key securely (key distribution).

In classical key distribution, i.e. where the key is stored in strings of classical bits and sent through standard communication channels, conceptually different protocols for key distribution have been developed. For instance, in public key distribution [128], the receiver Bob generates a private key from which he also generates a public key. The public key is distributed to the sender of the message, Alice, who uses it to encrypt the message. An efficient decryption also needs the knowledge of the private key, thus only Bob can decrypt the message in a reasonable amount of time. Security of public key distribution protocols is based on computational complexity of a given mathematical function. While decryption with the private key needs a small time to be performed, any eavesdropper not possessing this knowledge will take a longer time to break the code, and hopefully obtain the message when the

information contained in it has already lost its value. An example of a computationally complex problem is prime factorization of large input numbers for which any classical algorithm fails to be efficient, ensuring security of the key distribution protocol.

With the development of quantum computation, certain problems, considered unsolvable in useful time with classical computers, could be instead efficiently solved using quantum algorithms. In the case of factorization, for instance, Shore's quantum algorithm is able to factorize numbers in a time increasing only polynomially with the size of the input. [129]. Similar arguments can be formulated for other kinds of classical distribution protocols, e.g. the one time pad by Vernam [130]. An alternative to classical key distribution would then be useful in the future, since quantum computers would be able to break cryptographic protocols based, e.g. on large integer factorization.

These are some of the reasons for the birth of a quantum version of cryptography and in particular, key distribution. Quantum key distribution (QKD) aims at a secure distribution of encryption-decryption keys using quantum information sent through quantum communication channels. Security of these protocols is based on key principles of quantum mechanics, such as no-cloning theorem [131], entanglement [22] or unavoidable perturbations caused by quantum measurements [132]. The first protocol presented in the literature, known as BB84 [133], is based on transmission of single qubit states randomly prepared in certain pure states. An eavesdropper does not know in which states the qubits have been prepared and therefore makes a random choice of measurement basis which 50% of the times is wrong. After the transmission, the sender and the receiver, comparing their data, are able to conclude if there has been an external interference in the communication. More details can be found, e.g., in [134].

In general external interferences in the communication may come from two different sources. The first one is the natural noise level of the communication channels that, unavoidably, introduces errors in the information exchanged. The second is the presence of eavesdroppers that can actually exploit the noise of the channel to hide their presence and listen undetected. Quantum key distribution protocols need then to take into account these important sources of communication errors.

7.1 Coherent state protocol in noisy channels

In this section we concentrate on continuous variable QKD [27, 78, 135–138] and introduce one example of CV protocols based on transmission of coherent states of light [27, 78, 137]. The aim of coherent state protocols is to generate a common symmetric key using successful communication of coherent states.

The sender Alice generates pairs of real numbers (x_i, p_i) each distributed according to a Gaussian distribution with zero mean and variance Σ^2 . We assume for simplicity that the variance is the same for both elements of the pair. Each pair is then encoded in a coherent state $|\alpha_i\rangle$ of complex amplitude $\alpha_i = x_i + ip_i$ which is then sent to the receiver Bob through a noisy quantum channel. In the original version of coherent state QKD protocol, the channel noise is described by a Lindblad master equation of the form (4.16). Any coherent state evolves as follows

$$|\alpha_i\rangle \rightarrow |e^{-\gamma T} \alpha_i\rangle \equiv |\alpha_i \sqrt{\eta_M}\rangle, \quad (7.1)$$

