14

DISSERTATIO ACADEMICA

THEORIAM ÆQUATIONUM FUNCTIONALIUM DUARUM VARIABILIUM EJUSQUE IN DOCTRINA SERIERUM USUM EXHIBENS;

QUAM

CONSENSU AMPLISS. FACULTATIS PHILOSOPH.
AD IMPERIALEM ACAD. ABOËNSEM,

PRÆSIDE

Mag. NATH. G. AF SCHULTÉN,
Mathematum Professore Publ. & Ord.,
Acad. Imperialis Scientiarum Petropolitanæ
Socio Corresp.,

PRO GRADU PHILOSOPHICO

P. P.

JOHANNES EPHRAIM AHLSTEDT,
Satacundensis.

In Audit. Philos. die XVI Junii MDCCCXXVII.
horis a. m. solitis.

P. V.

ABOÆ, Typis FRENCKELLIANIS.

$$\Sigma^n \text{ Sin } (b+cx) = \frac{\mp 2^n \text{ Sin } \tfrac{1}{2} c^n . \text{ Sin } (b - nc + cx + \tfrac{n}{2}.c)}{2^{2n} \text{ Sin } \tfrac{1}{2} c^{2n}}$$

$$= \mp \frac{\text{Sin } (b + cx - \tfrac{n}{2}.c)}{2^n \text{ Sin } \tfrac{1}{2} c^n},$$

adhibito nimirum signo —, si n inter numeros est

2, 6, 10, 14, &c.,

+ vero, si inter

4, 8, 12, 16, &c.;

nec non, si n *impar*,

$$\Sigma^n \text{ Sin } (b+cx) = \frac{\mp 2^n \text{ Sin } \tfrac{1}{2} c^n . \text{ Cos } (b - nc + cx + \tfrac{n}{2}.c)}{2^{2n} \text{ Sin } \tfrac{1}{2} c^{2n}}$$

$$= \mp \frac{\text{Cos } (b + cx - \tfrac{n}{2}.c)}{2^n \text{ Sin } \tfrac{1}{2} c^n},$$

obtinente —, si n inter numeros est

1, 5, 9, 13, &c.,

+ vero, si inter

3, 7, 11, 15, &c. *)

 Obti-

---

*) Allatos istos valores simplicissimos seriei

$$\text{Sin } (b - nc + cx) - n \text{ Sin } (b - nc + cx + c) + \frac{n(n-1)}{1.2} \text{ Sin } (b - nc + cx + 2c) - \&c.$$

Obtinebit igitur tabella valorum particularium sequens, quam in promtu habere commodum saepe est

Σ

---

ex $n+1$ terminis compositæ, sequenti investigari modo observasse convenit. Posito brevitatis ergo

$$b - nc + cx = d,$$

nec non

$$\mathrm{Sin}\, d - \mathrm{Sin}\,(d+c) = -2\,\mathrm{Sin}\,\tfrac{1}{2}c\,.\,\mathrm{Cos}\,(d+\tfrac{1}{2}c) = \alpha_d$$

$$\mathrm{Cos}\,(d+\tfrac{1}{2}c) - \mathrm{Cos}\,(d+\tfrac{3}{2}c) = +2\,\mathrm{Sin}\,\tfrac{1}{2}c\,.\,\mathrm{Sin}\,(d+c) = \beta_d;$$

obtineri perspicuum facile est æquationes sequentes

$$\mathrm{Sin}\, d - \mathrm{Sin}\,(d+c) = -2\,\mathrm{Sin}\,\tfrac{1}{2}c\,.\,\mathrm{Cos}\,(d+\tfrac{1}{2}c) = \alpha_d,$$

$$\mathrm{Sin}\, d - 2\,\mathrm{Sin}\,(d+c) + \mathrm{Sin}\,(d+2c)$$

$$= \alpha_d - \alpha_{d+c} = -2\,\mathrm{Sin}\,\tfrac{1}{2}c\,(\mathrm{Cos}\,(d+\tfrac{1}{2}c) - \mathrm{Cos}\,(d+\tfrac{3}{2}c))$$