where $\eta_M = e^{-2\gamma T}$ is the overall channel transmission, and T is the propagation time of the coherent state. If light modes are used to convey the information, then $T = L$ ($c = 1$) with L being the physical channel length. For any received state Bob performs homodyne detection measuring randomly one of two orthogonal quadratures, e.g. \mathbf{x}_0 and $\mathbf{x}_{\pi/2}$. Due to channel noise, however, its results will be distributed around the point $(x_i \sqrt{\eta_M}, p_i \sqrt{\eta_M})$ with an uncertainty $\sigma_0^2 = 1/2$. In order to retrieve the original encoded value, Bob needs to rescale the measurement, i.e. to measure the quadratures $\mathbf{x}_0/\sqrt{\eta_M}$ and $\mathbf{x}_{\pi/2}/\sqrt{\eta_M}$. The results will be then statistically distributed around (x_i, p_i) , with an amplified noise $\sigma^2 = \sigma_0^2/\eta_M$. The key distribution ends with a post transmission communication between Alice and Bob through a classical channel, followed by reconciliation and privacy amplification protocols to extract a key depending on the channel overall noise [127, 139]. The key is given by a subset of all the pairs (x_i, p_i) , the higher the noise the smaller the subset. For highly noisy lines the number of transmitted pairs needs to be very large and the protocol becomes demanding from a resource point of view. Moreover, as we will see next, the higher the noise the higher is the risk for a successful eavesdropping attack.

7.1.1 Cloning machine attack

We now consider the coherent state protocol in the presence of an eavesdropper. Here we concentrate on a single Gaussian attack strategy performed locally at

a certain position L_E (or time t_E) along the line. We assume that the eavesdropper Eve possesses a complete knowledge about the channel properties, i.e. length and total transmission value, and can listen to any communication performed using classical information channels. Finally we also provide Eve with the power to substitute arbitrary parts of the transmission line. For instance Eve may decide to cut a certain part of the noisy channel and substitute it with a completely noiseless line. These extreme hypothesis are usually employed in QKD to develop the most secure protocol compatible with the laws of physics.

The best Gaussian attack that can be performed by Eve is obtained by implementing an asymmetric optimal cloning on each coherent state [140,141]. One of the clones is sent to Bob, while the other is preserved in a quantum memory, waiting for Alice and Bob post protocol communication, i.e. when the choice of quadrature measurements is revealed. In the case of noisy channels of the form (4.16), the cloning process is simulated using a simple beam splitter. After the attack, one of the clones is sent to Bob using a completely noiseless line. In order to hide her presence, Eve needs to tune the attack depending on her position on the line. If she attacks at time t_E , she has to carefully tune the beam splitter transmissivity to $\eta_E = e^{-2\gamma_M(T-t_E)}$, so that Bob always receives the same states that he would receive in the absence of any eavesdropper. Hence he cannot conclude that the communication was intercepted. The information collected by Eve's attack increases with decreasing values of η_E , meaning that the optimal attack consists in attacking at the beginning of the line ($t_E = 0$).

Once the communication is completed both Eve and Bob possess some information about the chosen pairs (x_i, p_i) . If the properties of the channel allow Eve to use a cloning machine with $\eta_E < 1/2$, then her information on the states sent by Alice is bigger than Bob's, i.e. the results of her measurements are more precise than those of Bob. However this is not equivalent to a successful attack as it can be demonstrated that the success depends on the chosen reconciliation strategy by Alice and Bob.

For a direct reconciliation protocol [137] the success depends on the overall transmission of the line: if $\eta_M \geq 1/2$ then $\eta_E \geq 1/2$ and therefore Eve, even if not detected, cannot access the same amount of information as Bob, and a key can be extracted securely. The key rate generation however approaches zero as $\eta_M \rightarrow 1/2$. For $\eta_M < 1/2$ (highly noisy lines), the direct reconciliation is not secure anymore because Eve possesses more information than Bob on Alice's states.

In this situation it is possible to adopt a reverse reconciliation protocol [142,

143], where both Alice and Eve have to guess the results of Bob measurements, with Alice always being advantaged in the process. The main problem of reverse reconciliation methods is that, even if they ensure the extraction of a key, they require a great amount of resources in terms of number of states to be communicated.

The coherent state protocol represents one of the main achievements of CV quantum key distribution, for its theoretical simplicity and easy technical implementation. For low noise lines ($\eta < 1/2$) its original version ensures a successful secure key generation with limited amount of resources needed. For higher noise lines, in our opinion, the protocol loses its feasibility and advantages compared to other practical key distribution methods. In the next section we briefly introduce a novel proposal, discussed in detail in paper V, where a scheme for eavesdropping detection and secure key distribution is devised exploiting the properties of non-Markovian noisy channels.

7.2 Quantum key distribution in non-Markovian channels

Lindblad channels of the form (4.16) are the simplest models of noisy dynamics in CV systems. This constitutes the main reason why in the literature losses and noise in QKD protocols have been considered to be determined by these equations. In paper V we consider the coherent state QKD protocol when the non-unitary dynamics is described by alternative models, such as Eqs. (4.18) and (4.14). This extension represents the first attempt in the direction of exploiting the non-Markovian properties of quantum channels in quantum key distribution. It is also one of the very few examples of non-Markovian open system dynamics in the field of quantum information and computation.

The coherent state protocol can be formulated in the same way in the presence of these alternative dynamical equations. First Alice encodes pairs of Gaussian distributed numbers into coherent states and sends them to Bob who performs homodyne detection of one of two orthogonal quadratures. In the case of Eq. (4.18) the dissipation is not uniform along the whole channel as in the Lindblad case. The channel is characterized by a total transmission $\eta_{NM}(T) = \exp\{-\Gamma(T)\}$, where $\Gamma(T) = 2 \int_0^T \gamma(t) dt$ and T is the transmission time. The best eavesdropping strategy is again to attack at the beginning of the channel with a beam splitter of transmissivity $\eta_E = \eta_{NM}(T)$.

If instead the dynamics is described by Eq. (4.14), the state undergoes both dissipation and decoherence, thus accumulating extra, or added, noise

which further reduces the fidelity of the communication. Also the eavesdropping attack needs to be adapted and performed using an appropriate cloning protocol in the presence of added noise [143].

Under the usual eavesdropping strategy assumptions, e.g. knowledge of the channel properties, the extension of the protocol in the case of new transmission channels does not bring any particular advantage to the existing techniques in coherent state QKD. The protocol is still secure when $\eta_{NM} \geq 1/2$ for direct reconciliation and for any value of η_{NM} when using a reverse reconciliation protocol.

In paper *V* we propose to add an element to the protocol which exploits the time-dependent dissipative properties of the channels (4.18) and (4.14). This element consists in the transmission of extra *reference* coherent states $|\alpha_0\rangle$ hidden in between the states containing the transmitted key, namely *key states*. While the key states travel along the standard channel of length T and dissipation $\gamma(t)$, the reference states are sent, randomly, along two different channels. The first is standard channel while the second is a channel of length $T + \Delta t$ obtained by elongating the line on Alice side, a place, we assume, not accessible to the eavesdropper (see Fig. 7.1). We also assume that only Alice knows about the channel choice and the sequence of states, forcing both Bob and Eve to treat every received state on an equal foot.

At the end of the transmission Alice communicates the sequence of states to Bob, who can study the statistics of the reference states, separating the results of the states sent along the two different channels. All the measurement distributions have the same width but a different mean value. For the channel (4.18), in the absence (presence) of an eavesdropper the mean value difference of the distributions of the states sent along the two channels denoted by δx_{NE} (δx_E), reads

$$\delta x_{NE} \simeq \alpha_0 \Delta t \gamma(T), \quad \delta x_E \simeq \alpha_0 \Delta t \gamma(t_E), \quad (7.2)$$

where t_E is the Eve's attacking time. If Bob is able to discriminate the difference $\delta x_{NE} - \delta x_E$, i.e. if Eve attacks when $\gamma(t_E) \neq \gamma(T)$, he concludes that an eavesdropper was listening to the communication and takes the necessary actions.