$$= -2\,\mathrm{Sin}\,\tfrac{1}{2}c\,.\,\beta_d = -2\,\mathrm{Sin}\,\tfrac{1}{2}c\,.+2\,\mathrm{Sin}\,\tfrac{1}{2}c\,\mathrm{Sin}\,(d+c)$$

$$= -2^2\,\mathrm{Sin}\,\tfrac{1}{2}c^2\,.\,\mathrm{Sin}\,(d+c) = \alpha'_d,$$

$$\mathrm{Sin}\, d - 3\,\mathrm{Sin}\,(d+c) + 3\,\mathrm{Sin}\,(d+2c) - \mathrm{Sin}\,(d+3c)$$

$$= \alpha'_d - \alpha'_{d+c} = -2^2\,\mathrm{Sin}\,\tfrac{1}{2}c^2\,(\mathrm{Sin}\,(d+c) - \mathrm{Sin}\,(d+2c))$$

$$\Sigma \text{ Sin } (b + cx) = - \frac{\text{Cos } (b+cx-\frac{1}{2}c)}{2 \text{ Sin } \frac{1}{2} c}$$

$$\Sigma^2 \text{ Sin } (b+cx) = - \frac{\text{Sin } (b+cx-c)}{4 \text{ Sin } \frac{1}{2} c^2}$$

Σ^3

$= -2^2 \text{ Sin } \frac{1}{2} c^2 \, . \, \alpha_{d+c} = -2^2 \text{ Sin } \frac{1}{2} c^2 \, . -2 \text{ Sin } \frac{1}{2} c \text{ Cos } (d+\frac{3}{2}c)$

$= + 2^3 \text{ Sin } \frac{1}{2} c^3 \text{ Cos } (d + \frac{3}{2} c) = \alpha''_d,$

$\text{Sin } d - 4 \text{ Sin}(d+c) + 6 \text{Sin}(d+2c) - 4 \text{Sin}(d+3c) + \text{Sin}(d+4c)$

$= \alpha''_d - \alpha''_{d+c} = + 2^3 \text{ Sin } \frac{1}{2} c^3 (\text{Cos } (d+\frac{3}{2}c) - \text{Cos } (d+\frac{5}{2}c))$

$= + 2^3 \text{ Sin } \frac{1}{2} c^3 . \beta_{d+c} = + 2^3 \text{ Sin } \frac{1}{2} c^3 . + 2 \text{ Sin } \frac{1}{2} c \text{ Sin } (d+2c)$

$= + 2^4 \text{ Sin } \frac{1}{2} c^4 \text{ Sin } (d+2c) = \alpha'''_d,$

$\text{Sin } d - 5 \text{ Sin } (d+c) + 10 \text{ Sin } (d+2c) - 10 \text{ Sin } (d+3c)$

$+ 5 \text{ Sin } (d+4c) - \text{Sin } (d+5c)$

$= \alpha'''_d - \alpha'''_{d+c} = + 2^4 \text{ Sin } \frac{1}{2} c^4 (\text{Sin } (d+2c) - \text{Sin } (d+3c))$

$= + 2^4 \text{ Sin } \frac{1}{2} c^4 . \alpha_{d+2c} = + 2^4 \text{ Sin } \frac{1}{2} c^4 . -2 \text{ Sin } \frac{1}{2} c \text{ Cos } (d+\frac{5}{2}c)$

$= -2^5 \text{ Sin } \frac{1}{2} c^5 \text{ Cos } (d+\frac{5}{2}c) = \alpha''''_d,$

$\text{Sin } d - 6 \text{ Sin } (d+c) + 15 \text{ Sin } (d+2c) - 20 \text{ Sin } (d+3c) +$