Coherent state QKD is performed experimentally using optical systems, i.e. photon states propagating in optical fibers. These situations are actually well described within Markovian, i.e. Lindblad form, open system models. To implement the non-Markovian extension just described one needs to rely on reservoir engineering techniques in order to modify the spectral properties

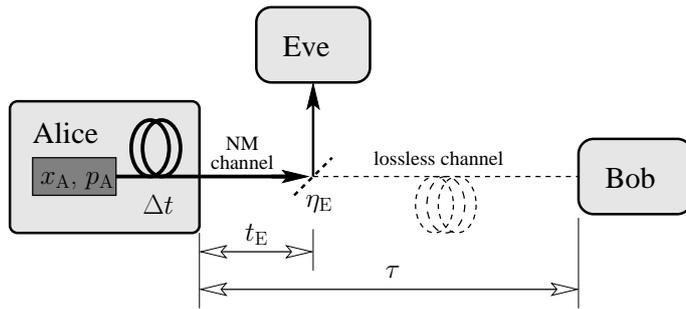


Figure 7.1: (a) Schematic of the QKD protocol in a non-Markovian damping channel. Eve attacks at a distance time $L_E = ct_E$ from Alice location, with a beam splitter of transmissivity η_E and substitutes the rest of the channel with a lossless line. Alice, instead, can choose on her own to elongate the channel transmission time of a quantity Δt , thus implementing the protection strategy.

of the environment and achieve an appropriate position dependent damping dynamics.

A further discussion on the main results of the paper can be found in the next chapter.

Chapter 8

Summary of results and conclusions

This chapter contains brief summaries of thesis papers contents, useful to guide the reader through the actual manuscripts. Last section is, instead, dedicated to final remarks about the thesis work.

8.1 Summary of papers results

8.1.1 Paper I

In this paper we investigate the dynamics of entanglement for a system of two non interacting modes coupled to independent reservoirs. In the weak coupling limit the dynamics is described by the master equation (5.24) with the time-dependent coefficients (4.9).

We consider different kind of environments, characterized by a mode spectral distribution of the form

$$J(\omega) \propto \omega^s e^{-\omega/\omega_c}, \quad (8.1)$$

with different s and in different thermal stationary states, at zero and high temperatures. The exponential function in (8.1) has the role of an ultraviolet cutoff, with a range determined by the cutoff frequency ω_c . We also concentrate on a particular class of entangled initial states, i.e. the twin beam (TWB) vacuum states, obtained by applying the two mode squeezing operator (3.18) to the vacuum state of the two modes.

The analysis conducted in the paper is focused on two main aspects. First we study the validity of the secular approximation, i.e. if it is possible or not to neglect fast oscillating terms appearing in the solution of the equation (see Eqs. 20a – d in the attached manuscript). The second aspect is the investigation of the dynamical features of entanglement for different reservoirs spectral functions (8.1) and temperature regimes. Entanglement is here measured using the Gaussian entanglement of formation (5.11).

The master equation (5.24), derived under the only assumptions of factorized initial condition and weak coupling, may describe a Markovian or non-Markovian dynamics depending on the values of the system reservoir parameters, and in particular on the ratio between the cut-off frequency and the free frequency, i.e. $x = \omega_c/\omega_0$. The results of this paper are precedent to the introduction of the non-Markovianity measure [16], hence we did not use the methods reported therein to derive conditions for non-Markovianity. However the Markovian/non-Markovian character of the dynamics can be partially established looking at the time evolution of entanglement. In Fig. 8.1 (a) we

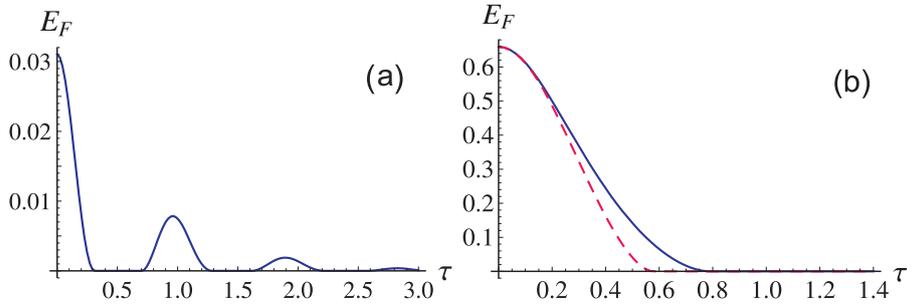


Figure 8.1: (a) Gaussian Entanglement of Formation for an Ohmic reservoir in the high-T regime, for $x = 0.15$ (out of resonance), $\alpha = 0.1$, $k_B T/\hbar\omega_c = 100$, as a function of $\tau = \omega_c t$. (b) Gaussian Entanglement of Formation for an Ohmic reservoir in high-T, for $x = 10$ (resonant case), $\alpha = 1$, $k_B T/\hbar\omega_c = 100$. The blue solid line corresponds to the exact solution, while the red dashed line gives the solution under the secular approximation.

show an example of entanglement evolution characterized by oscillations, sudden death and revivals, typical of the out of resonance regime $x \ll 1$, for an Ohmic reservoir at high temperatures. For independent reservoirs these oscillations witness a failure of the divisibility condition of the dynamical map (6.5), which is at basis of the non-Markovianity measure introduced in [125],

and it is a necessary condition for the measure introduced in [16]. On the other hand in the resonance regime $x \gg 1$, Fig. 8.1 (b), oscillations are not present and without further investigations we cannot infer anything about a possible non-Markovian behavior in this regime.

The results of the paper can be summarized as follows. First we prove that, regarding the entanglement dynamics, the secular approximation can be performed in the out of resonance regime only ($x \ll 1$). In the resonance regime instead the approximation fails, in agreement with what is reported in a previous publication [94]. We also find that entanglement is more robust in the case of Ohmic ($s = 1$) and Subohmic ($s = 1/2$) spectra, while it vanishes faster for a SuperOhmic ($s = 3$) spectrum. Also Gaussian entanglement oscillations [18] are evident in the out of resonance regimes only in the Ohmic and SubOhmic case, a result independent, qualitatively, on the temperature of the bath.

8.1.2 Paper II

In paper II we analyze the dynamics of entanglement, measured by logarithmic negativity (5.7), Gaussian quantum discord and intensity correlations in various QBM open system models. We consider both the independent (5.24) and common (5.26) reservoir scenarios in the case of an Ohmic spectral distribution with Lorentz-Drude cutoff [9]

$$J(\omega) = \frac{\omega_c^2}{\pi} \frac{\omega}{\omega^2 + \omega_c^2} \quad (8.2)$$

in the high temperature regime, exploiting the results of paper I about the limits of validity of the secular approximation.

The purpose of the work is to study the qualitative and quantitative differences in the dynamics of various indicators of quantum correlations. In the independent reservoir case, we consider initial two-mode thermal squeezed states, obtained by applying the two mode squeezing operator (3.18) to a product of thermal states for the two modes. We find that intensity correlations disappear faster than entanglement and quantum discord (see Fig. 8.2 (a)). Moreover quantum discord only disappears asymptotically in time due to the characterizing property of Gaussian discord which is zero only for completely uncorrelated states. Intensity correlations and entanglement instead disappear at a finite time (sudden death). These results are qualitatively independent on the value of the resonant parameter $x = \omega_c/\omega_0$, except for the presence of oscillations in the out of resonance case ($x \ll 1$).

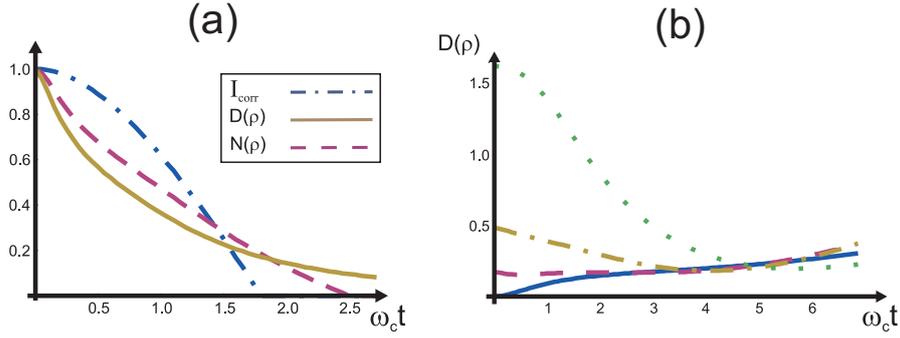


Figure 8.2: (a) Intensity correlations (dotted dashed blue line), Gaussian quantum discord (solid yellow line), Logarithmic Negativity (dashed red line) for an Ohmic reservoir at high-T, for $x = 10$, $k_B T / \hbar \omega_c = 100$ and $\alpha = 0.1$. (b) Gaussian quantum discord for an Ohmic reservoir at high-T, for $x = 10$, $k_B T / \hbar \omega_c = 100$ and $\alpha = 0.1$. The different curves correspond to different initial pure TWB states: $r = 0$ (solid blue), $r = 0.2$ (dashed red), $r = 0.4$ (dotted dashed yellow) and $r = 1$ (dotted green).