$15 \text{ Sin}(d+4c) - 6 \text{ Sin } (d+5c) + \text{Sin } (d+6c)$

$$\Sigma^3 \operatorname{Sin}(b+cx) = + \frac{\operatorname{Cos}(b+cx-\frac{3}{2}c)}{8 \operatorname{Sin}\frac{1}{2}c^3}$$

$$\Sigma^4 \operatorname{Sin}(b+cx) = + \frac{\operatorname{Sin}(b+cx-2c)}{16 \operatorname{Sin}\frac{1}{2}c^4}$$

Σ^5

$$= \alpha''''_d - \alpha''''_{d+c} = -2^5 \operatorname{Sin}\tfrac{1}{2}c^5 (\operatorname{Cos}(d+\tfrac{5}{2}c) - \operatorname{Cos}(d+\tfrac{7}{2}c))$$

$$= -2^5 \operatorname{Sin}\tfrac{1}{2}c^5 . \beta_{d+2c} = -2^5 \operatorname{Sin}\tfrac{1}{2}c^5 . + 2 \operatorname{Sin}\tfrac{1}{2}c \operatorname{Sin}(d+3c)$$

$$= -2^6 \operatorname{Sin}\tfrac{1}{2}c^6 \operatorname{Sin}(d+3c) = \alpha'''''_d,$$

sicque porro; unde cum perspiciatur seriem de qua agitur

$$\operatorname{Sin} d - n \operatorname{Sin}(d+c) + \frac{n(n-1)}{1.2} \operatorname{Sin}(d+2c) - \&c.,$$

ad $n+1$ scilicet terminos continuatam, pro $n = 2, 4, 6,$ &c., valores respective accipere particulares

$$-2^2 \operatorname{Sin}\tfrac{1}{2}c^2 \operatorname{Sin}(d+c)$$

$$+2^4 \operatorname{Sin}\tfrac{1}{2}c^4 \operatorname{Sin}(d+2c)$$

$$-2^6 \operatorname{Sin}\tfrac{1}{2}c^6 \operatorname{Sin}(d+3c)$$

. . . .,

nec non, pro $n = 1, 3, 5,$ &c., valores

$$-2 \operatorname{Sin}\tfrac{1}{2}c \operatorname{Cos}(d+\tfrac{1}{2}c)$$

$$+2^3 \operatorname{Sin}\tfrac{1}{2}c^3 \operatorname{Cos}(d+\tfrac{3}{2}c)$$

$$\Sigma^5 \text{Sin}(b+cx) = - \frac{\text{Cos}\,(b+cx-\frac{5}{2}c)}{32\ \text{Sin}\ \frac{1}{2}\ c^5}$$

$$\Sigma^6 \text{Sin}(b+cx) = - \frac{\text{Sin}\,(b+cx-3c)}{64\ \text{Sin}\ \frac{1}{2}\ c^6}$$

$$\Sigma^7 \text{Sin}(b+cx) = + \frac{\text{Cos}\,(b+cx-\frac{7}{2}c)}{128\ \text{Sin}\ \frac{1}{2}\ c^7}$$

$$\Sigma^8 \text{Sin}(b+cx) = + \frac{\text{Sin}\,(b+cx-4c)}{256\ \text{Sin}\ \frac{1}{2}\ c^8}$$

$$\Sigma^9 \text{Sin}(b+cx) = - \frac{\text{Cos}\,(b+cx-\frac{9}{2}c)}{512\ \text{Sin}\ \frac{1}{2}\ c^9}$$

$$\Sigma^{10} \text{Sin}(b+cx) = - \frac{\text{Sin}\,(b+cx-5c)}{1024\ \text{Sin}\ \frac{1}{2}\ c^{10}}$$

. . . .