In the common reservoir case, as expected, initially uncorrelated states may become correlated at later times, but, in the weak coupling limit, they remain separable. Quantum discord is always increasing starting from simple symmetric thermal state with a rate inversely proportional to the value of the temperature parameter of the state. We also studied the evolution of quantum discord in the case of initially correlated states (squeezed vacuum states) and shown that, independently from the initial value of quantum correlations, the open dynamics drives the system towards states with similar amount of discord (see Fig. 8.2 (b)). This property can be considered an effect of loss of information on the initial state due to the open system dynamics. In view of the results about non-Markovianity in CV systems (Chapter 6) initial different states become less and less distinguishable during the evolution and therefore show also similar amount of quantum correlations.

8.1.3 Paper III

In paper III we extend the analysis of paper I in the case of asymmetrical initial Gaussian states, for an asymmetric system reservoir coupling and for different oscillator frequencies. The environments are characterized by an

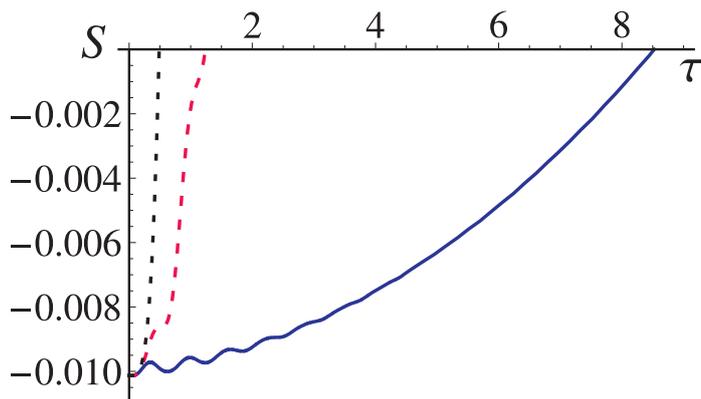


Figure 8.3: Separability as a function of $\tau = \omega_c t$ for $x_1 = \omega_c/\omega_1 = 0.1$, $k_B T/\hbar\omega_c = 100$ and $\alpha = 0.1$. $x_2 = 0.1$ (blue solid curve), $x_2 = 0.2$ (red dashed curve) and $x_2 = 0.3$ (black dotted curve).

Ohmic spectral structure with Lorentz-Drude cutoff function, and entanglement is witnessed by the Separability function, obtained by applying directly the Peres-Simon criterion [99].

The results obtained are in agreement with what has been presented in paper I. However, in this case we allow two different values for the resonance parameters x_1 and x_2 such that we can explore different dynamical regimes. The most interesting result is found when, for instance, the first oscillator is in resonance with the environment while the second oscillator is not. In this case we may still observe entanglement oscillations, and for certain initial states sudden death and revivals. However the intensity of these effects is reduced due to the choice of $x_1 \neq x_2$ as in Fig. 8.3.

This result can be interpreted in terms of non-Markovianity of the system dynamics, supposing that the total contribution is provided by single system-environment interaction. As one of the parameters x_i increases, the behavior of the corresponding oscillator becomes less and less non-Markovian. Hence signatures of non-Markovian behavior, e.g. entanglement oscillations, are less evident.

8.1.4 Paper IV

In paper IV we study the non-Markovianity properties of Gaussian dynamical maps using the concepts introduced in [16]. We concentrate on two examples

of single mode dynamical maps, described by the phenomenological master equation (4.18) and the microscopic model (4.14), for initial Gaussian states. For this purpose we first introduce, for the first time, an extension of the measure [16] based on the use of fidelity of two-mode Gaussian states, rather than on the original idea based on the trace distance.

The non-Markovianity condition coincides with divisibility property of the channels in the case of Eq. (4.18), i.e. the backflow of information is determined by the negativity periods of the damping coefficient $\gamma(t)$. On the contrary in the secular approximated QBM channel (4.14) case, the two properties do not coincide. We found, however, that for an Ohmic bath the fidelity essentially increases when the diffusion coefficient $\Delta(t)$ is negative, a result mathematically correct at the first order in the coupling constant. In the resonance case $\omega_0 < \omega_c$ the diffusion coefficient is never negative, therefore non-Markovian behavior is observed only in the opposite case. This results agrees with the previous studies on correlations dynamics (papers I-III) where we found that oscillations are present only in the off resonant case.

The non-Markovianity measure is then evaluated concentrating on subclasses of Gaussian states, and in particular pure coherent states and pure squeezed states. A first result shows that squeezed states are more sensitive to information backflow than coherent states, therefore the measure restricted to squeezed initial states is higher than that restricted to initial coherent states. In Fig. 8.4(a) we show this result for the damping channel (4.18). Moreover our results indicate that the maximization is obtained in the case of initial pure states.

Another interesting result valid for the QBM channel (4.14) is the presence of a saturation effect for the measure restricted to squeezed states, shown in Fig. 8.4(b). The value of the measure is independent of the coupling constant but only depends on the temperature of the bath and the relative strength of the free frequency ω_0 and cutoff frequency ω_c . This saturation effect implies the existence of a bound of information backflow from the environment to the system, due to the Gaussian structure of the map.

8.1.5 Paper V

In this paper we study the extension of continuous variable quantum key distribution protocols in the case of non trivial communication lines. We consider the coherent state protocol in the case of channels losses described by phenomenological equations like (4.18) and microscopic models like (4.14), and we prove that it is possible to enhance security of the protocol for both direct and

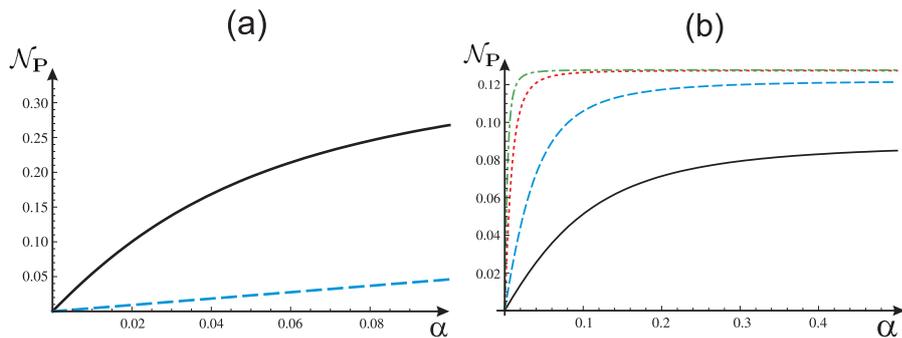


Figure 8.4: (a) Non-Markovianity measure for a damping channel as a function of the coupling constant α . The solid black line corresponds to the squeezed states measure while the blue dashed line is the measure for only coherent states. The form of the damping coefficient is given in Eq. (22) of paper IV. (b) Non-Markovianity measure for the secular approximated QBM channel as a function of the coupling α and restricted to squeezed states only. We choose $\omega_c = 0.2$, $\omega_0 = 1$ and the different curves correspond to different values of the temperature: $k_B T / \hbar \omega_0 = 0.3$ (black solid), $k_B T / \hbar \omega_0 = 0.9$ (blue dashed), $k_B T / \hbar \omega_0 = 4$ (red dotted) and $k_B T / \hbar \omega_0 = 8$ (green dotted dashed).

reverse reconciliation methods.

As reported in more detail in Sec. 7.2 the scheme consists in the sender Alice transmitting, together with the key signals, also some reference coherent states. These reference states are sent through two different channels, the usual one of length T and a longer channel $T + \Delta t$, and the information about the channel choice is revealed only at the end of the communication.