Per allata hactenus exhiberi igitur ope formulæ b) in terminis finitis possunt

$$\Sigma \frac{gx^p + hx^q + \&c.}{x(x+1)(x+2)\ldots(x+m)} \quad . \ . \ . \ \text{I)}$$

atque

$$\Sigma\,(gx^p + hx^q + \&c.).a^x\ \text{Sin}\ (b+cx) \quad . \ . \ . \ \text{II)},$$

ubi

$$-\ 2^5\ \text{Sin}\ \tfrac{1}{2}\,c^5\ \text{Cos}\,(d+\tfrac{5}{2}c)$$

. . . . ,

per inductionem allatæ supra formulæ facile colliguntur.

ubi observandum tamen, ut ex præcedentibus facile colligitur, est, in forma I) summam ipsius x potestatem in numeratore ad minimum *duabus* unitatibus numero factorum in denominatore inferiorem esse debere.

Generaliores vero quodammodo evadere adhuc posse ipsas I), II), per formulam *b*) determinandas, observasse operae est pretium.

Ad formam scilicet I) in genere reduci potest universalior ista

$$\Sigma \frac{gx^p + hx^q + \&c.}{(x+a)(x+b)(x+c)\ldots(x+l)} \ldots \text{III)}$$

ubi a, b, c, . . . l quantitates sunt quæcumque, sive positivæ sive negativæ (sive etiam aliqua earum = o), quæ numeris scilicet *integris* invicem differant, omnesque etiam inter se inæquales habeantur, obtinente tamen memorata nuperrime conditione summam ipsius x potestatem in numeratore numerumque factorum denominatoris respiciente.

Posita scilicet k *minima* ipsarum a, b, c, ... l, (si positivæ scilicet omnes habeantur, alias autem negativarum ponenda eadem est maxima) per assumtam

sumtam hasce inter quantitates relationem manifestum est haberi

$$\frac{gx^p + hx^q + \&c.}{(x+a)(x+b)(x+c)\ldots(x+l)} = \frac{gx^p + hx^q + \&c.}{(x+k)(x+k+a')(x+k+b')\ldots(x+k+k')},$$

ubi a', b', .. k', *numeri integri positivi* sunt. Facta igitur

$$x + k = y,$$

habebitur

$$\frac{gx^p + hx^q + \&c.}{(x+a)(x+b)(x+c)\ldots(x+l)} = \frac{g(y-k)^p + h(y-k)^q + \&c.}{y(y+a')(y+b')\ldots(y+k')},$$

positoque dein

$$\Sigma \frac{g(y-k)^p + h(y-k)^q + \&c.}{y(y+a')(y+b')\ldots(y+k')} = \psi y,$$

prodibit tandem

$$\Sigma \frac{gx^p + hx^q + \&c.}{(x+a)(x+b)(x+c)\ldots(x+l)} = \psi(x+k). \;{}^{*})$$

Ad

---

*) Quod quidem inde sequitur, quod, posito

$$\Sigma\chi x = \psi x,$$

habebitur necessario

Ad determinationem igitur functionis ψ revocatam videmus formam quæsitam III): determinari

$$\Sigma\chi(x+\alpha) = \psi(x+\alpha),$$

denotantibus χ, ψ functiones quascumque, et α quantitatem quamvis ab x non pendentem. Habetur scilicet, ob assumtam

$$\Sigma\chi x = \psi x,$$

per ideam signi Σ, æquatio

$$\chi x = \psi(x+1)-\psi x$$

pro omnibus ipsius x valoribus identica; hincque igitur

$$\chi(x+\alpha) = \psi(x+\alpha+1)-\psi(x+\alpha),$$

i. e.

$$\chi(x+\alpha) = \psi(x+1+\alpha)-\psi(x+\alpha),$$

identica etiam omnibus pro valoribus ipsius x; unde, per definitionem signi Σ, fiet

$$\Sigma\chi(x+\alpha) = \psi(x+\alpha).$$

Erit igitur vi æquationis

$$\Sigma\,\frac{g(y-k)^p + h(y-h)^q + \&c.}{y(y+a')(y+b')\ldots(y+k')} = \psi y,$$

necessario