By checking the distribution of measurements on the reference states it is possible to check the presence of an eavesdropping attack. The success of the method relies on the time/position dependence of the eavesdropping attack and on the impossibility to tune the attack power without knowing the Alice's choice of the channel used for the transmission of the reference states. In Fig. 8.5 we present an example of channel decay rate suitable to apply our eavesdropping detection protocol. As it can be seen, assuming ϵ to be the minimum detectable error in the protection strategy, we are able to find two attack regions, namely a danger zone and a safe zone. The threshold time τ between the danger and safe zone is defined by the condition

$$\gamma(T) - \gamma(\tau) = \frac{\epsilon}{|\alpha_0| \Delta t}, \quad (8.3)$$

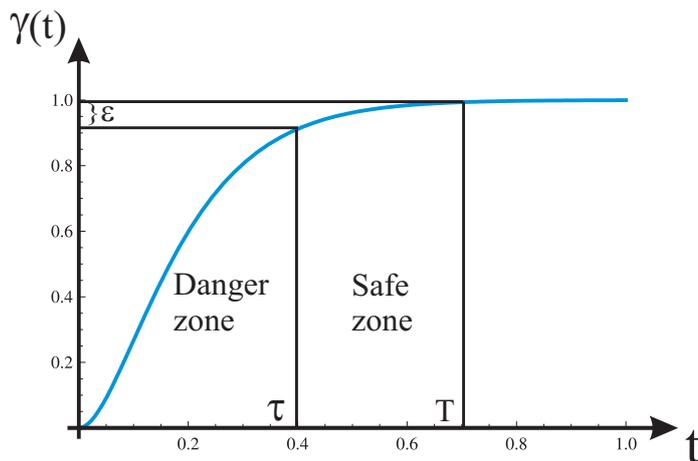


Figure 8.5: Example of suitable decay rate $\gamma(t)$ allowing the eavesdropping detection scheme. The transmission time is denoted by T , while τ denotes the threshold time separating the safe and the danger zone.

If the eavesdropper attacks in the danger zone then Bob's post transmission results are able to detect the attack. It follows that the eavesdropper is forced to attack in the safe zone and therefore to reduce the attack power, all in favor of an enhancement of the security of the protocol for both direct and reverse reconciliation. In the first case we can extend the security limit reported in [137] for channels of total transmission $\eta_{NM} > \exp\{-\Gamma(t_E)\}/2$, and in the second case we simply reduce the amount of resources needed to perform a secure communication for any value of the total channel transmission.

8.2 Conclusion

In this thesis we explored the concept and the properties of non-Markovian dynamics in continuous variable systems. Non-Markovianity turns out to be a dynamical feature which leaves signatures of its presence in many observable quantities. It is, for instance, responsible for oscillating dynamics, death and revivals of entanglement and more general quantum correlations. In the case of light beams it manifests its presence in the experimental data of joint photodetection. We have shown that non-Markovianity can be exploited as a resource in quantum communication protocols, as proven by the eavesdropping

detection scheme for QKD with coherent states.

The definition of non-Markovianity as a resource requires, indeed, a measure for its degree. We have therefore extended to the realm of continuous variable Gaussian states the general definition introduced in [16]. We have demonstrated that different states are more or less sensible to the backflow of information, and therefore some class of states, the squeezed states, may be more useful in detecting non-Markovianity and exploiting its properties.

The exploration of non-Markovian dynamical systems is still, in many regards, at its infancy. Only few examples of the possible use of memory effects as a resource have been studied so far, our being the first result in quantum cryptography. In many optical systems non-Markovianity still turns out to be a very elusive property, as its effect can be seen only at the beginning of the evolution, i.e. in a non-Markovian time scale. In these systems, in order to enhance non-Markovian effects, one needs to modify the properties and the interaction between the system and the environment in the spirit of reservoir engineering techniques. Examples of this strategy can be already foreseen, e.g. using properly engineered photonic band gap materials.

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